Czechoslovak Mathematical Journal, 60 (135) (2010), 655–667

A COHOMOLOGICAL STEINNESS CRITERION FOR HOLOMORPHICALLY SPREADABLE COMPLEX SPACES

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(Received January 8, 2009)

Abstract. Let X be a complex space of dimension n, not necessarily reduced, whose cohomology groups $H^1(X,\mathcal{O}),\dots,H^{n-1}(X,\mathcal{O})$ are of finite dimension (as complex vector spaces). We show that X is Stein (resp., 1-convex) if, and only if, X is holomorphically spreadable (resp., X is holomorphically spreadable at infinity).

This, on the one hand, generalizes a known characterization of Stein spaces due to Siu, Laufer, and Simha and, on the other hand, it provides a new criterion for 1-convexity.

Keywords: Stein space, 1-convex space, branched Riemannian domain, holomorphically spreadable complex space, structurally acyclic space

MSC 2010: 32E10, 32L20, 32C35, 32C15

1. Introduction

Let $X=(X,\mathcal{O}_X)$ be a complex space, not necessarily reduced. A coherent sheaf $\mathcal{F}$ on X is called Φ-*acyclic* if the cohomology groups $H^j(X,\mathcal{F})$, $j\geqslant 1$, are of finite dimension (as complex vector spaces). If $\mathcal{O}_X$ is Φ-acyclic (resp., acyclic), then we call X *structurally* Φ-*acyclic* (resp., *structurally acyclic*). For instance, every Stein space is structurally acyclic; but there are such non Stein spaces like the mixed product $\mathbb{P}^k\times\mathbb{C}^l$.

Gunning ([9], p. 157) raised the question to characterize structurally acyclic complex spaces. This belongs to the circle of ideas going back to Serre's characterization of Steinness of open sets in $\mathbb{C}^n$ precisely when they are structurally acyclic ([16]).

Our main results, which are listed below, partially answer the above question. We prove:

Theorem 1. *Let X be a complex space of dimension n. Then X is Stein if, and only if, the following two conditions hold:*

(a) *The cohomology groups $H^1(X, \mathcal{O}), \ldots, H^{n-1}(X, \mathcal{O}_X)$ are of finite dimension (as complex vector spaces).*

(b) *There is a holomorphic map with fibres Stein $f\colon X \longrightarrow S$ from X into a holomorphically spreadable complex space S.*

Proposition 1. *Let X be a complex space of dimension n such that the cohomology groups $H^1(X, \mathcal{O}), \ldots, H^{n-1}(X, \mathcal{O})$ are of finite dimension. Then X is Stein if, and only if, X is holomorphically spreadable.*

In particular, X is Stein if X can be realized as a branched Riemann domain over another Stein space.

This proposition (see also the subsequent remark 1) improves several similar results from [2], [12], [17] [18] (resp., [11]) where the case of Riemann domains over $\mathbb{C}^n$ or Stein manifolds (resp., Stein spaces) is treated. It is perhaps important to point out that all of the papers quoted above have used essentially the fact that X is a non-branched Riemann domain.

Theorem 2. *Let X be a complex space of dimension n such that the cohomology groups $H^j(X, \mathcal{O})$, $j = 1, \ldots, n-1$, are of finite dimension. Then X is 1-convex if, and only if, X is holomorphically spreadable at infinity.*

Remark 1. Here we mention some general results related to the hypotheses of our theorems. Let Y be a complex space of dimension n. Let $\mathcal{F}$ be a coherent sheaf $\mathcal{F}$ on Y. Then, for any integer $k \geqslant 1$, the complex dimension of $H^k(Y, \mathcal{F})$ is either finite or uncountable ([18]). Besides $H^k(Y, \mathcal{F})$ vanishes if $k > n$, and $H^n(Y, \mathcal{F})$ vanishes if Y has no compact irreducible component of dimension n, and has finite dimension if there are finitely many compact irreducible components of dimension n (see [19]).

Therefore in our hypotheses we could have relaxed the cohomology condition of finiteness by asking that they are at most of countable dimension and only in the range less than $\dim X$.

