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ON THE MEAN VALUE OF THE GENERALIZED DIRICHLET L-FUNCTIONS

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Abstract. Let $q \geqslant 3$ be an integer, let χ denote a Dirichlet character modulo q. For any real number $a \geqslant 0$ we define the generalized Dirichlet L-functions

$$L(s, \chi, a) = \sum_{n=1}^{\infty} \frac{\chi(n)}{(n+a)^s},$$

where $s = \sigma + it$ with $\sigma > 1$ and t both real. They can be extended to all s by analytic continuation. In this paper we study the mean value properties of the generalized Dirichlet L-functions especially for $s = 1$ and $s = \frac{1}{2} + it$, and obtain two sharp asymptotic formulas by using the analytic method and the theory of van der Corput.

Keywords: generalized Dirichlet L-functions, mean value properties, functional equation, asymptotic formula

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1. Introduction

Let $q \geqslant 3$ be an integer and χ a Dirichlet character modulo q. For any real number $a \geqslant 0$ we consider the mean value properties of the generalized Dirichlet L-functions defined by

$$L(s, \chi, a) = \sum_{n=1}^{\infty} \frac{\chi(n)}{(n+a)^s},$$

where $s = \sigma + it$ with $\sigma > 1$ and t both real. We can also extend temt to all s by analytic continuation. Conserning the generalized Dirichlet series, Bruce C. Berndt (see [1]–[3]) studied many identical properties satisfying restrictive conditions. It is

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well known that for χ a nonprincipal, primitive character modulo q, the Dirichlet L-function $L(s,\chi)$ satisfies the functional equation

$$R(s,\chi)=\Big(\frac{\pi}{q}\Big)^{-\frac{1}{2}(s+b)}\Gamma\Big(\frac{1}{2}(s+b)\Big)L(s,\chi)=\frac{\tau(\chi)}{i^b\sqrt{q}}R(1-s,\bar{\chi}),$$

where

$$b=\begin{cases}0, & \chi(-1)=1,\\ 1, & \chi(-1)=-1.\end{cases}$$

and $|\varepsilon(\chi)|=1$. For $\sigma>\frac{1}{2}-m$ with m a positive integer, B. C. Berndt (see [3]) derived

$$L(s,\chi,a)=\frac{a^{-s}}{\Gamma(s)}\bigg(\sum_{j=0}^{m-1}\frac{(-1)^j\Gamma(s+j)L(-j,\chi)}{j!\,a^j}+G(s)\bigg),$$

where $G(s)$ is an analytic function. When n is a nonpositive integer, we can easily calculate $L(n,\chi,a)$, in particular, $L(0,\chi,a)=L(0,\chi)$. A lot of scholars also studied the mean value properties of Dirichlet L-functions. D. R. Heath-Brown (see [4]) studied the square mean value properties of Dirichlet L-functions on the line $\sigma=\frac{1}{2}$. Wenpeng Zhang (see [5] and [6]) got different kinds of the mean value of Dirichlet L-functions with weight or not. In addition, R. Balasubramanian (see [8]) got the asymptotic formula

$$\sum_{\chi \bmod q}\Big|L\Big(\frac{1}{2}+\mathrm{i}t,\chi\Big)\Big|^2=\frac{\varphi^2(q)}{q}\log(qt)+O(q(\log\log q)^2)+O(t\mathrm{e}^{10\sqrt{\log q}})$$
$$+O(\sqrt{q}t^{\frac{2}{3}}\mathrm{e}^{10\sqrt{\log q}}),$$

which is satisfied for $t\geqslant 3$ and for all q. In this paper, taking $s=1$, $a\geqslant 1$, we are interested in the mean value properties of the generalized Dirichlet L-functions, that is, we want to get an asymptotic formula for

$$\sum_{\chi\neq\chi_0}|L(1,\chi,a)|^2,$$

where χ is the Dirichlet character modulo q and χ_0 is the principal character. On the other hand, the General Riemann hypothesis points out that all nontrivial zeros of Dirichlet L-functions lie on the line $\sigma=\frac{1}{2}$. It has attracted attention of many eminent mathematicians and a great deal has been discovered about the distribution of the zeros of Dirichlet L-functions. Therefore, we are also interested in the mean value properties of the generalized Dirichlet L-functions especially in the critical region $\sigma=\frac{1}{2}$, that is, we want to get another asymptotic formula for

$$\sum_{\chi \bmod q}\Big|L\Big(\frac{1}{2}+\mathrm{i}t,\chi,a\Big)\Big|^2,$$

where χ is the Dirichlet character modulo q, $0 \leqslant a \leqslant 1$. About the mean value properties of the generalized Dirichlet L-functions, we know very little at present. At least we have not found it in any reference that available. However, the problem is very interesting because it can help us to find some relationships between Dirichlet L-functions and the general case. That is, we shall prove

Theorem 1.1. *Let $q \geqslant 3$ be an integer and let χ denote the Dirichlet character modulo q. The Hurwitz zeta function $\zeta(s,\alpha)$ ($s = \sigma + \mathrm{i}t$, $\alpha > 0$) is defined for $\sigma > 1$ by the series*

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}.$$

Then for any positive real number $a \geqslant 1$ we have the asymptotic formula

$$\sum_{\chi \neq \chi_0} |L(1,\chi,a)|^2 = \varphi(q) \sum_{d|q} \frac{\mu(d)}{d^2} \zeta\Big(2, \frac{a}{d}\Big) - \frac{4\varphi(q)}{a} \sum_{d|q} \frac{\mu(d)}{d} \sum_{k=1}^{[a/d]} \frac{1}{k} + O\Big(\frac{\varphi(q)\log q}{\sqrt{q}}\Big),$$

where $\varphi(q)$ is the Euler function, $\mu(d)$ is the Möbius function and the O constant only depends on a.

For the general case of the $2k$-th ($k \geqslant 2$) power mean value of the generalized Dirichlet L-functions

$$\sum_{\chi \neq \chi_0} |L(1,\chi,a)|^{2k},$$

the authors can also get the asymptotic formula by similar methods, but the course of the calculation is quite complex, so we have not given the general conclusions in this paper.

Theorem 1.2. *Let $q \geqslant 3$ be an integer, a real number, $t \geqslant 3$, and χ a Dirichlet character modulo q. Then for any positive real number $0 \leqslant a \leqslant 1$, we have the asymptotic formula*

$$\begin{aligned}\sum_{\chi \bmod q} \Big|L\Big(\frac{1}{2} + \mathrm{i}t, \chi, a\Big)\Big|^2 = \frac{\varphi^2(q)}{q}\Big(\log\Big(\frac{qt}{2\pi}\Big) + 2\gamma + \sum_{p|q} \frac{\log p}{p-1}\Big) - \varphi(q) \sum_{d|q} \frac{\mu(d)}{d} \beta_{a/d} \\ + O(qt^{-\frac{1}{12}}) + O(t^{\frac{5}{6}} \log^3 t\, 2^{\omega(q)}) + O(q^{\frac{1}{2}} t^{\frac{5}{12}} \log t\, 2^{\omega(q)}),\end{aligned}$$

where $\varphi(q)$ is the Euler function, $\beta_x = \sum\limits_{n=1}^{\infty} x/n(n+x)$ a computable constant only depending on x and $\omega(q)$ denotes the number of distinct prime divisors of q.

Obviously, we can get the mean value of the Dirichlet L-function by taking $a=0$. So this result is brought up for a universal construction. For the general case of the $2k$-th ($k \geqslant 2$) power mean value of the generalized Dirichlet L-functions

$$\sum_{\chi \bmod q} \Big| L\Big(\frac{1}{2}+\mathrm{i}t, \chi, a\Big)\Big|^{2k},$$

the authors cannot get the asymptotic formula for the generalized Dirichlet L-functions because there is no good method to solve this problem. Even for $k=2$, we have not grasped a better asymptotic formula owing to the confinements of the method in existence.

2. On theorem 1.1

To complete the proof of Theorem 1.1 we need the following several lemmas. First, we make an identity of the Dirichlet L-functions and the generalized form.

Lemma 2.1. *Let $q \geqslant 3$ be an integer and let χ denote a Dirichlet character modulo q. $L(s,\chi)$ denotes the Dirichlet L-functions corresponding to χ, and $L(s,\chi,a)$ denotes the generalized Dirichlet L-functions. Then for any real number $a \geqslant 0$ we have*

$$L(1,\chi,a) = L(1,\chi) - a\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}.$$

Proof. According to the definition of the Dirichlet L-functions and the generalized form, we have

$$\begin{aligned} L(1,\chi,a) - L(1,\chi) &= \sum_{n=1}^{\infty}\frac{\chi(n)}{n+a} - \sum_{n=1}^{\infty}\frac{\chi(n)}{n} = \sum_{n=1}^{\infty}\Big(\frac{\chi(n)}{n+a} - \frac{\chi(n)}{n}\Big) \\ &= -a\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}, \end{aligned}$$

so by transposition we get

$$L(1,\chi,a) = L(1,\chi) - a\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}.$$

This proves Lemma 2.1. □

Lemma 2.2. *Let q be an integer with $q \geqslant 3$ and χ a Dirichlet character modulo q. Denote $A(y,\chi) = \sum\limits_{N \leqslant n \leqslant y} \chi(n)d(n)$. Then we have the estimate*

$$\sum_{\chi \neq \chi_0} |A(y,\chi)|^2 \ll y\varphi^2(q).$$

P r o o f. See Lemma 4, and let $k = 2$ (see [6]). □

Lemma 2.3. *Let $q \geqslant 3$ be an integer and χ a Dirichlet character modulo q. Then, we have*

$$\sum_{\chi \neq \chi_0} |L(1,\chi)|^2 = \varphi(q)\zeta(2) \prod_{p|q} \Bigl(1 - \frac{1}{p^2}\Bigr) + O(q^{\varepsilon}),$$

where $\varphi(q)$ is the Euler function, $\prod\limits_{p|q}$ denotes the product over all different prime divisors of q, and ε is a fixed positive constant.

