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ON DENJOY TYPE EXTENSIONS OF THE PETTIS INTEGRAL

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Abstract. In this paper two Denjoy type extensions of the Pettis integral are defined and studied. These integrals are shown to extend the Pettis integral in a natural way analogous to that in which the Denjoy integrals extend the Lebesgue integral for real-valued functions. The connection between some Denjoy type extensions of the Pettis integral is examined.

Keywords: scalar derivative, approximate scalar derivative, absolute continuity, bounded variation, VBG function, ACG function, Pettis integral, Denjoy-Pettis integral

MSC 2010: 26A39, 26B30, 46G10

1. Introduction

This paper deals with some function classes that arise naturally in the context of vector-valued integration [6], [11]. In the case of real-valued functions, the Denjoy integrals (referred to as $\mathscr{D}^*$ and $\mathscr{D}$ in Saks [13]) obviously extend the Lebesgue integral from the descriptive point of view. The classical result elucidates the nature of this extension: a Denjoy integrable real-valued function must be Lebesgue integrable on some portion of each perfect set [13]. The Pettis integral is the widest among the classical integrals of vector-valued functions [8]. The reader should refer to Talagrand's monograph [14] for the general theory of the Pettis integral. It is important to point out that the Pettis integral is equivalent to the Lebesgue integral for real-valued functions. Some Denjoy type extensions of the Pettis integral have already been proposed (see [2], [6] and the references therein). However, those integrals are actually too general. For example, it can be shown that the corresponding indefinite integrals may fail to be continuous (see Example 4.1). As a further result of this generality, neither of those integrals inherits the classical extension property of the real-valued Denjoy integrals. These difficulties have led us to demand other extensions. To this end we introduce and study classes of vector-valued functions that are VB^*, VB, AC^*, or AC in a weak sense. In particular, in the situation in which the Banach space involved contains no isomorphic copy of c_0, the space of all sequences of reals that tend to 0, we obtain a characterization of the relationship between the AC and VB properties by means of an analogue of the Banach-Zarecki Theorem. In the concluding section, it will be demonstrated how these classes can become the basis for descriptive definitions of two Denjoy type extensions of the Pettis integral, the Henstock-Kurzweil-Pettis* and Denjoy-Pettis* integrals, having the classical extension property. In addition, we examine the connection between the Denjoy-Pettis* integral and Gordon's Denjoy-Pettis integral [6].

2. Notation and Preliminaries

First of all, we set our notation and recall basic definitions. Throughout this paper $[a, b]$ will denote a fixed non-degenerate interval of the real line and I (or J) its closed non-degenerate subinterval. X denotes a real Banach space and X^* its dual. The closed unit ball of X is denoted by $B(X)$. Given $F\colon [a, b] \to X$, $\Delta F(I)$ denotes the *increment* of F on I. Finally, if E is a subset of the real line, then $\overline{E}$, ∂E, χ_E, and $\mu(E)$ will denote the *closure* of E, the *boundary* of E, the *characteristic function* of E, and the *Lebesgue measure* of E, respectively.

In what follows, we will need some standard notions related to the integration and differentiation of vector-valued functions. They are summarized below for the reader's convenience.

We first define scalar derivatives and approximate scalar derivatives [12].

Definition 2.1. Let $F\colon [a, b] \to X$.

(a) Let $t \in (a, b)$. A vector w in X is the *approximate derivative* of F at t if there exists a measurable set $E \subset [a, b]$ that has t as a point of density such that

$$\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = w.$$

We write $F'_{\mathrm{ap}}(t)$ to represent the vector w.

(b) Let $E \subset [a, b]$. A function $f\colon E \to X$ is a *scalar derivative* (an *approximate scalar derivative*) of F on E if for each x^* in X^* the function x^*F is differentiable (approximately differentiable) almost everywhere on E and $(x^*F)' = x^*f$ $((x^*F)'_{\mathrm{ap}} = x^*f)$ almost everywhere on E (the exceptional set may vary with x^*).

Next we define the classical Dunford and Pettis integrals. It should be noted that Theorem 19 of [6] guarantees the existence of the Dunford integral.

Definition 2.2. Let $f\colon [a,b] \to X$.