Suppose now that Y is holomorphically spreadable. Alessandrini [1] showed the following: Let q be an integer $\geqslant 1 - \mathrm{prof}_Y \mathcal{F}$. If $H^{q+r}(Y, \mathcal{F}) = 0$ for all $r = 1, 2, \ldots$, then $H^q(Y, \mathcal{F})$ is either zero or has infinite dimension. (Note that in [1] this theorem was stated for Y holomorphically separable; but her proof adapts easily to this more general case. However, there are examples of holomorphically spreadable complex spaces for which global holomorphic functions do not separate points; see, for instance, [10].)

This implies that if Y is structurally Φ-acyclic and holomorphically spreadable, then Y is, in fact, structurally acyclic.

2. Preliminaries

Let $X = (X, \mathcal{O}_X)$ be a complex space, not necessarily reduced. A curve, surface, *etc.*, will be a complex space of the appropriate pure dimension.

We say that X is *holomorphically spreadable at a point* $x_0 \in X$ if there are finitely many holomorphic functions $f_1, \ldots, f_k$ on X (k might depend on x_0) such that the analytic set $\{f_1 = f_1(x_0), \ldots, f_k = f(x_0)\}$ contains x_0 as an isolated point.

It can be readily seen that the set Σ of all points $x \in X$ such that X is not holomorphically spreadable at x is analytic.

Following the standard pattern, the space X is said to be *holomorphically spreadable* (resp., *holomorphically spreadable at infinity*) if Σ is the empty set (resp., Σ is compact).

Lemma 1 ([22]). *If X is holomorphically spreadable at infinity, then Σ is exceptional*[1] *in X.*

Remark 2. If X is holomorphically spreadable and $\{A_i\}_{i \in I}$ is a locally finite family of irreducible analytic sets of positive dimension, then there exists a global holomorphic function f on X such that, for all i, $f|_{A_i}$ is not the constant function. (As a matter of fact, the set of all such functions f is dense in $\mathcal{O}(X)$ with respect to the canonical topology; but we shall not need this fact.)

A complex space X is said to be a *branched Riemann domain over* another complex space S if there is a holomorphic map $\pi\colon\ X \longrightarrow S$ with fibres discrete. If π is locally biholomorphic, then (X, π) is called a *Riemann domain* (or a *spread*) over S.

A deep theorem due to Grauert [6] states that any holomorphically spreadable complex space X of dimension n can be realized as a branched Riemann domain over $\mathbb{C}^n$.

Examples. (i) Any non-singular Stein curve may be realized as a Riemann domain over $\mathbb{C}$ (see [7]).

(ii) The smooth Stein surface

$$X = \{(x : y : z) \in \mathbb{P}^2;\ x^2 + y^2 + z^2 \neq 0\}$$

cannot be realized as a non-branched Riemann domain over $\mathbb{C}^2$ (see [5]).

[1] A compact analytic set A without isolated points in a complex space X is called *exceptional* if A admits a holomorphically convex open neighborhood in which A is the maximally compact analytic set.

Because we are dealing with not necessarily reduced structures, we shall need the subsequent characterization of injectivity of multiplication by a holomorphic function. First, let us recall the singular sets of a coherent sheaf $\mathcal{F}$ on a complex space X. For a non-negative integer k, consider the set $S_k(\mathcal{F}) := \{x \in X\,;\ \operatorname{prof} \mathcal{F}_x \leqslant k\}$. Then each $S_k(\mathcal{F})$ is an analytic set in X of dimension $\leqslant k$ (see [15]). In particular $S_0(\mathcal{F})$ is discrete in X.

Proposition 2 ([21]). *For a holomorphic function f on a complex space X the following statements are equivalent:*

- *For each $x \in X$, the germ f_x is not a zero-divisor in $\mathcal{F}_x$.*
- *For each non-negative integer k, $\dim(\{f = 0\} \cap S_k(\mathcal{F})) < k$.*

This proposition will be used in conjuction with Remark 2.

Recall also that the topology of the cohomology group $H^0(X, \mathcal{F})$ is defined by a locally finite Stein covering $\{U_i\}$ of X and presentations

$$\mathcal{O}_{U_i}^{p_i} \longrightarrow \mathcal{F}_{|U_i} \longrightarrow 0.$$

From [4] we quote Lemma 2.3.2.