P r o o f. See Lemma 4, and let $m = 2$ (see [6]). □

Lemma 2.4. *Let $q \geqslant 3$ be an integer and χ the Dirichlet character modulo q. Then for any positive real number $a \geqslant 1$ we have*

$$\sum_{\chi \neq \chi_0} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n(n+a)} L(1,\chi) = \frac{\varphi(q)}{a}\zeta(2) \prod_{p|q} \Bigl(1 - \frac{1}{p^2}\Bigr) + \frac{\varphi(q)}{a^2} \sum_{d|q} \frac{\mu(d)}{d} \sum_{k=1}^{[a/d]} \frac{1}{k} + O\Bigl(\frac{\varphi(q)\log q}{\sqrt{q}}\Bigr).$$

P r o o f. First, applying Abel's identity, by analytic continuation we have

$$L(1,\chi) = \sum_{n=1}^{q} \frac{\chi(n)}{n} + \int_q^{+\infty} \frac{A(\chi,y)}{y^2}\,\mathrm{d}y,$$

$$\sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n(n+a)} = \sum_{n=1}^{N} \frac{\bar{\chi}(n)}{n(n+a)} + \int_N^{+\infty} \frac{(2y+a)A(\bar{\chi},y)}{y^2(y+a)^2}\,\mathrm{d}y,$$

where $A(\chi, y) = \sum\limits_{q<n<y} \chi(n)$, and $N > q$ is an integer. Further,

$$
\begin{aligned}
&\sum_{\chi \neq \chi_0} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n(n+a)} L(1, \chi) \\
&= \sum_{\chi \neq \chi_0} \left(\sum_{n=1}^{N} \frac{\bar{\chi}(n)}{n(n+a)} + \int_N^{+\infty} \frac{(2y+a)A(\bar{\chi}, y)}{y^2(y+a)^2} \, \mathrm{d}y \right) \left(\sum_{n=1}^{q} \frac{\chi(n)}{n} + \int_q^{+\infty} \frac{A(\chi, y)}{y^2} \, \mathrm{d}y \right) \\
&= \sum_{\chi \neq \chi_0} \sum_{n=1}^{N} \frac{\bar{\chi}(n)}{n(n+a)} \sum_{m=1}^{q} \frac{\chi(m)}{m} + \sum_{\chi \neq \chi_0} \sum_{n=1}^{N} \frac{\bar{\chi}(n)}{n(n+a)} \int_q^{+\infty} \frac{A(\chi, y)}{y^2} \, \mathrm{d}y \\
&\quad + \sum_{\chi \neq \chi_0} \sum_{n=1}^{q} \frac{\chi(n)}{n} \int_N^{+\infty} \frac{(2y+a)A(\bar{\chi}, y)}{y^2(y+a)^2} \, \mathrm{d}y \\
&\quad + \sum_{\chi \neq \chi_0} \int_N^{+\infty} \frac{(2y+a)A(\bar{\chi}, y)}{y^2(y+a)^2} \, \mathrm{d}y \int_q^{+\infty} \frac{A(\chi, z)}{z^2} \, \mathrm{d}z \\
&\equiv A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

We will estimate each of the summands. First, we estimate A_1. From Cauchy's inequality and Lemma 2.3 we have

$$
\begin{aligned}
A_1 &= \sum_{\chi \neq \chi_0} \sum_{n=1}^{N} \frac{\bar{\chi}(n)}{n(n+a)} \sum_{m=1}^{q} \frac{\chi(m)}{m} \\
&= \sum_{\chi \bmod q} \sum_{n=1}^{N} \sum_{m=1}^{q} \frac{\chi(m)\bar{\chi}(n)}{mn(n+a)} + O(\log q) \\
&= \varphi(q) \sideset{}{'}\sum_{\substack{n=1 \\ m \equiv n \pmod{q}}}^{N} \sideset{}{'}\sum_{m=1}^{q} \frac{1}{mn(n+a)} + O(\log q) \\
&= \varphi(q) \sideset{}{'}\sum_{m=1}^{q} \frac{1}{m^2(m+a)} + \varphi(q) \sideset{}{'}\sum_{k=1}^{N/q} \sideset{}{'}\sum_{m=1}^{q} \frac{1}{m(kq+m)(kq+m+a)} + O(\log q) \\
&= \varphi(q) \sum_{d|q} \mu(d) \sum_{\substack{m=1 \\ d|m}}^{q} \frac{1}{m^2(m+a)} + O(\log q) \\
&= \varphi(q) \sum_{d|q} \frac{\mu(d)}{d^3} \sum_{m=1}^{q/d} \frac{1}{m^2(m+a/d)} + O(\log q)
\end{aligned}
$$

$$= \varphi(q) \sum_{d|q} \frac{\mu(d)}{d^3} \sum_{m=1}^{\infty} \frac{1}{m^2(m+a/d)}$$

$$+ O\Big(\varphi(q) \sum_{d|q} \frac{\mu(d)}{d^3} \sum_{m=q/d}^{\infty} \frac{1}{m^2(m+a/d)}\Big) + O(\log q)$$

$$= \varphi(q) \sum_{d|q} \frac{\mu(d)}{d^3} \Big(\frac{1}{(a/d)^2} \sum_{m=1}^{\infty} \Big(\frac{a/d}{m^2} + \frac{1}{m+a/d} - \frac{1}{m} \Big) \Big) + O(\log q)$$

$$= \frac{\varphi(q)}{a} \sum_{d|q} \frac{\mu(d)}{d^2} \zeta(2) + \frac{\varphi(q)}{a^2} \sum_{d|q} \frac{\mu(d)}{d} \sum_{k=1}^{[a/d]} \frac{1}{k} + O(\log q)$$

$$= \frac{\varphi(q)}{a} \zeta(2) \prod_{p|q} \Big(1 - \frac{1}{p^2}\Big) + \frac{\varphi(q)}{a^2} \sum_{d|q} \frac{\mu(d)}{d} \sum_{k=1}^{[a/d]} \frac{1}{k} + O(\log q).$$

Now, we will estimate A_2, A_3 and A_4. According to Lemma 2.2 and Cauchy's inequality we have

$$|A_2| = \sum_{\chi \neq \chi_0} \sum_{n=1}^{N} \frac{\bar{\chi}(n)}{n(n+a)} \int_q^{+\infty} \frac{A(\chi, y)}{y^2} \, \mathrm{d}y \leqslant \int_q^{\infty} \frac{1}{y^2} \sum_{n=1}^{N} \frac{1}{n^2} \sum_{\chi \neq \chi_0} |A(\chi, y)| \, \mathrm{d}y$$

$$\ll \frac{1}{N} \int_q^{\infty} \frac{1}{y^2} \Big(\sum_{\chi \neq \chi_0} 1 \Big)^{\frac{1}{2}} \Big(\sum_{\chi \neq \chi_0} |A(\chi, y)|^2 \Big)^{\frac{1}{2}} \mathrm{d}y \ll \frac{\varphi(q) \log q}{\sqrt{q}},$$

$$|A_3| = \sum_{\chi \neq \chi_0} \sum_{n=1}^{q} \frac{\chi(n)}{n} \int_N^{+\infty} \frac{(2y+a) A(\bar{\chi}, y)}{y^2 (y+a)^2} \, \mathrm{d}y$$

$$\ll \log q \int_N^{\infty} \frac{2y+a}{y^2(y+a)^2} \sum_{\chi \neq \chi_0} |A(\bar{\chi}, y)| \, \mathrm{d}y \ll \frac{\varphi^{\frac{3}{2}}(q) \log q}{N \sqrt{N}},$$

$$|A_4| = \sum_{\chi \neq \chi_0} \int_N^{+\infty} \frac{(2y+a) A(\bar{\chi}, y)}{y^2 (y+a)^2} \, \mathrm{d}y \int_q^{+\infty} \frac{A(\chi, z)}{z^2} \, \mathrm{d}z$$

$$\leqslant \int_N^{\infty} \int_q^{\infty} \frac{2y+a}{y^2(y+a)^2 z^2} \sum_{\chi \neq \chi_0} |A(\bar{\chi}, y)| |A(\chi, z)| \, \mathrm{d}y \, \mathrm{d}z \ll \frac{\varphi^2(q)}{N \sqrt{N} \sqrt{q}},$$

where we have used the common estimate. Taking $N = q^2$ we get

$$\sum_{\chi \neq \chi_0} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n(n+a)} L(1, \chi)$$

$$= \frac{\varphi(q)}{a} \zeta(2) \prod_{p|q} \Big(1 - \frac{1}{p^2}\Big) + \frac{\varphi(q)}{a^2} \sum_{d|q} \frac{\mu(d)}{d} \sum_{k=1}^{[a/d]} \frac{1}{k} + O\Big(\frac{\varphi(q) \log q}{\sqrt{q}}\Big).$$