(a) The function f is *Dunford integrable* on $[a,b]$ if for each x^* in X^* the function x^*f is Lebesgue integrable on $[a,b]$. In this case, the *Dunford integral* of f on a measurable set $E \subset [a,b]$ is the vector x_E^{**} in X^{**} such that $x_E^{**}(x^*) = (\mathrm{L}) \int_E x^*f$ for all x^* in X^*.

(b) The function f is *Pettis integrable* on $[a,b]$ if f is Dunford integrable on $[a,b]$ and $x_E^{**} \in X$ (X is identified with its canonical image in X^{**}) for each measurable set E in $[a,b]$.

As usual, we say that the function f is Dunford or Pettis integrable on a set $E \subset [a,b]$ if the function $f\chi_E$ is Dunford or Pettis integrable on $[a,b]$, respectively. In either case, it will be convenient to use the phrase *'indefinite integral'* to mean the function $F(t) = \int_a^t f$. Then it is easy to verify that if (D) $\int_I f \in X$ for each interval I in $[a,b]$, then the function f is a scalar derivative of its indefinite Dunford integral on $[a,b]$.

3. Vector-valued functions of bounded variation

We begin with the notions of bounded variation and absolute continuity on a set. Let $F\colon [a,b] \to X$ and let E be a non-empty subset of $[a,b]$.

Definition 3.1. F is said to be VB or VB^* on E if there exists a positive number M such that

$$\Big\| \sum_{k=1}^{K} \Delta F(I_k) \Big\| \leqslant M \tag{1}$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \subset E$ or $\partial I_k \cap E \neq \emptyset$, respectively. We denote by $\mathbf{V}(F,E)$ or $\mathbf{V}_*(F,E)$ the lower bound of those M.

Definition 3.2. F is said to be AC or AC^* on E if for each positive number ε there exists a positive number η such that

$$\Big\| \sum_{k=1}^{K} \Delta F(I_k) \Big\| < \varepsilon \tag{2}$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \subset E$ or $\partial I_k \cap E \neq \emptyset$, respectively, and $\sum_{k=1}^{K} \mu(I_k) < \eta$.

Further, we say that F is VBG (ACG, VBG^*, ACG^*) on E if E can be written as a countable union of sets on each of which F is VB (AC, VB^*, AC^*). Throughout it will be convenient to say that F is *scalarly* VB (*scalarly* AC, VB^*, AC^*, VBG, ACG, VBG^*, ACG^*) on E if for each x^* in X^* the function x^*F is VB (AC, VB^*, AC^*, VBG, ACG, VBG^*, ACG^*, respectively) on E. The following lemma illustrates the usefulness of this notion.

Lemma 3.1. *Let $F\colon [a,b] \to X$ and let E be a non-empty subset of $[a,b]$. F is VB or VB^* on E if and only if F is respectively scalarly VB or scalarly VB^* on E.*

Proof. We will prove the VB case. Suppose that F is scalarly VB on E. For each positive integer m let $V_m = \{x^* \in B(X^*)\colon \mathbf{V}(x^*F, E) \leqslant m\}$. Then $B(X^*) = \bigcup_m V_m$ and we next show that each V_m is closed.

Let $x_i^* \in V_m$ and $\|x_i^* - x^*\| \to 0$ as $i \to \infty$. Fix a finite collection of non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \subset E$ and compute

$$\left|\sum_{k=1}^{K} \Delta(x^*F)(I_k)\right| = \lim_i \left\{\left|\sum_{k=1}^{K} \Delta(x_i^*F)(I_k)\right|\right\} \leqslant m.$$

This means that $x^* \in V_m$.

By the Baire Category Theorem there exist M, x_0^*, and $r > 0$ such that $\{x^*\colon \|x^* - x_0^*\| \leqslant r\} \subset V_M$. For each $x^* \in B(X^*)$ we have

$$\begin{aligned}\mathbf{V}(x^*F, E) &= r^{-1}\mathbf{V}(rx^*F + x_0^*F - x_0^*F, E)\\ &\leqslant r^{-1}\{\mathbf{V}((rx^* + x_0^*)F, E) + \mathbf{V}(x_0^*F, E)\} \leqslant \frac{2M}{r}.\end{aligned}$$

The necessity part of the lemma is obvious. □

The next theorem establishes the basic properties of the VB, VB^*, AC, and AC^* function classes.