Lemma 2. *Let X be a Stein space and $K \subset X$ compact. Let $\Omega \Subset X$ be a Runge open set containing K. Put $L := \overline{\Omega}$. Let also $\mathcal{F}$, $\mathcal{G}$ be coherent sheaves on X, and $\mu\colon\ \mathcal{F} \longrightarrow \mathcal{G}$ a surjective $\mathcal{O}_X$-morphism.*

Then there is a constant $C > 0$ such that, for any $s \in H^0(X, \mathcal{G})$ and any constant $\tau > 0$, there is a section $\tilde{s} \in H^0(X, \mathcal{F})$ such that $\mu(\tilde{s}) = s$ and

$$\|\tilde{s}\|_K \leqslant C\|s\|_L + \tau.$$

Lemma 3. *Let X be a Stein space and D a Stein open set in X. Then the pair (X, D) is Runge if it is so with respect to the reduced structure.*

Proof. First we recall a fact about Fréchet spaces whose proof is left to the reader (a standard exercise in functional analysis).

Let $\{(E_k, \alpha_k)\}_k$ be a projective system of Fréchet spaces such that each $\alpha_k\colon E_{k+1} \longrightarrow E_k$ is continuous and surjective; let E be its projective limit endowed with the projective limit topology. Then E is a Fréchet space.

Similarly, consider $\{(F_k, \beta_k)\}_k$ and F. Suppose that there are continuous mappings $u_k\colon\ E_k \longrightarrow F_k$ with dense images such that, for all k, $\beta_k \circ u_{k+1} = u_k \circ \alpha_k$. Let $u\colon\ E \longrightarrow F$ be the canonical induced map. Then u is continuous and has dense image.

Now, taking into account the known fact that a complex space is Stein if and only if its reduction is Stein, and since X and D may be written as limits of increasing sequences $\{X_\nu\}$, $\{D_\nu\}$ of Stein open subsets such that, for all ν, the pair (X_ν, D_ν) is Runge with respect to $\operatorname{red}\mathcal{O}_X$, granting the above fact we may assume that there exists a positive integer m such that $\mathcal{N}^{m+1} = 0$, where $\mathcal{N}$ is the ideal sheaf of germs of nilpotent functions. Notice also that, for integers $j = 1, 2, \ldots$, the sheaves $\mathcal{N}^j/\mathcal{N}^{j+1}$ are coherent with respect to the reduced structure on X, and from the exact sequences

$$0 \longrightarrow \mathcal{N}^j/\mathcal{N}^{j+1} \longrightarrow \mathcal{O}/\mathcal{N}^{j+1} \longrightarrow \mathcal{O}/\mathcal{N}^j \longrightarrow 0,$$

by decreasing reccurence (start with $j = m$) and some further standard facts on Fréchet spaces, we obtain that the restriction maps

$$H^0(X, \mathcal{O}/\mathcal{N}^j) \longrightarrow H^0(D, \mathcal{O}/\mathcal{N}^j)$$

have dense image. Hence $H^0(X, \mathcal{O}_X) \longrightarrow H^0(D, \mathcal{O}_X)$ has dense image in view of the above discussion. $\square$

Corollary 1. *Let X be a holomorphically convex space and $\pi\colon X \longrightarrow \mathbb{C}^n$ a holomorphic map. Let $r > 0$ and consider Ω to be a union of connected components of the open set $\{\|\pi\| < r\}$. Then, for any coherent sheaf $\mathcal{F}$ on X, the restriction map $H^0(X, \mathcal{F}) \longrightarrow H^0(\Omega, \mathcal{F})$ has dense image.*

Proof. Here $\|\pi\| = \max(|\pi_1|, \ldots, |\pi_n|)$, where $\pi = (\pi_1, \ldots, \pi_n)$. Then the proof follows immediately from Lemma 3 using the Remmert reduction and Grauert's Coherence Theorem. $\square$

Finally, we mention that X is said to be 1-*convex* if X is holomorphically convex with a maximally compact analytic set.

Thanks to Narasimhan ([13]), 1-convexity of X is equivalent to each of the following two statments:

($\bullet$) The space X is a "proper modification of a Stein space Y in a finite number of points", *i.e.* there exists a proper holomorphic map $\pi\colon X \longrightarrow Y$ with $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ (in particular π is surjective and has connected fibers) and a finite set $B \subset Y$ such that π induces a biholomorphism between $X \setminus \pi^{-1}(B)$ and $Y \setminus B$.