This proves Lemma 2.4. $\square$

Lemma 2.5. *Let $q \geqslant 3$ be an integer and χ the Dirichlet character modulo q. Then for any positive real number $a \geqslant 1$ we have*

$$\sum_{\chi\neq\chi_0}\Big|\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}\Big|^2=\frac{\varphi(q)}{a^2}\zeta(2)\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)+\frac{\varphi(q)}{a^2}\sum_{d\mid q}\frac{\mu(d)}{d^2}\zeta\Big(2,\frac{a}{d}\Big)$$
$$-\frac{2\varphi(q)}{a^3}\sum_{d\mid q}\frac{\mu(d)}{d}\sum_{k=1}^{[a/d]}\frac{1}{k}+O\Big(\frac{\varphi(q)\log q}{q}\Big).$$

P r o o f. Let $N > q$ be any integer, then we have

$$\sum_{\chi\neq\chi_0}\Big|\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}\Big|^2=\sum_{\chi\neq\chi_0}\Big(\sum_{n=1}^{N}\frac{\chi(n)}{n(n+a)}+\int_N^{+\infty}\frac{(2y+a)A(\chi,y)}{y^2(y+a)^2}\,\mathrm{d}y\Big)$$
$$\times\Big(\sum_{m=1}^{N}\frac{\bar{\chi}(m)}{m(m+a)}+\int_N^{+\infty}\frac{(2z+a)A(\bar{\chi},z)}{z^2(z+a)^2}\,\mathrm{d}z\Big)$$
$$=\varphi(q)\sum_{\substack{n=1\\ m\equiv n \pmod q}}^{N}{}'\sum_{m=1}^{N}{}'\frac{1}{mn(m+a)(n+a)}+O\Big(\frac{\varphi^{\frac32}(q)}{N^{\frac32}}\Big)+O(1)$$
$$=\varphi(q)\sum_{n=1}^{N}{}'\frac{1}{n^2(n+a)^2}+2\varphi(q)\sum_{\substack{n=1\\ m\equiv n \pmod q,\, m>n}}^{N}{}'\sum_{m=1}^{N}{}'\frac{1}{mn(m+a)(n+a)}+O(1)$$
$$=\varphi(q)\sum_{n=1}^{N}{}'\frac{1}{n^2(n+a)^2}+O\Big(\frac{\varphi(q)\log N}{q}\Big)+O(1)$$
$$=\varphi(q)\sum_{d\mid q}\frac{\mu(d)}{d^4}\sum_{n=1}^{N/d}\frac{1}{n^2(n+a/d)^2}+O\Big(\frac{\varphi(q)\log N}{q}\Big)$$
$$=\varphi(q)\sum_{d\mid q}\frac{\mu(d)}{d^4}\sum_{n=1}^{\infty}\frac{1}{n^2(n+a/d)^2}+O\Big(\frac{\varphi(q)\log N}{q}\Big)$$
$$=\varphi(q)\sum_{d\mid q}\frac{\mu(d)}{d^4}\Big(\frac{d^3}{a^3}\sum_{n=1}^{\infty}\Big(\frac{a/d}{(n+a/d)^2}+\frac{a/d}{n^2}+\frac{2}{n+a/d}-\frac{2}{n}\Big)\Big)+O\Big(\frac{\varphi(q)\log N}{q}\Big)$$
$$=\frac{\varphi(q)}{a^2}\zeta(2)\prod_{p\mid q}\Big(1-\frac{1}{p^2}\Big)+\frac{\varphi(q)}{a^2}\sum_{d\mid q}\frac{\mu(d)}{d^2}\zeta\Big(2,\frac{a}{d}\Big)$$
$$-\frac{2\varphi(q)}{a^3}\sum_{d\mid q}\frac{\mu(d)}{d}\sum_{k=1}^{[a/d]}\frac{1}{k}\quad+O\Big(\frac{\varphi(q)\log N}{q}\Big).$$

Taking $N = q^2$ we have

$$\sum_{\chi\neq\chi_0}\left|\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}\right|^2 = \frac{\varphi(q)}{a^2}\zeta(2)\prod_{p|q}\Big(1-\frac{1}{p^2}\Big) + \frac{\varphi(q)}{a^2}\sum_{d|q}\frac{\mu(d)}{d^2}\zeta\Big(2,\frac{a}{d}\Big) - \frac{2\varphi(q)}{a^3}\sum_{d|q}\frac{\mu(d)}{d}\sum_{k=1}^{[a/d]}\frac{1}{k} + O\Big(\frac{\varphi(q)\log q}{q}\Big).$$

This proves Lemma 2.5. □

Next, we shall complete the proof of Theorem 1.1. From Lemma 2.1, Lemma 2.4 and Lemma 2.5, we can easily get Theorem 1.1. For any $q \geqslant 3$ we have

$$\begin{aligned}
&\sum_{\chi\neq\chi_0}|L(1,\chi,a)|^2\\
&= \sum_{\chi\neq\chi_0}\left|L(1,\chi) - a\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}\right|^2\\
&= \sum_{\chi\neq\chi_0}|L(1,\chi)|^2 - a\sum_{\chi\neq\chi_0}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n(n+a)}L(1,\chi)\\
&\quad - a\sum_{\chi\neq\chi_0}\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}L(1,\bar{\chi}) + a^2\sum_{\chi\neq\chi_0}\left|\sum_{n=1}^{\infty}\frac{\chi(n)}{n(n+a)}\right|^2\\
&= \varphi(q)\sum_{d|q}\frac{\mu(d)}{d^2}\zeta\Big(2,\frac{a}{d}\Big) - \frac{4\varphi(q)}{a}\sum_{d|q}\frac{\mu(d)}{d}\sum_{k=1}^{[a/d]}\frac{1}{k} + O\Big(\frac{\varphi(q)\log q}{\sqrt{q}}\Big).
\end{aligned}$$

This completes the proof of Theorem 1.1.

3. On theorem 1.2

First, we should introduce one of the most important methods. That is the so-called saddle-point method that van der Corput (See [9]–[11]) found to deal with exponential integrals and exponential sums which occur in a large number of problems whose solutions ultimately depend on asymptotic formulas or good O-bounds in the 1920s. In the proof of Theorem 1.2, we usually use the following propositions.

Proposition 3.1. *Let $f(x)$, $f'(x)$, $g(x)$, $g'(x)$ be real-valued functions on the interval $[a,b]$, let all of them be continuous and monotonic on $[a,b]$, and let $|f'(x)| \leqslant$*

$\delta < 1$, $|g(x)| \leqslant h_0$, $|g'(x)| \leqslant h_1$. *Then we have*

$$\sum_{a<n\leqslant b} g(n)e(f(n)) = \int_a^b g(x)e(f(x))\,\mathrm{d}x + O\Big(\frac{h_0+h_1}{1-\delta}\Big), \tag{1}$$

where $e(f(n)) = \exp(2\pi i f(x))$, *and the* O*-constant is absolute.*

Proposition 3.2. *Under the conditions of Proposition* 1, *let* $g(x)/f'(x)$ *be monotonic on the interval* $[a,b]$ *and such that* $f'(x)/g(x) \geqslant m > 0$ *or* $-f'(x)/g(x) \geqslant m > 0$. *Then we have*

$$\Big|\int_a^b g(x)\mathrm{e}^{\mathrm{i}f(x)}\,\mathrm{d}x\Big| \leqslant \frac{4}{m}. \tag{2}$$

Combining Proposition 3.1 and Proposition 3.2, we can get

Proposition 3.3. *By virtue of* (1) *and* (2) *we have*

$$\Big|\sum_{a<n\leqslant b} g(n)e(f(n))\Big| \ll \frac{1}{m} + \frac{h_0+h_1}{1-\delta}. \tag{3}$$

This method is simpler than the theory of exponent pairs which is, nevertheless, fairly simple to use in practice. We will also use the theory of the exponent pairs in the proof of Theorem 1.2. For example,

Proposition 3.4. *Let* $b-a \geqslant 1$, *let* $f''(x)$ *be continuous on the interval* $[a,b]$ *and such that*

$$0 < \lambda \leqslant |f''(x)| \leqslant h\lambda.$$

Then we have

$$\sum_{a<n\leqslant b} e(f(n)) = O(h(b-a)\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}),$$

where the O*-constant is absolute. The theory of exponent pairs also can be found in references* [9]–[11].

In addition to the above method, we must have another important formula onfor the Hurwitz-zeta function, that is, the famous approximate functional equation.