Theorem 3.1. *Let $F\colon [a,b] \to X$ and let E be a non-empty subset of $[a,b]$.*

(a) *If F is AC on E, then F is VB on E.*
(b) *If F is both AC^* on E and bounded on $[a,b]$, then F is VB^* on E.*
(c) *If F is VB^* on E, then F is VB^* on $\overline{E}$.*
(d) *Suppose that $F|_{\overline{E}}$ is weakly continuous on $\overline{E}$. If F is VB on E, then F is VB on $\overline{E}$.*
(e) *Suppose that $F|_{\overline{E}}$ is continuous on $\overline{E}$. If F is AC (resp. AC^*) on E, then F is AC (resp. AC^*) on $\overline{E}$.*

(f) *Suppose that E is closed with $a, b \in E$ and let G be the linear extension of F from E to $[a,b]$. If F is VB or AC on E, then G is respectively VB or AC on $[a,b]$. In either case, if f is an approximate scalar derivative of F on E, then G is scalarly differentiable on $[a,b]$ to a function g such that $g|_E = f$.*

Proof. Standard arguments show that parts (a), (b), and (e) are valid.

Part (c) (and part (d)) result from Lemma 3.1 and [9, Lemma 5.3.9] (and part (d) of Theorem 6.2 of [7]).

The the VB case of part (f) follows from Lemma 3.1 and [7, Exercise 6.2].

We will prove the AC case of part (f). Let $\{J_n\}_{n=1}^{\infty}$ be the sequence of intervals contiguous to E and let $w_n = \Delta F(J_n)/\mu(J_n)$ for each n. Fix $\varepsilon > 0$. Choose $\eta_1 > 0$ that corresponds to $\varepsilon/6$ in Definition 3.2. Since $\sum\limits_{n=1}^{\infty} \mu(J_n) < \infty$, there exists a positive integer N such that $\sum\limits_{n=N}^{\infty} \mu(J_n) < \eta_1$. Choose $M > 0$ such that $\|w_n\| \leqslant M$ for all $1 \leqslant n < N$. If $1 \leqslant n < N$, then $\|\Delta G(I)\| \leqslant M\mu(I)$ for each $I \subset J_n$. Let $\eta = \min\{\eta_1, \varepsilon/3M\}$. Suppose that $\{I_k\}_{k=1}^{K}$ is a finite collection of pairwise non-overlapping intervals with $\sum\limits_{k=1}^{K} \mu(I_k) < \eta$. By partitioning each interval if necessary, we may assume that for each k either $\partial I_k \subset E$ or I_k is a proper subset of J_n for some n. Let π_0 be the set of all k such that $\partial I_k \subset E$ and let π_n be the set of all k such that I_k is a proper subset of J_n. We make note of the fact that $\sum\limits_{k\in\pi_n} \Delta G(I_k) = \lambda_n \Delta F(J_n)$ for each n and for some $\lambda_n \in [0,1]$. Let σ be the set of all $n \geqslant N$ such that $\pi_n \neq \emptyset$. Choose x^* in X^* such that $\|x^*\| = 1$ and $\Big\|\sum\limits_{n\in\sigma} \lambda_n \Delta F(J_n)\Big\| = x^*\Big(\sum\limits_{n\in\sigma} \lambda_n \Delta F(J_n)\Big)$. Finally, we let σ_+ and σ_- denote the set of all $n \in \sigma$ for which $\Delta(x^*F)(J_n) > 0$ and $\Delta(x^*F)(J_n) < 0$, respectively. Then