($\bullet$) Every coherent sheaf $\mathcal{F}$ on X is Φ-acyclic.

3. An auxiliary result

In this section we prove the following proposition that will be used in the proof of Theorem 1. Note that, although it is close to results in [4], it requires a careful analysis and for its proof we need the generalized Remmert reduction theorem due to Wiegmann [23].

Proposition 3. *Let X be a complex space which has a holomorphic function $f\colon X \longrightarrow \mathbb{C}$ with fibers 1-convex. If $H^1(X,\mathcal{O})$ has finite dimension, then X is holomorphically convex.*

Remark 3. For $X = \mathbb{C}\times\mathbb{P}^1$ and $f\colon X \longrightarrow \mathbb{C}$ induced by the first projection one checks readily that $H^1(X,\mathcal{O}) = 0$ and f has fibres compact, *a fortiori* 1-convex. Clearly X is holomorphically convex but it fails to be an increasing union of 1-convex open subspaces. This shows that we cannot improve the conclusion of the above proposition.

Before starting the proof of the proposition, we note the following. For a complex space $X = (X,\mathcal{O}_X)$ and a complex subalgebra B of $H^0(X,\mathcal{O}_X)$ we say that X is B-*convex* if $\widehat{K}^{\mathrm{B}}$ is compact whenever $K \subset X$ is compact, where

$$\widehat{K}^{\mathrm{B}} := \Big\{x \in X\,;\ \forall f \in \mathrm{B},\ |f(x)| \leqslant \max_{y\in K} |f(y)|\Big\}.$$

Standard holomorphic convexity is recovered as $H^0(X,\mathcal{O}_X)$-convexity. It is perhaps important to notice that if X is holomorphically convex and B has finite codimension in $H^0(X,\mathcal{O}_X)$, then X is B-convex.

Also for $K = \{x_1,\ldots,x_m\}$, it is straightforward to check that $\widehat{K}^{\mathrm{B}}$ is analytic and in fact it equals

$$\widehat{K}^{\mathrm{B}} = \bigcap_{f\in\mathrm{B}}\Big(\bigcup_{j=1}^{m}\{f = f(x_j)\}\Big).$$

Theorem 3 ([23])**.** *Let X be B-convex. Then there exists (up to isomorphism) a unique Stein space $(Y,\mathcal{O}_Y)$ and a holomorphic morphism*

$$(p,p^*)\colon\ (X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$$

such that p is proper, surjective, and the induced algebra homomorphism

$$\sigma\colon\ H^0(Y,\mathcal{O}_Y) =: \mathrm{A} \longrightarrow H^0(X,\mathcal{O}_X)$$

is continuous and $\mathrm{B}\subset\sigma(\mathrm{A})$. Besides, if B is closed in $H^0(X,\mathcal{O}_X)$, then $\mathrm{B} = \sigma(\mathrm{A})$.

P r o o f of Proposition, beginning. For this it will be convenient to use the notation that if $t \in \mathbb{C}$, $X_t := \{f = t\}$ (regarded as an analytic set in X) and $\mathcal{O}_{X_t}$ is the analytic sheaf restriction of $\mathcal{O}_X/(f-t)$ to X_t. Thus $(X_t, \mathcal{O}_{X_t})$ is the full fiber $f^{-1}(t)$.

Let $\mu_t\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X$ denote the morphism induced by the multiplication by $f-t$; its kernel $\mathcal{K}_t$ becomes a coherent $\mathcal{O}_{X_t}$-module. Thus, $A_t := \mathrm{Supp}(\mathcal{K}_t)$ is analytic and contained in X_t. In fact

$$A_t = \{x \in X\,;\ (f-t)_x \text{ is a zero divisor in } \mathcal{O}_{X,x}\}.$$

Since X_t is 1-convex and $\mathcal{K}_t$ is $\mathcal{O}_{X_t}$-coherent, $\mathcal{K}_t$ is Φ-acyclic; in particular $H^2(X_t, \mathcal{K}_t)$ has finite dimension. Now granting the short exact sequence

$$0 \longrightarrow \mathcal{K}_t \longrightarrow \mathcal{O}_X \longrightarrow (f-t) \longrightarrow 0$$

and because $H^1(X, \mathcal{O}_X)$ has finite dimension by hypothesis, we deduce easily that $H^1(X, (f-t))$ has finite dimension, too. Furthermore, from the short exact sequence,

$$0 \longrightarrow (f-t) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/(f-t) \longrightarrow 0$$

one gets that the image $\mathcal{B}_t$ of $\gamma_t\colon H^0(X, \mathcal{O}_X) \longrightarrow H^0(X_t, \mathcal{O}_{X_t})$ has finite codimension; hence X_t is $\mathcal{B}_t$-convex.