Proposition 3.5. *Let $s=\sigma+\mathrm{i}t$, and let real numbers $x,y>C>0$ be such that $2\pi xy=t$. Then for $0<\sigma_1\leqslant\sigma\leqslant\sigma_2<1$, $t\geqslant 0$ we have*

$$\zeta(s,a)=\sum_{0\leqslant n\leqslant x-a}\frac{1}{(n+a)^s}+A(s)\sum_{1\leqslant v\leqslant y}\frac{e(-av)}{v^{1-s}} + O\big(x^{-\sigma}\log(y+2)+y^{\sigma-1}(|t|+2)^{\frac12-\sigma}\big), \tag{4}$$

where the O-constant depends on σ_1, σ_2, and

$$A(s)=\Big|\frac{t}{2\pi}\Big|^{\frac12-\sigma}\exp\Big(-\mathrm{i}\Big(t\log\Big|\frac{t}{2\pi e}\Big|-\frac{\pi}{4}\Big)\Big)\Big(1+O\Big(\frac{1}{|t|}\Big)\Big)\qquad(t\to+\infty).$$

For $t\leqslant 0$, $e(-av)$ is replaced by $e(av)$ in (4), and

$$A(s)=\Big|\frac{t}{2\pi}\Big|^{\frac12-\sigma}\exp\Big(-\mathrm{i}\Big(t\log\Big|\frac{t}{2\pi e}\Big|+\frac{\pi}{4}\Big)\Big)\Big(1+O\Big(\frac{1}{|t|}\Big)\Big)\qquad(t\to-\infty).$$

Especially, let $s=\frac12+\mathrm{i}t$. Then for $t\geqslant 0$ we have

$$\zeta\Big(\frac12+\mathrm{i}t,a\Big) = \sum_{0\leqslant n\leqslant x-a}\frac{1}{(n+a)^{\frac12+\mathrm{i}t}}+\Big(\frac{2\pi}{t}\Big)^{\mathrm{i}t}\mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})}\sum_{1\leqslant v\leqslant y}\frac{e(-av)}{v^{\frac12-\mathrm{i}t}}+O(x^{-\frac12}\log t). \tag{5}$$

Next, we need the following several lemmas.

Lemma 3.1. *Let q be a positive integer we have*

$$\sum_{d\mid q}\frac{\mu(d)\log d}{d}=-\frac{\varphi(q)}{q}\sum_{p\mid q}\frac{\log p}{p-1}.$$

P r o o f. See Lemma 2 of [5]. □

Lemma 3.2. *Let q be a positive integer, let $0\leqslant a\leqslant 1$, $t\geqslant 2$ be real numbers and χ a Dirichlet character modulo q. Then we have the identity*

$$\sum_{\chi \bmod q}\Big|L\Big(\frac12+\mathrm{i}t,\chi,a\Big)\Big|^2=\frac{\varphi(q)}{q}\sum_{d\mid q}\mu(d)\sum_{b=1}^{q/d}\Big|\zeta\Big(\frac12+\mathrm{i}t,\frac{b+a/d}{q/d}\Big)\Big|^2,$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta function which is analytic for all s except for $s=1$.

P r o o f. Let $s=\sigma+\mathrm{i}t$ and $\sigma>1$. The generalized Dirichlet L-functions converge absolutely, so we have

$$L(s,\chi,a)=\sum_{n=1}^{\infty}\frac{\chi(n)}{(n+a)^s}=\sum_{b=1}^{q}\sum_{n=0}^{\infty}\frac{\chi(b)}{(nq+b+a)^s}=\frac{1}{q^s}\sum_{b=1}^{q}\chi(b)\zeta\Bigl(s,\frac{b+a}{q}\Bigr). \tag{6}$$

By using the orthogonality relationship for character sums modulo q, we obtain from (6) that

$$\begin{aligned}\sum_{\chi \bmod q}|L(s,\chi,a)|^2 &=\frac{1}{q^{2\sigma}}\sum_{\chi \bmod q}\Bigl|\sum_{b=1}^{q}\chi(b)\zeta\Bigl(s,\frac{b+a}{q}\Bigr)\Bigr|^2\\ &=\frac{1}{q^{2\sigma}}\sum_{b_1=1}^{q}\zeta\Bigl(s,\frac{b_1+a}{q}\Bigr)\sum_{b_2=1}^{q}\bar\zeta\Bigl(s,\frac{b_2+a}{q}\Bigr)\sum_{\chi \bmod q}\chi(b_1)\bar\chi(b_2)\\ &=\frac{\varphi(q)}{q^{2\sigma}}\mathop{\sum_{\substack{b_1=1\\(b_1,q)=1}}^{q}\ \sum_{\substack{b_2=1\\(b_2,q)=1}}^{q}}_{b_1\equiv b_2 \pmod q}\zeta\Bigl(s,\frac{b_1+a}{q}\Bigr)\bar\zeta\Bigl(s,\frac{b_2+a}{q}\Bigr)\\ &=\frac{\varphi(q)}{q^{2\sigma}}\sum_{\substack{b=1\\(b,q)=1}}^{q}\Bigl|\zeta\Bigl(s,\frac{b+a}{q}\Bigr)\Bigr|^2\\ &=\frac{\varphi(q)}{q^{2\sigma}}\sum_{b=1}^{q}\Bigl|\zeta\Bigl(s,\frac{b+a}{q}\Bigr)\Bigr|^2\sum_{d\mid(b,q)}\mu(d)\\ &=\frac{\varphi(q)}{q^{2\sigma}}\sum_{d\mid q}\mu(d)\sum_{b=1}^{q/d}\Bigl|\zeta\Bigl(s,\frac{b+a/d}{q/d}\Bigr)\Bigr|^2,\end{aligned}$$

and our lemma follows immediately by analytic continuation. □

In the following, we will put $a'=a/d$, $q'=q/d$ for brevity.

Lemma 3.3. *Let $q'\geqslant 2$ be an integer and $0\leqslant a'\leqslant 1$, $t\geqslant 3$. Then we have*

$$\Bigl|\sum_{b=1}^{q'}\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}\Bigl(\zeta\Bigl(\frac12-\mathrm{i}t,\frac{b+a'}{q'}\Bigr)-\frac{1}{(\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}}\Bigr)\Bigr|\ll q'^{\frac12}t^{\frac{5}{12}}\log t+q't^{-\frac14}\log t, \tag{7}$$

where the $\ll$ constant depends on a'.

Proof. From Proposition 3.5 we have

$$\zeta\Big(\frac12-\mathrm{i}t,\frac{b+a'}{q'}\Big)-\frac{1}{\big(\frac{b+a'}{q'}\big)^{\frac12-\mathrm{i}t}}$$
$$=\sum_{1\leqslant n\leqslant x}\frac{1}{(n+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}}+\Big(\frac{2\pi}{t}\Big)^{-\mathrm{i}t}\mathrm{e}^{\mathrm{i}(-t+\frac{\pi}{4})}\sum_{1\leqslant v\leqslant y}\frac{e\big(-\frac{b+a'}{q'}v\big)}{v^{\frac12+\mathrm{i}t}}+O(x^{-\frac12}\log t),$$

so we immediately get

$$\Big|\sum_{b=1}^{q'}\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}\Big(\zeta\Big(\frac12-\mathrm{i}t,\frac{b+a'}{q'}\Big)-\frac{1}{(\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}}\Big)\Big| \tag{8}$$
$$=\sum_{b=1}^{q'}\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}\Big(\sum_{1\leqslant n\leqslant x}\frac{1}{(n+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}}$$
$$+\Big(\frac{2\pi}{t}\Big)^{-\mathrm{i}t}\mathrm{e}^{\mathrm{i}(-t+\frac{\pi}{4})}\sum_{1\leqslant v\leqslant y}\frac{e\big(-\frac{b+a'}{q'}v\big)}{v^{\frac12+\mathrm{i}t}}+O(x^{-\frac12}\log t)\Big)$$
$$=\sum_{1\leqslant n\leqslant x}\sum_{b=1}^{q'}\frac{\big(\frac{n+(b+a')/q'}{(b+a')/q'}\big)^{\mathrm{i}t}}{\big(\frac{b+a'}{q'}\big)^{\frac12}\big(n+\frac{b+a'}{q'}\big)^{\frac12}}$$
$$+\Big(\frac{2\pi}{t}\Big)^{-\mathrm{i}t}\mathrm{e}^{\mathrm{i}(-t+\frac{\pi}{4})}\sum_{1\leqslant v\leqslant y}\sum_{b=1}^{q'}\frac{e\big(-\frac{b+a'}{q'}v\big)\big(\frac{b+a'}{q'}v\big)^{-\mathrm{i}t}}{(\frac{b+a'}{q'})v^{\frac12}}$$
$$+O(q'x^{-\frac12}\log t)$$
$$=M_1+M_2+O(q'x^{-\frac12}\log t),$$

where $2\pi xy=t$, and $0<C<x,y<t$ is to be determined later.

In M_1, we know that

$$\Bigg(\frac{n+\frac{b+a'}{q'}}{\frac{b+a'}{q'}}\Bigg)^{\mathrm{i}t}=\Big(\frac{nq'+b+a'}{b+a'}\Big)^{\mathrm{i}t}=\mathrm{e}^{\mathrm{i}t\log\frac{nq'+b+a'}{b+a'}}=\mathrm{e}^{2\pi\mathrm{i}f(b)},$$

where we denote $f(b)=\frac{t}{2\pi}\log\frac{nq'+b+a'}{b+a'}$. Also, let $g(b)=1/(\frac{b+a'}{q'})^{\frac12}(n+\frac{b+a'}{q'})^{\frac12}$; we will apply Proposition 3.3 and the theory of the exponent pairs to estimate A_1. So we have

$$f'(b)=\frac{t}{2\pi}\frac{-nq'}{(b+a')(nq'+b+a')}<0, \tag{9}$$

$$g'(b)=\frac{-q'(nq'+2b+2a')}{(b+a')(nq'+b+a')}. \tag{10}$$

When $b+a'>t$, we have

$$(11)\quad |f'(b)| \leqslant \frac{1}{2\pi} < 1,$$

$$(12)\quad |g(b)| \leqslant q' t^{-\frac{1}{2}} (nq')^{-\frac{1}{2}} = q'^{\frac{1}{2}} (nt)^{-\frac{1}{2}},$$

$$(13)\quad |g'(b)| \ll q t^{-1},$$

$$(14)\quad -\frac{f'(b)}{g(b)} = \frac{tnq'}{2\pi(b+a')(nq'+b+a')} \frac{(b+a')^{\frac{1}{2}}(nq'+b+a')^{\frac{1}{2}}}{q'}$$
$$= \frac{tn}{2\pi(b+a')^{\frac{1}{2}}(nq'+b+a')^{\frac{1}{2}}} \geqslant \frac{tn}{2\pi(q'+1)^{\frac{1}{2}}(nq'+q'+1)^{\frac{1}{2}}} \geqslant \frac{tn^{\frac{1}{2}}}{q'}.$$