$$\begin{aligned}
\Big\|\sum_{k=1}^{K} \Delta G(I_k)\Big\| &\leqslant \Big\|\sum_{k\in\pi_0} \Delta F(I_k)\Big\| + \sum_{1\leqslant n<N}\sum_{k\in\pi_n} \|\Delta G(I_k)\| + \Big\|\sum_{n\in\sigma}\sum_{k\in\pi_n} \Delta G(I_k)\Big\| \\
&< \varepsilon/6 + \varepsilon/3 + \sum_{n\in\sigma} \lambda_n \Delta(x^*F)(J_n) \\
&\leqslant \varepsilon/2 + \sum_{n\in\sigma} |\Delta(x^*F)(J_n)| \\
&= \varepsilon/2 + \Big|\sum_{n\in\sigma_+} \Delta(x^*F)(J_n)\Big| + \Big|\sum_{n\in\sigma_-} \Delta(x^*F)(J_n)\Big| \\
&\leqslant \varepsilon/2 + \Big\|\sum_{n\in\sigma_+} \Delta F(J_n)\Big\| + \Big\|\sum_{n\in\sigma_-} \Delta F(J_n)\Big\| \\
&< \varepsilon/2 + \varepsilon/6 + \varepsilon/6 < \varepsilon.
\end{aligned}$$

Now suppose that f is an approximate scalar derivative of F on E. Fix x^* in X^*. It is clear that G has a scalar derivative on each interval where it is linear. Since G is VB on $[a,b]$, the function x^*G is differentiable almost everywhere on $[a,b]$. Clearly, we have $(x^*F)'_{\text{ap}} = x^*f$ almost everywhere on E. On the other hand, the equality $F|_E = G|_E$ implies $(x^*G)' = (x^*F)'_{\text{ap}}$ almost everywhere on E. It follows that $(x^*G)' = x^*f$ almost everywhere on E. Thus G is scalarly differentiable on $[a,b]$ to a function g such that $g|_E = f$. The proof is complete. □

The next theorem gives the Denjoy-Lusin definition of the VBG, ACG, VBG^*, and ACG^* properties. The proof follows the same lines as the proof in Gordon [7, Theorem 6.10].

Theorem 3.2. *Let $F\colon [a,b] \to X$ and let E be a non-empty closed subset of $[a,b]$.*

(a) *Suppose that F is bounded on $[a,b]$. Then F is VBG^* on E if and only if each perfect set in E contains a portion on which F is VB^*.*

(b) *Suppose that $F|_E$ is weakly continuous on E. Then F is VBG on E if and only if each perfect set in E contains a portion on which F is VB.*

(c) *Suppose that $F|_E$ is continuous on E. Then F is ACG or ACG^* on E if and only if each perfect set in E contains a portion on which F is AC or AC^*, respectively.*

The following theorem provides a useful characterization of the VBG and VBG^* properties. Our proof patterned after Gordon's proof [6, Lemma 29] is included for completeness.

Theorem 3.3. *Let $F\colon [a,b] \to X$ and let E be a non-empty closed subset of $[a,b]$. Suppose that $F|_E$ is weakly continuous on E. If F is scalarly VBG or scalarly VBG^* on E, then F is VBG or VBG^*, respectively, on E.*

Proof. We will prove the VBG case. Let P be a perfect set in E and let $\{J_n\}$ be the sequence of all open intervals in (a,b) such that $\partial J_n \subset \mathbb{Q}$ and $J_n \cap P \neq \emptyset$. For each pair of positive integers m and n let $A_m^n = \{x^* \in X^*\colon \mathbf{V}(x^*F, J_n \cap P) \leqslant m\}$. It follows from part (b) of Theorem 3.2 that $X^* = \bigcup\limits_m \bigcup\limits_n A_m^n$. We claim that each of the sets A_m^n is closed.

Let $x_i^* \in A_m^n$ and $\lim\limits_i \|x_i^* - x^*\| = 0$. Fix a finite collection of non-overlapping intervals $\{I_k\}_{k=1}^K$ with $\partial I_k \subset J_n \cap P$ and compute

$$\Bigl|\sum_{k=1}^K \Delta(x^*F)(I_k)\Bigr| = \lim_i \Bigl\{\Bigl|\sum_{k=1}^K \Delta(x_i^*F)(I_k)\Bigr|\Bigr\} \leqslant m.$$

This means that $x^* \in A_m^n$.