Notice also that, if P is a non-constant holomorphic polynomial in one complex variable, then $P(f)$ has the same properties as f, that is its fibers are 1-convex. Moreover, we can choose P such that $P(f)H^1(X, \mathcal{O}_X) = 0$. So, from now on we assume that f, besides the hypothesis of Proposition 3, annihilates $H^1(X, \mathcal{O}_X)$. $\square$

Lemma 4. *Let K be compact subset of X. Then there exists a finite set Λ in $\mathbb{C}$ and a compact neighborhood L of K in X with the following property. For any $t \in \mathbb{C} \setminus \Lambda$ there exists $C_t > 0$ such that: for any $h \in H^0(X_t, \mathcal{O}_{X_t})$ and any $\tau > 0$ there exists $H \in H^0(X, \mathcal{O}_X)$ that extends fh and such that*

$$\|H\|_K \leqslant \tau + C_t \|fh\|_{L \cap X_t}.$$

P r o o f (Sketch). This follows as in ([4], Lemma 2.5) or ([22], Lemma 6) with some changes which we briefly mention.

First, as $H^1(X, \mathcal{O}_X)$ has finite dimension, corresponding to a locally finite covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X by relatively compact Stein open sets, one gets finitely many 1-cocycles $\{\xi_{ij}^{(r)}\}_{ij} \in Z^1(\mathcal{U}, \mathcal{O}_X)$, $r = 1, \ldots, m$, inducing a base of cohomology classes for the complex vector space $H^1(\mathcal{U}, \mathcal{O}_X)$. This induces a continuous surjective map of Fréchet spaces

$$\alpha\colon\ C^0(\mathcal{U}, \mathcal{O}_X) \oplus \mathbb{C}^m \longrightarrow Z^1(\mathcal{U}, \mathcal{O}_X),$$

given by

$$\alpha(\{g_i\}_i, \lambda) = \delta(\{g_i\}_i) + \sum_k \lambda_k \xi_{ij}^{(k)},$$

where δ is the usual coboundary map.

The second point to be taken care of is, because $\mathcal{O}_X$ might have nilpotents, to use the singular sets of $\mathcal{O}_X$ since multiplying by f does not induces an injective endomorphism of $\mathcal{O}_X$. The finite set Λ comes from this fact and is constructed as follows. Let D be the union of those U_α with $U_\alpha \cap K \neq \emptyset$. Clearly D is relatively compact in X. Now, for each integer $m \geqslant 0$ let $\{A_{mj}\}_{j \in J_m}$ be the irreducible components of dimension m of $S_m(\mathcal{O}_X)$ which intersects D on which $\mathrm{red} f$ is constant, say t_{mj}. The set Λ is the union of all t_{mj}. $\square$

P r o o f of the Proposition, concluded. Let K be a compact set in X. We show that $\widehat{K}$ (computed with respect to $H^0(X, \mathcal{O}_X)$) is compact. By Lemma 4 there exist a finite set Λ in $\mathbb{C}$ containing 0 (otherwise we add 0 to Λ) and a compact neighborhood L of K such that, for any $t \in \mathbb{C} \setminus \Lambda$ one has:

$$\widehat{K}^{\mathcal{O}(X)} \cap X_t \subseteq \widehat{L_t}^{\mathcal{B}_t}, \quad \text{where } L_t := L \cap X_t. \tag{4}$$

Moreover, since $\widehat{K}^{\mathcal{O}(X)} \cap X_t$ is compact for all $t \in \mathbb{C}$, enlarging L, if necessary, we may suppose that (4) holds true for all complex numbers t.