When $b+a'<t$, for $N<b<N+h<2N$, and $|f'(b)| \ll t/N$, if we take $(\frac{1}{6}, \frac{2}{3})$ as the exponent pair, we obtain

$$(15)\quad \Big|\sum_{N\leqslant b\leqslant N+h\leqslant 2N} e(f(b))\Big| \ll \Big(\frac{t}{N}\Big)^{\frac{1}{6}} N^{\frac{2}{3}} \ll t^{\frac{1}{6}} N^{\frac{1}{2}}.$$

Therefore, from Proposition 3.3 and (11)–(15) we have

(16)

$$|M_1| \leqslant \sum_{1\leqslant n\leqslant x} \Big|\sum_{b+a'>t} g(b)e(f(b))\Big| + \sum_{1\leqslant n\leqslant x} \Big|\sum_{b+a'\leqslant t} g(b)e(f(b))\Big|$$
$$\ll \sum_{1\leqslant n\leqslant x} \Big|\frac{q'}{n^{\frac{1}{2}}t} + (nt)^{-\frac{1}{2}} q'^{\frac{1}{2}}\Big| + \sum_{1\leqslant n\leqslant x} \Big|\sum_{j\ll \log t} \sum_{2^{j-1}<b<2^j} g(b)e(f(b))\Big|$$
$$\ll \sum_{1\leqslant n\leqslant x} \frac{q'}{n^{\frac{1}{2}}t} + O\Big(q'^{\frac{1}{2}} t^{-\frac{1}{2}} x^{\frac{1}{2}}\Big) + \sum_{1\leqslant n\leqslant x} \Big|\sum_{j\ll\log t} \max_{1\leqslant N\leqslant t-a'} \Big|\sum_{N<b<2N} g(b)e(f(b))\Big|\Big|$$
$$\ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{2}}$$
$$+ \sum_{1\leqslant n\leqslant x} \Big|\log t \max_{1\leqslant N\leqslant t-a'} \frac{1}{(\frac{N+a'}{q'})^{\frac{1}{2}}} \frac{1}{(n+\frac{N+a'}{q'})^{\frac{1}{2}}} \max_{1\leqslant N\leqslant t-a'} \Big|\sum_{N<b<N+h<2N} e(f(b))\Big|\Big|$$
$$\ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{2}} + \sum_{1\leqslant n\leqslant x} \log t \Big|\max_{1\leqslant N\leqslant t-a'} \frac{q'}{\sqrt{N}\sqrt{n}\sqrt{q'}}\Big|\sum_{N<b<N+h<2N} e(f(b))\Big|\Big|$$
$$\ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{2}} + \sum_{1\leqslant n\leqslant x} \frac{\sqrt{q'}}{\sqrt{n}} \log t \max_{1\leqslant N\leqslant t-a'} \frac{1}{\sqrt{N}} \Big|\sum_{N\leqslant b\leqslant N+h\leqslant 2N} e(f(b))\Big|$$
$$\ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{2}} + \sum_{1\leqslant n\leqslant x} \frac{\sqrt{q'}}{\sqrt{n}} \log t \max_{1\leqslant N\leqslant t-a'} \frac{1}{\sqrt{N}}\, t^{\frac{1}{6}} N^{\frac{1}{2}}$$
$$\ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{2}} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{\frac{1}{6}} \log t$$
$$\ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{3}} \log t.$$

Similarly, we can also establish estimates of M_2,

$$M_2 \ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{3}} \log t. \tag{17}$$

Therefore, from (8), (16) and (17) we have

$$\begin{aligned}\Big|\sum_{b=1}^{q'} \frac{1}{(\frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}}\Big(\zeta\Big(\frac{1}{2}-\mathrm{i}t, \frac{b+a'}{q'}\Big) - \frac{1}{\big(\frac{b+a'}{q'}\big)^{\frac{1}{2}-\mathrm{i}t}}\Big)\Big| \\ \ll q' x^{\frac{1}{2}} t^{-1} + q'^{\frac{1}{2}} x^{\frac{1}{2}} t^{-\frac{1}{3}} \log t + q' x^{-\frac{1}{2}} \log t.\end{aligned}$$

Taking $x = y = \sqrt{t/(2\pi)}$, we have

$$\Big|\sum_{b=1}^{q'} \frac{1}{(\frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}}\Big(\zeta\Big(\frac{1}{2}-\mathrm{i}t, \frac{b+a'}{q'}\Big) - \frac{1}{(\frac{b+a'}{q'})^{\frac{1}{2}-\mathrm{i}t}}\Big)\Big| \ll q'^{\frac{1}{2}} t^{\frac{5}{12}} \log t + q' t^{-\frac{1}{4}} \log t.$$

This proves Lemma 3.3. □

Lemma 3.4. *Let $q' \geqslant 2$ be an integer and $0 \leqslant a' \leqslant 1$, $t \geqslant 3$. Then we have*

$$\begin{aligned}&\sum_{b=1}^{q'} \Big|\Big(\zeta\Big(\frac{1}{2}+\mathrm{i}t, \frac{b+a'}{q'}\Big) - \frac{1}{(\frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}}\Big)\Big|^2 \\ &\qquad = q'\Big(\log\frac{q't}{(q'+a')2\pi} + \gamma\Big) + O(q' t^{-\frac{1}{12}}) + O(t^{\frac{5}{6}} \log^3 t) + O(\sqrt{q'} \log t),\end{aligned}$$

where γ is the Euler constant, and the O-constant depends on a.

P r o o f. From Proposition 3.5, by analytic continuation we have

(18)

$$\begin{aligned}&\sum_{b=1}^{q'} \Big|\Big(\zeta\Big(\frac{1}{2}+\mathrm{i}t, \frac{b+a'}{q'}\Big) - \frac{1}{(\frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}}\Big)\Big|^2 \\ &= \sum_{b=1}^{q'} \Big| \sum_{1\leqslant n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}} + \Big(\frac{2\pi}{t}\Big)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant v\leqslant y} \frac{e\big(-\frac{b+a'}{q'}v\big)}{v^{\frac{1}{2}-\mathrm{i}t}} + O(x^{-\frac{1}{2}} \log t)\Big|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{b=1}^{q'} \Bigl| \sum_{1\leqslant n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \Bigr|^2 + \sum_{b=1}^{q'} \Bigl| \sum_{1\leqslant v\leqslant y} \frac{e(-\frac{b+a'}{q'}v)}{v^{\frac12-\mathrm{i}t}} \Bigr|^2 \\
&\quad + \sum_{b=1}^{q'} \sum_{1\leqslant n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \Bigl(\frac{2\pi}{t}\Bigr)^{-\mathrm{i}t} \mathrm{e}^{-\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant v\leqslant y} \frac{e(\frac{b+a'}{q'}v)}{v^{\frac12+\mathrm{i}t}} \\
&\quad + \sum_{b=1}^{q'} \sum_{1\leqslant n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}} \Bigl(\frac{2\pi}{t}\Bigr)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant v\leqslant y} \frac{e(-\frac{b+a'}{q'}v)}{v^{\frac12-\mathrm{i}t}} \\
&\quad + O\Biggl(x^{-\frac12}\log t \sum_{1\leqslant b\leqslant q'} \Bigl| \sum_{1\leqslant n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} + \Bigl(\frac{2\pi}{t}\Bigr)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant v\leqslant y} \frac{e(-\frac{b+a'}{q'}v)}{v^{\frac12-\mathrm{i}t}} \Bigr| \Biggr) \\
&\quad + O\left(x^{-1}\log^2 t\right) \\
&= A_1 + A_2 + A_3 + A_4 + A_5 + O\left(x^{-1}\log^2 t\right).
\end{aligned}
$$

We will estimate each term of (18).