By the Baire Category Theorem there exist M, N, x_0^*, and $r > 0$ such that $\{x^*\colon \|x^* - x_0^*\| \leqslant r\} \subset A_M^N$. For each $x^* \in X^* \setminus \{0\}$ we have

$$\begin{aligned}\mathbf{V}(x^*F, J_N \cap P) &= \frac{\|x^*\|}{r}\mathbf{V}\Big(\frac{r}{\|x^*\|}x^*F + x_0^*F - x_0^*F, J_N \cap P\Big)\\ &\leqslant \frac{\|x^*\|}{r}\Big\{\mathbf{V}\Big(\Big(\frac{r}{\|x^*\|}x^* + x_0^*\Big)F, J_N \cap P\Big) + \mathbf{V}(x_0^*F, J_N \cap P)\Big\}\\ &\leqslant \frac{2M}{r}\|x^*\|.\end{aligned}$$

Hence, F is VB on $P \cap J_N$ and it follows from part (b) of Theorem 3.2 that F is VBG on E. □

In passing we point out that, by part (a) of Theorem 3.2, Theorem 3.3 is valid for the VBG^* case even if the function $F|_E$ is not weakly continuous.

4. Banach-Zarecki Type Theorems

In the first theorem of this section we will find a sufficient condition for the Dunford integrability of a scalar derivative in the situation in which no restriction is placed on the Banach space involved.

Theorem 4.1. *Let $F\colon [a,b] \to X$ be scalarly AC on $[a,b]$. If $f\colon [a,b] \to X$ is a scalar derivative of F on $[a,b]$, then f is Dunford integrable on $[a,b]$ and $\Delta F(I) =$ (D) $\int_I f$ for each interval I in $[a,b]$.*

P r o o f. For each positive integer n let

$$f_n(t) = \frac{\Delta F\big(\overline{\Delta_n^k}\big)}{\mu\big(\Delta_n^k\big)}$$

whenever $t \in \Delta_n^k = \big[a + (k-1)n^{-1}(b-a), a + kn^{-1}(b-a)\big)$ for some $k \in \{1, \dots, n\}$.

Fix x^* in X^*. As the function x^*F is VB on $[a,b]$, it follows from Lebesgue's Theorem [13, Chapter IV, Theorem 5.4] that $\{x^*f_n\}$ converges to x^*f almost everywhere on $[a,b)$ and

$$\int_a^b |x^*f_n| \leqslant 2\mathbf{V}(x^*F, [a,b]) \leqslant 2\mathbf{V}(F, [a,b]).$$

By Fatou's Lemma we have

$$\int_a^b |x^*f| \leqslant 2\mathbf{V}(F, [a,b])$$

and it follows that f is Dunford integrable on $[a,b]$. Since the function x^*F is AC on $[a,b]$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$, we have $x^*(\Delta F(I)) = \int_I x^*f$ for each interval I in $[a,b]$. Hence, $\Delta F(I) = (\mathrm{D})\int_I f$ for each interval I in $[a,b]$. This completes the proof. □

Now suppose that the Banach space X does not contain an isomorphic copy of c_0. In this context, it can easily be seen that the same hypotheses are in fact sufficient for the Pettis integrability of a scalar derivative.

Corollary 4.1. *Suppose that X does not contain an isomorphic copy of c_0 and let $F\colon [a,b] \to X$ be scalarly AC on $[a,b]$. If $f\colon [a,b] \to X$ is a scalar derivative of F on $[a,b]$, then f is Pettis integrable on $[a,b]$ and $\Delta F(I) = (\mathrm{P})\int_I f$ for each interval I in $[a,b]$.*

Proof. By the preceding theorem, $\Delta F(I) = (\mathrm{D})\int_I f \in X$ for each interval I in $[a,b]$. The Pettis integrability of f follows from [6, Theorem 23], and the equality $\Delta F(I) = (\mathrm{P})\int_I f$ is obvious. □

Recall that a function $F\colon E \to \mathbb{R}$ is said to satisfy *condition* (N) on $E \subset [a,b]$ if $\mu^*(F(A)) = 0$ for each Lebesgue negligible set $A \subset E$. Here $\mu^*(A)$ represents the *Lebesgue outer measure* of the set A. A function $F\colon E \to X$ satisfies *scalar condition* (N) on E if for each x^* in X^* the function x^*F satisfies condition (N) on E. A further consequence of Theorem 4.1 reads as follows.