Now we claim that, for any $t_0 \in \mathbb{C}$, there exist a compact set F in X and $\varepsilon > 0$ such that:

$$\widehat{L_t}^{\mathcal{B}_t} \subseteq F, \quad \text{for all } t \in \mathbb{C} \text{ with } |t - t_0| \leqslant \varepsilon. \tag{5}$$

Clearly this claim concludes the proof.

In order to verify (5), there is no loss of generality if we take $t_0 \in \Lambda$, say $t_0 = 0$. (For $t_0 \notin \Lambda$ the argument is similar and somewhat easier so we omit the checking in this case.)

Then, because X_0 is $\mathcal{B}_0$-convex, $\widehat{L_0}^{B_0}$ is compact. By Theorem 3 there exist a Stein space Y and a proper morphism of complex spaces

$$(p, p^*)\colon\ (X_0, \mathcal{O}_{X_0}) \longrightarrow (Y, \mathcal{O}_Y)$$

such that p is proper, surjective, and $\sigma(H^0(Y, \mathcal{O}_Y)) = \mathcal{B}_0$, where

$$\sigma\colon\ H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(X_0, \mathcal{O}_{X_0})$$

is induced by p^*. Since $(Y, \mathcal{O}_Y)$ is Stein, there exists (see [8]) an almost-proper[2] holomorphic map $\theta\colon\ Y \longrightarrow \mathbb{C}^n$, $\theta = (\theta_1, \dots, \theta_n)$, where $n = \dim(Y)$.

Let $r > 0$ be such that $\theta(p(K \cap X_0)) \subset \Delta(r)$, where $\Delta(r) := \{z \in \mathbb{C}^n;\ \|z\| < r\}$. Since $\theta_1 \circ p, \dots, \theta_n \circ p$ belong to $\mathcal{B}_0$, there is a holomorphic map $\pi\colon\ X \longrightarrow \mathbb{C}^n$ such that $\theta \circ p = \pi|_{X_0}$. Pictorially we have a diagram:

$$\begin{tikzcd} X \supset X_0 \arrow[r, "p"] \arrow[dr, "\pi|_{X_0}"'] & Y \arrow[d, "\theta"] \\ & \mathbb{C}^n \end{tikzcd}$$

Further, from [8] (see p. 220) there are finitely many compact components of $A' := \theta^{-1}(\overline{\Delta(r)})$ meeting $p(L_0)$; let A'_0 be their union. By standard topological arguments (see [14], pp. 111–112) there exists a relatively compact open neighborhood W of A'_0 in Y with $A' \cap \partial W = \emptyset$. Put $V_0 := p^{-1}(W)$. Then V_0 is a relatively compact open set in X_0 containing $A := \theta^{-1}(A')$ and such that $A \cap \partial V_0 = \emptyset$. Since p is surjective, we deduce that $p(\partial V_0) \subset \partial W$. Let V be a relatively compact open set in X such that $V \cap X_0 = V_0$ and $A \cap \partial V = \emptyset$. Thus $|\pi| > r$ on $X_0 \cap \partial V$ and $L_0 \subset V$. Take $\varepsilon > 0$ such that $|\pi| > r$ on $X_t \cap \partial V$ and $L_t \subset V$ for $|t| < \varepsilon$. Thus, for such t, the open set $\Omega_t \subset X_t$,

$$\Omega_t := \{x \in X_t \cap V;\ \|\pi\| < r\}$$

contains L_t and equals a union of connected components of the open set $\{x \in X_t;\ \|\pi\| < r\} \subset X_t$. Note that $\Omega_t \subset V$ for t as above. By Corollary 1, the restriction map

$$H^0(X_t, \mathcal{O}_{X_t}) \longrightarrow H^0(\Omega_t, \mathcal{O}_{X_t})$$

has dense image, hence $\widehat{L_t}^{\mathcal{O}(X_t)} \subset \Omega_t$. Finally, setting $F := \overline{V} \cup \widehat{L_0}^{B_0}$ concludes the claim; hence the proof of the proposition. $\square$

[2] A continuous map $\pi\colon\ X \longrightarrow Y$ between locally compact topological spaces is said to be *almost-proper* (see [8]) if every connected component of $\pi^{-1}(K)$ is compact whenever $K \subset Y$ is compact.

Proof of Theorem 1. Let $\pi\colon X \longrightarrow S$ be a holomorphic map with fibres Stein such that S is holomorphically spreadable. The proof is divided into three steps.