(i) First, we estimate A_1. From (18) we have

$$
\begin{aligned}
(19)\qquad A_1 &= \sum_{b=1}^{q'} \Bigl| \sum_{1\leqslant n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \Bigr|^2 \\
&= \sum_{b=1}^{q'} \sum_{1\leqslant n\leqslant x} \sum_{1\leqslant m\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \frac{1}{(m+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}} \\
&= \sum_{b=1}^{q'} \sum_{1\leqslant n\leqslant x} \frac{1}{n+\frac{b+a'}{q'}} + \sum_{b=1}^{q'} \sum_{1\leqslant m<n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \frac{1}{(m+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}} \\
&\quad + \sum_{b=1}^{q'} \sum_{1\leqslant n<m\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \frac{1}{(m+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}} \\
&= q' \sum_{b=1}^{q'} \sum_{1\leqslant n\leqslant x} \frac{1}{nq'+b+a'} + \sum_{b=1}^{q'} \sum_{1\leqslant m<n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \frac{1}{(m+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}} \\
&\quad + \sum_{b=1}^{q'} \sum_{1\leqslant m<n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}} \frac{1}{(m+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \\
&= q' A_{11} + A_{12} + A_{13}.
\end{aligned}
$$

Apparently, A_{13} is the conjugate of A_{12}. So we can only estimate A_{11} and A_{12}. Since

$$
\sum_{n=1}^{q} \frac{1}{n+a} = \sum_{n=1}^{q} \frac{1}{n} - \sum_{n=1}^{q} \frac{a}{n(n+a)} = \log q + \gamma - \sum_{n=1}^{\infty} \frac{a}{n(n+a)} + O(q^{-1}),
$$

we have

$$(20)\qquad A_{11}=\sum_{b=1}^{q'}\sum_{1\leqslant n\leqslant x}\frac{1}{nq'+b+a'}$$
$$=\sum_{1\leqslant n\leqslant q'([x]+1)}\frac{1}{n+a'}-\sum_{1\leqslant n\leqslant q'}\frac{1}{n+a'}$$
$$=\log(q'([x]+1))+\gamma-\beta_{a'}+O\Big(\frac{1}{q'x}\Big)-\Big(\log q'+\gamma-\beta_{a'}+O\Big(\frac{1}{q'}\Big)\Big)$$
$$=\log x+O\Big(\frac{1}{q'}\Big),$$

where $\beta_{a'}=\sum\limits_{n=1}^{\infty}a'/n(n+a')$. Next we will estimate A_{12}. First, we have

$$(21)\qquad A_{12}=\sum_{b=1}^{q'}\sum_{1\leqslant m<n\leqslant x}\frac{\Big(\frac{m+(b+a')/q'}{n+(b+a')/q}\Big)^{\mathrm{i}t}}{(n+\frac{b+a'}{q'})^{\frac12}(m+\frac{b+a'}{q'})^{\frac12}}$$
$$=\sum_{1\leqslant m<n\leqslant x}\sum_{b=1}^{q'}\frac{\mathrm{e}^{\mathrm{i}t\log{(mq'+b+a')/(nq'+b+a')}}}{(n+\frac{b+a'}{q'})^{\frac12}(m+\frac{b+a'}{q'})^{\frac12}}.$$

Let

$$(22)\qquad f(b)=\frac{t}{2\pi}\log\frac{mq'+b+a'}{nq'+b+a'},$$

$$(23)\qquad g(b)=\Big(n+\frac{b+a'}{q'}\Big)^{-\frac12}\Big(m+\frac{b+a'}{q'}\Big)^{-\frac12}$$

apparently, $g(b)$ is monotonic decreasing and we have

$$(24)\qquad f'(b)=\frac{t}{2\pi}\frac{(n-m)q'}{(mq'+b+a')(nq'+b+a')}>0,$$

$$(25)\qquad f''(b)=\frac{-t(n-m)q'((n+m)q'+2(b+a'))}{2\pi(mq'+b+a')^2(nq'+b+a')^2}<0,$$

$$(26)\qquad g'(b)=-\frac{q'}{2}\frac{q'(m+a')+2(b+a')}{(nq'+b+a')^{\frac32}(mq'+b+a')^{\frac32}}.$$

Therefore, $f(b)$ is monotonic increasing, and $f'(b)$ is monotonic decreasing. Since $t \leqslant mnq'/(n-m)$, we have

$$(27)\qquad |f'(b)| = \frac{t}{2\pi}\frac{(n-m)q'}{(mq'+b+a')(nq'+b+a')} \leqslant \frac{1}{2\pi} < 1,$$

$$(28)\qquad |g(b)| = \frac{1}{\left(n+\frac{b+a'}{q'}\right)^{\frac{1}{2}}\left(m+\frac{b+a'}{q'}\right)^{\frac{1}{2}}} \leqslant \frac{1}{\sqrt{mn}},$$

$$(29)\qquad |g'(b)| = \frac{q'\left(q'(m+n)+2(b+a')\right)}{2\left(nq'+b+a'\right)^{\frac{3}{2}}\left(mq'+b+a'\right)^{\frac{3}{2}}} \leqslant \frac{1}{\sqrt{mn}},$$

$$(30)\qquad \frac{f'(b)}{g(b)} = \frac{t(n-m)}{2\pi(mq'+b+a')^{\frac{1}{2}}(nq'+b+a')^{\frac{1}{2}}} \gg \frac{(n-m)t}{\sqrt{mn}q'}.$$

Therefore, according to Proposition 3.3 and by virtue of (19), (27)–(30) we have

$$(31)\qquad \begin{aligned} |A_{12}| &\ll \sum_{1\leqslant m<n\leqslant x}\left(\frac{1}{(n-m)t/\sqrt{mn}q'}+\frac{1}{\sqrt{mn}}\right) \\ &\ll q't^{-1}\sum_{1\leqslant m<n\leqslant x}\frac{\sqrt{mn}}{n-m}+x = q't^{-1}\sum_{k=1}^{x-1}\sum_{m=1}^{x-1}\frac{\sqrt{m(m+k)}}{k}+x \\ &\ll q't^{-1}x^2\log t. \end{aligned}$$

Since $t > mnq'/(n-m)$, from (25), we have

$$\frac{t(n-m)}{m^2q'^2n} \ll |f''(b)| \ll \frac{t(n-m)}{m^2q'^2n},$$

hence Propositions 3.4 and (21) yield

$$(32)\qquad \begin{aligned} |A_{12}| &\ll \sum_{1\leqslant m<n\leqslant x}\frac{\log q}{\left(n+\frac{1+a'}{q'}\right)^{\frac{1}{2}}\left(m+\frac{1+a'}{q'}\right)^{\frac{1}{2}}}\max_{1\leqslant N\leqslant q'}\left|\sum_{N<b<N+h<2N}\mathrm{e}^{\mathrm{i}t\log\frac{mq'+b+a'}{nq'+b+a'}}\right| \\ &\ll \sum_{1\leqslant m<n\leqslant x}\frac{\log t}{\left(n+\frac{1+a'}{q'}\right)^{\frac{1}{2}}\left(m+\frac{1+a'}{q'}\right)^{\frac{1}{2}}}\max_{1\leqslant v\leqslant q'}\left|q'\left(\frac{t(n-m)}{m^2q'^2n}\right)^{\frac{1}{2}}+\left(\frac{m^2q'^2n}{t(n-m)}\right)^{\frac{1}{2}}\right| \\ &\ll \sum_{1\leqslant m<n\leqslant x}\frac{\log t\sqrt{t}}{\sqrt{n}\sqrt{m^3}}+\sum_{1\leqslant m<n\leqslant x}\frac{\log t\sqrt{t}}{\sqrt{nm}} \\ &\ll t^{\frac{1}{2}}x\log t+q'x^2t^{-1}\log t. \end{aligned}$$

Therefore, (21), (31) and (32) imply

$$(33)\qquad |A_{12}| \ll \left(q't^{-1}x^2+t^{\frac{1}{2}}x\right)\log t.$$

So, from (19), (20) and (33) we have

$$A_1 = q' \log x + O\big((q' t^{-1} x^2 + t^{\frac{1}{2}} x) \log t\big). \tag{34}$$

In addition, according to Cauchy's inequality, we have

$$\begin{aligned}\sum_{b=1}^{q'} \Bigg| \sum_{1 \leqslant n \leqslant x} \frac{1}{(n + \frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}} \Bigg| &\leqslant \Big(\sum_{b=1}^{q'} 1^2 \Big)^{\frac{1}{2}} \Big(\sum_{b=1}^{q'} \Bigg| \sum_{1 \leqslant n \leqslant x} \frac{1}{(n + \frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}} \Bigg|^2 \Big)^{\frac{1}{2}} \\ &\ll q' \log^{\frac{1}{2}} t. \end{aligned} \tag{35}$$

(ii) For A_2, since $\sum_{1 \leqslant u \leqslant y} 1/u = \log y + \gamma + O(1/q')$, we have

$$\begin{aligned} A_2 &= \sum_{b=1}^{q'} \Bigg| \sum_{1 \leqslant v \leqslant y} \frac{e(-\frac{b+a'}{q'} v)}{v^{\frac{1}{2}-\mathrm{i}t}} \Bigg|^2 = \sum_{1 \leqslant u \leqslant y} \sum_{1 \leqslant v \leqslant y} \sum_{b=1}^{q'} \frac{\mathrm{e}^{2\pi i \frac{b+a'}{q'}(v-u)}}{u^{\frac{1}{2}-\mathrm{i}t} v^{\frac{1}{2}+\mathrm{i}t}} \\ &= q' \sum_{1 \leqslant u \leqslant y} \frac{1}{u} + O\Bigg(q' \sum_{\substack{1 \leqslant u < v \leqslant y \\ u \equiv v \pmod{q'}}} \frac{1}{u^{\frac{1}{2}} v^{\frac{1}{2}}} \Bigg) \\ &= q'(\log y + \gamma + O(q'^{-1})) + O\Bigg(q' \sum_{1 \leqslant k \leqslant [y/q']} \sum_{1 \leqslant u \leqslant y} \frac{1}{\sqrt{u}\sqrt{kq' + u}} \Bigg) \\ &= q'(\log y + \gamma) + O\Bigg(q'^{\frac{3}{4}} \sum_{1 \leqslant k \leqslant [y/q']} \frac{1}{k^{\frac{1}{4}}} \sum_{1 \leqslant u \leqslant y} \frac{1}{u^{\frac{3}{4}}} \Bigg) \\ &= q'(\log y + \gamma) + O(y). \end{aligned} \tag{36}$$