Corollary 4.2. *Suppose that X does not contain an isomorphic copy of c_0 and let $F\colon [a,b] \to X$ be VB and weakly continuous on $[a,b]$, satisfy scalar condition (N) on $[a,b]$ and have a scalar derivative on $[a,b]$. Then F is AC on $[a,b]$.*

Proof. The Banach-Zarecki Theorem [7, Theorem 6.16] implies that F is scalarly AC on $[a,b]$. Now Corollary 4.1 applies to F. Thus, F is an indefinite Pettis integral and, by Proposition 2B of [4], F is AC on $[a,b]$. □

Theorem 4.2. *Suppose that X does not contain an isomorphic copy of c_0 and E is a non-empty closed subset of $[a,b]$. Let $F\colon [a,b] \to X$ have an approximate scalar derivative on E and let $F|_E$ be weakly continuous on E. Then F is AC on E if and only if F is VB on E and satisfies scalar condition (N) on E.*

Proof. With no loss of generality, we may assume that E contains a and b. Suppose first that F is VB on E and satisfies scalar condition (N) on E. We seek to prove that F is AC on E. Let G denote the linear extension of F from E to $[a,b]$ and let f be an approximate scalar derivative of F on E. By part (f) of Theorem 3.1, G is both VB and scalarly differentiable on $[a,b]$. Furthermore, the function G is weakly continuous on $[a,b]$ and satisfies scalar condition (N) on $[a,b]$. Thus Corollary 4.2 applies to G.

The necessity part of the theorem is a compilation of part (a) of Theorem 3.1 and [7, Theorem 6.12]. $\square$

Corollary 4.3. *Suppose that X does not contain an isomorphic copy of c_0 and E is a non-empty closed subset of $[a,b]$. Let $F\colon [a,b] \to X$ be VBG on E, have an approximate scalar derivative on E, satisfy scalar condition* (N) *on E, and let $F|_E$ be weakly continuous on E. Then E can be written as a countable union of closed sets on each of which F is AC.*

P r o o f. Suppose that F is VBG on E and satisfies scalar condition (N) on E. Since E is closed and since $F|_E$ is weakly continuous on E, it follows from part (d) of Theorem 3.1 that E can be written as a countable union of closed sets E_n on each of which F is VB. By the preceding theorem, F is AC on each E_n. $\square$

Corollary 4.4. *Suppose that X does not contain an isomorphic copy of c_0 and E is a non-empty closed subset of $[a,b]$. Let $F\colon [a,b] \to X$ be VBG^* on E, have a scalar derivative on E, satisfy scalar condition* (N) *on E, and let $F|_E$ be weakly continuous on E. Then E can be written as a countable union of closed sets on each of which F is both VB^* and AC.*

P r o o f. The proof is completely similar to that of Corollary 4.3. $\square$

We conclude our discussion of Banach-Zarecki type theorems with two examples showing that the principal results of this section are complete in their own terms.

Example 4.1 (Russ Gordon [6]). Let $\{I_n\}_{n=1}^{\infty}$ be a fixed sequence of intervals in $[a,b]$ such that $b_n = \max I_n < \min I_{n+1}$ for each n, $\lim\limits_n b_n = b$, and let $\{e_n\}_{n=1}^{\infty}$ denote the standard unit vector basis of c_0. We write φ_n to represent the function

$$\frac{\chi_{I_{2n-1}}}{2\mu(I_{2n-1})} - \frac{\chi_{I_{2n}}}{2\mu(I_{2n})}.$$

Define a function $f\colon [a,b] \to c_0$ by $f = \sum\limits_n \varphi_n e_n$. Then f is Dunford integrable on $[a,b]$ and

$$(\mathrm{D})\int_E f = \left(\frac{\mu(E\cap I_{2n-1})}{2\mu(I_{2n-1})} - \frac{\mu(E\cap I_{2n})}{2\mu(I_{2n})}\right)_{n=1}^{\infty} \in c_0^{**}$$

for each Lebesgue measurable set $E \subset [a,b]$ [5, pp. 128–129]. Moreover, $F(t) = (\mathrm{D})\int_a^t f \in c_0$ for all t in $[a,b)$ and $F(b) = (\mathrm{D})\int_a^b f = 0$. For each n, we have

$$\|F(b) - F(b_{2n-1})\| = \|F(b_{2n-1})\| = 1/2.$$

Thus the indefinite Dunford integral of f is not continuous at b.