Step 1. Here we recall a standard fact on Koszul's complex. Let R be a commutative ring with unit $\mathbf{1}$. Let $a_1, \dots, a_k$ be elements of R. They give the Koszul complex

$$0 \longrightarrow \Lambda^0 R^k \xrightarrow{\alpha_1} \Lambda^1 R^k \xrightarrow{\alpha_2} \dots \longrightarrow \Lambda^{k-1} R^k \xrightarrow{\alpha_k} \Lambda^k R^k \longrightarrow 0.$$

Note that $\Lambda^0 R^k \simeq R$ and $\Lambda^k R^k \simeq R$. The above complex is given as follows: Let $e_1, \dots, e_k$ be the canonical basis of R^k as R-module, $e_1 = (\mathbf{1}, 0, \dots, 0)$, *etc.* Put $\omega = a_1 e_1 + \dots + a_k e_k$. Then a basis of $\Lambda^j R^k$ consists of the wedge products $e_{i_1} \wedge \dots \wedge e_{i_j}$ where $1 \leqslant i_1 < \dots < e_{i_j} \leqslant k$ and $\alpha_{j+1}(\eta) = \omega \wedge \eta$.

We claim that $\operatorname{Ker} \alpha_j / \operatorname{Im} \alpha_{j-1}$ is an $R/(a_1, \dots, a_k)$-module.

Indeed, we put $\alpha_0 = 0$. It suffices to show that if $\xi \in \operatorname{Ker} \alpha_j$, then $a_1 \xi \in \operatorname{Im} \alpha_{j-1}$. Write $\xi = \xi' + e_1 \wedge \xi''$, where $\xi' \in \Lambda^j R^k$ and $\xi'' \in \Lambda^{j-1} R^k$ does not contain e_1. The condition $\alpha_j(\xi) = 0$ means that $\omega \wedge \xi = 0$, or $(a_1 e_1 + \dots + a_k e_k) \wedge (\xi' + e_1 \wedge \xi'') = 0$, which gives $a_1 \xi' = (a_2 e_2 + \dots + a_k e_k) \wedge \xi''$. Therefore $a_1 \xi = \omega \wedge \xi''$ as desired.

Step 2. Here we show that X is holomorphically spreadable.

Observe that, for each $s_0 \in S$, there is a holomorphic map $g\colon S \longrightarrow \mathbb{C}^k$ such that $g^{-1}(g(s_0))$ is discrete in S. This follows readily from the definition and standard arguments. Clearly we may assume that $g(s_0) = 0$. Let $f := g \circ \pi$. Thus $f^{-1}(0)$ is Stein; we regard this as a complex space $(Z, \mathcal{O}_Z)$. In fact, $Z := \{f = 0\}$ and $\mathcal{O}_Z := (\mathcal{O}_X/\mathcal{I})|_Z$, where $\mathcal{I}$ is the ideal subsheaf of $\mathcal{O}_X$ generated by $f_1, \dots, f_k$. Now, to conclude this step it suffices to show that, for any point $a \in Z$, there exist holomorphic functions $h_1, \dots, h_r$ on X such that a is isolated in the fibre of $(h_1|_Z, \dots, h_r|_Z)$ over $h(a)$.

To verify this we consider the associated Koszul complex of sheaves induced by $f_1, \dots, f_k$:

$$0 \longrightarrow \Lambda^0 \mathcal{O}_X^k \xrightarrow{\alpha_1} \Lambda^1 \mathcal{O}_X^k \xrightarrow{\alpha_2} \dots \longrightarrow \Lambda^{k-1} \mathcal{O}_X^k \xrightarrow{\alpha_k} \Lambda^k \mathcal{O}_X^k \longrightarrow 0.$$

In a canonical way, we view α_k as the morphism from $\mathcal{O}_X^k$ into $\mathcal{O}_X$ induced by $f_1, \dots, f_k$, that is $\alpha_k(g_1, \dots, g_k) = f_1 g_1 + \dots + f_k g_k$. Hence $\operatorname{Im} \alpha_k$ can be identified with the ideal subsheaf of $\mathcal{O}_X$ generated by $f_1, \dots, f_k$.