From Cauchy's inequality we have

$$\sum_{b=1}^{q'} \Bigg| \sum_{1 \leqslant v \leqslant y} \frac{e(-\frac{b+a'}{q'} v)}{v^{\frac{1}{2}-\mathrm{i}t}} \Bigg| \leqslant \Big(\sum_{b=1}^{q'} 1^2 \Big)^{\frac{1}{2}} \Big(\sum_{b=1}^{q'} \Bigg| \sum_{1 \leqslant v \leqslant y} \frac{e(-\frac{b+a'}{q'} v)}{v^{\frac{1}{2}-\mathrm{i}t}} \Bigg|^2 \Big)^{\frac{1}{2}} \ll q' \log^{\frac{1}{2}} t. \tag{37}$$

(iii) Let $x < y$, $2\pi xy = t$. Then Proposition 3.5 yields

$$\begin{aligned} &\sum_{0 \leqslant n \leqslant x} \frac{1}{(n + \frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}} + \Big(\frac{2\pi}{t}\Big)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1 \leqslant v \leqslant y} \frac{e(-\frac{b+a'}{q'} v)}{v^{\frac{1}{2}-\mathrm{i}t}} + O\big(x^{-\frac{1}{2}} \log t\big) \\ &= \sum_{0 \leqslant n \leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{(n + \frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}} + \Big(\frac{2\pi}{t}\Big)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1 \leqslant v \leqslant \sqrt{\frac{t}{2\pi}}} \frac{e(-\frac{b+a'}{q'} v)}{v^{\frac{1}{2}-\mathrm{i}t}} + O\big(t^{-\frac{1}{4}} \log t\big). \end{aligned}$$

Therefore,

(38)

$$
\begin{aligned}
&\left(\frac{2\pi}{t}\right)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant v\leqslant y} \frac{e(-\frac{b+a'}{q'}v)}{v^{\frac12-\mathrm{i}t}}\\
&= \left(\frac{2\pi}{t}\right)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant v\leqslant \sqrt{\frac{t}{2\pi}}} \frac{e(-\frac{b+a'}{q'}v)}{v^{\frac12-\mathrm{i}t}} + \sum_{x<n\leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} + O\big(x^{-\frac12}\log t\big).
\end{aligned}
$$

The definition of A_3 together with (38), (35) implies that

$$
\begin{aligned}
(39)\qquad A_3 &= \sum_{b=1}^{q'} \sum_{1\leqslant n\leqslant x} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}} \Bigg(\left(\frac{2\pi}{t}\right)^{-\mathrm{i}t} \mathrm{e}^{-\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant v\leqslant \sqrt{\frac{t}{2\pi}}} \frac{e(\frac{b+a'}{q'}v)}{v^{\frac12+\mathrm{i}t}}\\
&\quad + \sum_{x\leqslant n\leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{(n+\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}} + O(x^{-\frac12}\log t)\Bigg)\\
&= \left(\frac{2\pi}{t}\right)^{-\mathrm{i}t} \mathrm{e}^{-\mathrm{i}(t+\frac{\pi}{4})} \sum_{1\leqslant n\leqslant x} \sum_{1\leqslant v\leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{v^{\frac12+\mathrm{i}t}} \sum_{b=1}^{q'} \frac{e(\frac{b+a'}{q'}v)}{\left(n+\frac{b+a'}{q'}\right)^{\frac12+\mathrm{i}t}}\\
&\quad + \sum_{1\leqslant n\leqslant x} \sum_{x\leqslant m\leqslant \sqrt{\frac{t}{2\pi}}} \sum_{b=1}^{q'} \frac{1}{\left(n+\frac{b+a'}{q'}\right)^{\frac12+\mathrm{i}t}\left(m+\frac{b+a'}{q'}\right)^{\frac12-\mathrm{i}t}}\\
&\quad + O(\sqrt{q'}\log t)\\
&= A_{31} + A_{32} + O(\sqrt{q'}\log t).
\end{aligned}
$$

For A_{32}, the estimate can be got by the same method as that of A_1, so we have

$$
(40)\qquad |A_{32}| \ll q' x^{\frac32} t^{-\frac34} + t^{\frac34} x^{\frac12}\log t.
$$

For A_{31}, from (39), we have

$$
\begin{aligned}
(41)\qquad |A_{31}| &\leqslant \sum_{1\leqslant n\leqslant x} \sum_{1\leqslant v\leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{v^{\frac12+\mathrm{i}t}} \sum_{b=1}^{q'} \frac{\mathrm{e}^{2\pi i\frac{b+a'}{q'}v}\left(n+\frac{b+a'}{q'}\right)^{-\mathrm{i}t}}{\left(n+\frac{b+a'}{q'}\right)^{\frac12}}\\
&\leqslant \sum_{1\leqslant n\leqslant x} \sum_{1\leqslant v\leqslant \sqrt{\frac{t}{2\pi}}} \frac{1}{v^{\frac12}} \left| \sum_{b=1}^{q'} \frac{\mathrm{e}^{2\pi i\frac{b+a'}{q'}v - \mathrm{i}t\log\left(n+\frac{b+a'}{q'}\right)}}{\left(n+\frac{b+a'}{q'}\right)^{\frac12}} \right|.
\end{aligned}
$$

Let

$$
\begin{aligned}
f(b) &= \frac{b+a'}{q'}v - \frac{t}{2\pi}\log\Big(n+\frac{b+a'}{q'}\Big),\\
g(b) &= \Big(n+\frac{b+a'}{q'}\Big)^{-\frac12}.
\end{aligned}
$$

Obviously, $g(b)$ is monotonic decreasing, therefore, we have

$$f'(b)=\frac{v}{q'}-\frac{t}{2\pi(nq'+b+a')},\qquad f''(b)=\frac{t}{2\pi(nq'+b+a')^2},$$
$$g'(b)=-\frac{1}{2q'\big(n+(b+a')/q'\big)^{\frac{3}{2}}}.$$

Thus, for $t\leqslant nq'$,

$$\text{(42)}\quad |f'(b)|=\Big|\frac{v}{q'}-\frac{t}{2\pi(nq'+b+a')}\Big|\leqslant\frac{t}{2\pi(nq'+b+a')}\leqslant\frac{t}{2\pi nq'}\leqslant\frac{1}{2\pi}<1,$$

$$\text{(43)}\quad |g(b)|\leqslant\Big|\Big(n+\frac{b+a'}{q'}\Big)^{-\frac{1}{2}}\Big|\leqslant\frac{1}{\sqrt{n}},$$

$$\text{(44)}\quad |g'(b)|\leqslant\frac{1}{\sqrt{n}},$$

$$\text{(45)}\quad -\frac{f'(b)}{g(b)}=\Big(\frac{t}{2\pi(nq'+b+a')}-\frac{v}{q'}\Big)\Big(n+\frac{b+a'}{q'}\Big)^{\frac{1}{2}}$$
$$>\Big(\frac{t}{2\pi(nq'+b+a')}-\frac{v}{q'}\Big)\Big(n+\frac{1+a'}{q'}\Big)^{\frac{1}{2}}$$
$$\gg\frac{t}{nq'}\sqrt{n}\gg\frac{t}{q'\sqrt{n}}.$$

Now, Proposition 3.3 and (41)–(45) imply

$$\text{(46)}\quad |A_{31}|\ll\sum_{1\leqslant n\leqslant x}\sum_{1\leqslant v\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{v^{\frac{1}{2}}}\Big(\frac{1}{\frac{t}{q'\sqrt{n}}}+\frac{1}{\sqrt{n}}\Big)$$
$$\ll q't^{-1}\sum_{1\leqslant n\leqslant x}\sqrt{n}\sum_{1\leqslant v\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{\sqrt{v}}$$
$$\ll q't^{-\frac{3}{4}}x^{\frac{3}{2}}.$$

When $t>nq'$ we obtain $t/n^2q'^2\ll|f''(b)|\ll t/n^2q'^2$, so applying Proposition 3.4 and (41), we have

$$\text{(47)}\quad |A_{31}|\ll\sum_{1\leqslant n\leqslant x}\sum_{1\leqslant v\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{v^{\frac{1}{2}}}\frac{\log q'}{\sqrt{n}}\max_{1<N<q'}\Big|\sum_{N<b<N+h<2N}\mathrm{e}^{2\pi if(b)}\Big|$$
$$\ll\sum_{1\leqslant n\leqslant x}\sum_{1\leqslant v\leqslant\sqrt{\frac{t}{2\pi}}}\frac{1}{v^{\frac{1}{2}}}\frac{\log q'}{\sqrt{n}}\Big[q'\Big(\frac{t}{n^2q'^2}\Big)^{\frac{1}{2}}+\Big(\frac{n^2q'^2}{t}\Big)^{\frac{1}{2}}\Big]$$
$$\ll\sum_{1\leqslant n\leqslant x}\sum_{1\leqslant v\leqslant\sqrt{\frac{t}{2\pi}}}\frac{\log t}{\sqrt{nv}}\Big(\frac{\sqrt{t}}{n}+\frac{nq'}{\sqrt{t}}\Big)\ll t^{\frac{3}{4}}x^{\frac{1}{2}}\log t+q't^{-\frac{1}{4}}x^{\frac{3}{2}}\log t.$$