The above Koszul complex can be splitted into short exact sequences,

$$0 \longrightarrow \operatorname{Ker} \alpha_\nu \longrightarrow \Lambda^{\nu-1} \mathcal{O}_X^k \longrightarrow \operatorname{Im} \alpha_\nu \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} \alpha_{\nu-1} \longrightarrow \operatorname{Ker} \alpha_\nu \longrightarrow \operatorname{Ker} \alpha_\nu / \operatorname{Im} \alpha_{\nu-1} \longrightarrow 0$$

for $\nu = 1, 2, \ldots, k$ (with the convention that $\alpha_0 = 0$).

Now from step 1 and since $\operatorname{Ker} \alpha_1$ and $\operatorname{Ker} \alpha_\nu / \operatorname{Im} \alpha_{\nu-1}$ are $\mathcal{O}_Z$-coherent sheaves the Cartan's vanishing theorem for Stein spaces, we deduce readily in a standard way that $\operatorname{Im} \alpha_\nu$ is Φ-acyclic for all ν. In particular $H^1(X, \operatorname{Im} \alpha_k)$ has finite dimension.

Hence the image B of the restriction map $H^0(X, \mathcal{O}_X) \longrightarrow H^0(Z, \mathcal{O}_Z)$ has finite codimension in $H^0(Z, \mathcal{O}_Z)$. Since Z is Stein, *a fortiori* holomorphically convex, the space Z results B-convex. Therefore the holomorphically convex hull of any point of Z computed with respect to B is a compact analytic set, hence it is a finite set; whence step 2.

Step 3. Here we conclude the proof of Theorem 1 by induction over the dimension n of X.

First note that granting Remark 2 and Proposition 2, there exists a discrete set T in X and a holomorphic function h on X such that the germ $(h - \lambda)_x$ is not a zero divisor in $\mathcal{O}_{X,x}$, for every $x \in X \setminus T$ and $\lambda \in \mathbb{C}$.

Now fix $t \in \mathbb{C}$ and let $Y = (Y, \mathcal{O}_Y)$ be the complex space given by $Y = \{h = t\}$ (as analytic set) and structural sheaf $\mathcal{O}_Y := (\mathcal{O}_X/(h-t))|_Y$. Note that $\dim(Y) < n$.

Let $\mu\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X$ be given by multiplication with $h - t$. Then $\operatorname{Ker} \mu$ has discrete support (contained in T). It follows that $(h-t)$, the ideal sheaf generated by $h - t$ in $\mathcal{O}_X$, is Φ-acyclic.

On the one hand, from this, remark 1 and the short exact sequence

$$0 \longrightarrow (h-t) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/(h-t) \longrightarrow 0$$

we get immediately that Y is structurally Φ-acyclic. On the other hand π induces a holomorphic map from Y into S with fibers Stein; the inductive step follows applying Proposition 3, whence the proof of the theorem. $\square$

Proof of Theorem 2. Since Σ_X is exceptional in X, there is a complex space $(Y, \mathcal{O}_Y)$ and a proper holomorphic map $\pi\colon X \longrightarrow Y$ with $\pi_\star(\mathcal{O}_X) \simeq \mathcal{O}_Y$, $A := \pi(\Sigma_X)$ is a finite set of points in Y, and π induces a biholomorphism between $X \setminus \Sigma_X$ and $Y \setminus A$. Clerly this implies that Y is holomorphically spreadable.

We claim that the cohomology groups $H^j(Y, \mathcal{O}_Y)$, $j = 1, \ldots, n-1$, are of finite dimension. To see this, let V be a Stein open neighborhood of A in Y (which exists since A is a finite set). Put $U = \pi^{-1}(V)$. Since $U' := X \setminus \Sigma_X$ is biholomorphic to $V' := Y \setminus A$ via π, the Mayer-Vietoris sequence applied to $Y = V' \cup V$ and $X = U' \cup U$ gives a commutative diagram with exact rows (coefficients in $\mathcal{O}_Y$ and

$\mathcal{O}_X$, respectively) from which we infer readily by the "Five lemma" that the canonical map $H^j(Y, \mathcal{O}_Y) \longrightarrow H^j(X, \mathcal{O}_X)$ is injective for $j = 1$ and bijective for $j > 1$, whence the the above claim. Thus Y is Stein by Theorem 1, whence Theorem 2. $\square$

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