Combining (46) and (47) leads to

$$
\begin{aligned}
(48) \qquad |A_{31}| &\ll q' t^{-\frac{3}{4}} x^{\frac{3}{2}} + t^{\frac{3}{4}} x^{\frac{1}{2}} \log t + q' t^{-\frac{1}{4}} x^{\frac{3}{2}} \log t \\
&\ll q' x^{\frac{3}{2}} t^{-\frac{1}{4}} \log t + t^{\frac{3}{4}} x^{\frac{1}{2}} \log t.
\end{aligned}
$$

Therefore, from (39), (40) and (48) we have

$$
(49) \qquad |A_3| \ll q' x^{\frac{3}{2}} t^{-\frac{3}{4}} + t^{\frac{3}{4}} x^{\frac{1}{2}} \log t.
$$

We know that A_4 is the conjugate of A_3, so they have the same estimates, i.e.

$$
(50) \qquad |A_4| \ll q' x^{\frac{3}{2}} t^{-\frac{3}{4}} + t^{\frac{3}{4}} x^{\frac{1}{2}} \log t.
$$

(iv) We will estimate A_5. This will be done also by using the above estimates. From (18), (35) and (37) we have
(51)

$$
\begin{aligned}
|A_5| &\ll x^{-\frac{1}{2}} \log t \sum_{1 \leqslant b \leqslant q'} \left| \sum_{1 \leqslant n \leqslant x} \frac{1}{(n + \frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}} + \left(\frac{2\pi}{t}\right)^{\mathrm{i}t} \mathrm{e}^{\mathrm{i}(t+\frac{\pi}{4})} \sum_{1 \leqslant v \leqslant y} \frac{e(-\frac{b+a'}{q'} v)}{v^{\frac{1}{2}-\mathrm{i}t}} \right| \\
&\leqslant x^{-\frac{1}{2}} \log t \Bigg(\sum_{1 \leqslant b \leqslant q'} \left| \sum_{1 \leqslant n \leqslant x} \frac{1}{(n + \frac{b+a'}{q'})^{\frac{1}{2}+\mathrm{i}t}} \right| + \sum_{1 \leqslant b \leqslant q'} \left| \sum_{1 \leqslant v \leqslant y} \frac{e(-\frac{b+a'}{q'} v)}{v^{\frac{1}{2}-\mathrm{i}t}} \right| \Bigg) \\
&\leqslant q' x^{-\frac{1}{2}} \log^{\frac{3}{2}} t.
\end{aligned}
$$

Combining the estimates in (i), (ii), (iii) and (iv), we immediately obtain

$$
\begin{aligned}
(52) \qquad A = {} & q' \log x + q'(\log y + \gamma) + O(q' t^{-1} x^{\frac{1}{2}} \log t) \\
& + O\big(t^{\frac{1}{2}} x \log t\big) + O(y) + O\big(q' x^{\frac{3}{2}} t^{-\frac{3}{4}}\big) \\
& + O\big(t^{\frac{3}{4}} x^{\frac{1}{2}} \log t\big) + O\big(q' x^{-\frac{1}{2}} \log^{\frac{3}{2}} t\big) + O(x^{-1} \log^2 t).
\end{aligned}
$$

Let $x = t^{\frac{1}{6}} \log^3 t$, $y = t/(2\pi x)$. We get

$$
(53) \qquad A = q' \log \frac{t}{2\pi} + q'\gamma + O(q' x^{-\frac{1}{12}}) + O(t^{\frac{5}{6}} \log^{\frac{5}{2}} t).
$$

This proves Lemma 3.4. □

In the end, we shall complete the proof of Theorem 1.2. From Lemma 3.2 we easily get

$$\begin{aligned}
\sum_{\chi \bmod q} \left|L\Big(\frac{1}{2}+\mathrm{i}t,\chi,a\Big)\right|^2 &= \frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\sum_{b=1}^{q'}\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t,\frac{b+a'}{q'}\Big)\right|^2\\
&= \frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\sum_{b=1}^{q'}\left|\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}+\left(\zeta\Big(\frac{1}{2}+\mathrm{i}t,\frac{b+a'}{q'}\Big)-\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}\right)\right|^2\\
&= \frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\sum_{b=1}^{q'}\frac{q'}{b+a'}\\
&\quad+\frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\sum_{b=1}^{q'}\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}\left(\zeta\Big(\frac{1}{2}-\mathrm{i}t,\frac{b+a'}{q'}\Big)-\frac{1}{(\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}}\right)\\
&\quad+\frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\sum_{b=1}^{q'}\frac{1}{(\frac{b+a'}{q'})^{\frac12-\mathrm{i}t}}\left(\zeta\Big(\frac{1}{2}+\mathrm{i}t,\frac{b+a'}{q'}\Big)-\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}\right)\\
&\quad+\frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\sum_{b=1}^{q'}\left|\left(\zeta\Big(\frac{1}{2}+\mathrm{i}t,\frac{b+a'}{q'}\Big)-\frac{1}{(\frac{b+a'}{q'})^{\frac12+\mathrm{i}t}}\right)\right|^2.
\end{aligned}$$

Using Lemmas 3.3, 3.4 and 3.1, we conclude for any real number $t \geqslant 3$

$$\begin{aligned}
&\sum_{\chi \bmod q} \left|L\Big(\frac{1}{2}+\mathrm{i}t,\chi,a\Big)\right|^2\\
&= \frac{\varphi(q)}{q}\sum_{d|q}\mu(d)q'\left(\log(q')+\gamma-\beta_{a'}+O\left(q'^{-1}\right)\right)\\
&\quad+\frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\Big(q'\Big(\log\frac{t}{2\pi}+\gamma\Big)+O(q't^{-\frac{1}{12}}+t^{\frac56}\log^3 t+\sqrt{q'}\log t)\Big)\\
&\quad+O\bigg(\frac{\varphi(q)}{q}\sum_{d|q}|\mu(d)|(q'^{\frac12}t^{\frac{5}{12}}\log t+q't^{-\frac14}\log t)\bigg)\\
&= \varphi(q)\sum_{d|q}\frac{\mu(d)}{d}\Big(\log\Big(\frac{qt}{2\pi}\Big)+2\gamma\Big)-\varphi(q)\sum_{d|q}\frac{\mu(d)}{d}\beta_{a/d}-\varphi(q)\sum_{d|q}\frac{\mu(d)\log d}{d}\\
&\quad+O\bigg(\varphi(q)t^{-\frac{1}{12}}\sum_{d|q}\frac{|\mu(d)|}{d}\bigg)+O\bigg(\frac{\varphi(q)}{q}t^{\frac56}\log^3 t\sum_{d|q}|\mu(d)|\bigg)\\
&\quad+O\bigg(\frac{\varphi(q)}{q^{\frac12}}t^{\frac{5}{12}}\log t\sum_{d|q}\frac{|\mu(d)|}{d^{\frac12}}\bigg)
\end{aligned}$$

$$= \frac{\varphi^2(q)}{q}\left(\log\left(\frac{qt}{2\pi}\right) + 2\gamma + \sum_{p|q}\frac{\log p}{p-1}\right) - \varphi(q)\sum_{d|q}\frac{\mu(d)}{d}\beta_{a/d}$$
$$+ O(qt^{-\frac{1}{12}}) + O(t^{\frac{5}{6}}\log^3 t2^{\omega(q)}) + O(q^{\frac{1}{2}}t^{\frac{5}{12}}\log t2^{\omega(q)}),$$

where $\beta_x = \sum_{n=1}^{\infty} x/n(n+x)$ is a computable constant only depending on x, and $\omega(q)$ denotes the number of distinct prime divisors of q. Here we have also used $\sum_{d|q}|\mu(d)|/d \ll q/\varphi(q)$. This completes the proof of Theorem 1.2.

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References

[1] *B. C. Berndt*: Generalized Dirichlet series and Hecke's functional equation. Proc. Edinburgh Math. Soc. *15* (1966/67), 309–313.
[2] *B. C. Berndt*: Identities involving the coefficients of a class of Dirichlet series. III. Trans. Amer. Math. Soc. *146* (1969), 323–342.
[3] *B. C. Berndt*: Identities involving the coefficients of a class of Dirichlet series. IV. Trans. Amer. Math. Soc. *149* (1970), 179–185.
[4] *D. R. Heath-Brown*: An asymptotic series for the mean value of Dirichlet L-functions. Comment. Math. Helvetici *56* (1981), 148–161.
[5] *W. P. Zhang*: On the second mean value of Dirichlet L-functions. Chinese Annals of Mathematics *11A* (1990), 121–127.
[6] *W. P. Zhang, Y. Yi and X. L. He*: On the $2k$-th power mean of Dirichlet L-functions with the weight of general Kloosterman sums. Journal of Number Theory *84* (2000), 199–213.
[7] *Y. Yi and W. P. Zhang*: On the $2k$-th power mean of Dirichlet L-functions with the weight of Gauss sums. Advances in Mathematics *31* (2002), 517–526.
[8] *R. Balasubramanian*: A note on Dirichlet L-functions. Acta Arith. *38* (1980), 273–283.
[9] *E. C. Titchmarsh*: The Theory of the Riemannn Zeta-function. Oxford, 1951.
[10] *A. Ivic*: The Riemann zeta-function. The Theory of the Riemann Zeta-Function with Applications, New York: Wiley, 1985.
[11] *C. D. Pan and C. B. Pan*: Elements of the Analytic Number Theory. Science Press, Beijing, 1991. (In Chinese.)

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