Now we modify Gordon's example, showing that a continuous indefinite Dunford integral is not necessarily AC.

Example 4.2. Under the notation of the above example, define a sequence $\{x_n\}$ in c_0 by

$$e_1,\ \frac{e_2}{2},\ \frac{e_2}{2},\ \frac{e_3}{3},\ \frac{e_3}{3},\ \frac{e_3}{3},\ \frac{e_4}{4},\ \ldots$$

Note that $\|x_1\| \geqslant \|x_2\| \geqslant \ldots$, $\lim_n \|x_n\| = 0$, and the series $\sum_n x_n$ diverges. By [3, Chapter V, Theorem 6], $\sum_n x_n$ is *weakly unconditionally Cauchy* (wuC in short). Define a function $g\colon [a,b] \to c_0$ by $g = \sum_n \varphi_n x_n$. Since $\sum_n x_n$ is wuC, we have

$$\int_a^b |x^* g| \leqslant \sum_n |x^*(x_n)| \int_a^b \Bigl(\frac{\chi_{I_{2n-1}}}{2\mu(I_{2n-1})} + \frac{\chi_{I_{2n}}}{2\mu(I_{2n})}\Bigr) = \sum_n |x^*(x_n)| < \infty$$

for all x^* in c_0^*. It follows that g is Dunford integrable on $[a,b]$. Evidently we have $G(t) = (\mathrm{D}) \int_a^t g \in c_0$ for all t in $[a,b)$ and $G(b) = (\mathrm{D}) \int_a^b g = 0$. It is clear that G is continuous on $[a,b)$. Fix a positive number ε. Choose a positive integer N such that $\|x_N\| < 2\varepsilon$. We have

$$\|G(b) - G(t)\| = \|G(t)\| \leqslant \|x_N\|/2 < \varepsilon$$

whenever $t \geqslant \min I_{2N-1}$. Hence G is continuous at b as well. As the series

$$\sum_n \Delta G(I_{2n-1}) = \sum_n \frac{x_n}{2}$$

diverges, G is not AC on $[a,b]$. It should be noted that nevertheless G is ACG^* on $[a,b]$.

5. Some Denjoy type extensions of the Pettis integral

We begin by describing indefinite Pettis integrals. The next theorem extends Pettis' classical result [12, §8] from weakly sequentially complete spaces to the context of arbitrary Banach spaces.

Theorem 5.1. *A function $F\colon [a,b] \to X$ is an indefinite Pettis integral if and only if F has a scalar derivative on $[a,b]$ and is AC on $[a,b]$. In this case, the function F is the indefinite Pettis integral of any of its scalar derivatives.*

P r o o f. Suppose first that F is AC on $[a,b]$ and f is a scalar derivative of F on $[a,b]$. We seek to prove that f is Pettis integrable on $[a,b]$. Theorem 4.1 yields the Dunford integrability of f on $[a,b]$ and the equality $\Delta F(I) = (\mathrm{D}) \int_I f$ for each interval I in $[a,b]$. Let $\{I_n\}_{n=1}^{\infty}$ be an arbitrary sequence of pairwise non-overlapping intervals in $[a,b]$. Fix a positive number ε and let $\eta > 0$ correspond to ε in Definition 3.2. Since $\sum\limits_{n=1}^{\infty} \mu(I_n) < \infty$, there exists a positive integer N such that $\sum\limits_{n=N}^{\infty} \mu(I_n) < \eta$. Then $\Big\| \sum\limits_{n\in\sigma} \Delta F(I_n) \Big\| < \varepsilon$ for each finite set $\sigma \subset \{N, N+1, \ldots\}$. By [10, Proposition 1.c.1], the series $\sum\limits_{n=1}^{\infty} \Delta F(I_n)$ is unconditionally convergent in X. Now, by [4, Proposition 2B], f is Pettis integrable on $[a,b]$ and F is its indefinite Pettis integral.

The necessity part of the theorem follows easily from the definition of the Pettis integral and [4, Proposition 2B]. □

Corollary 5.1. *Suppose that E is a non-empty closed subset of $[a,b]$. Let $F\colon [a,b] \to X$ be AC on E. If f is an approximate scalar derivative of F on E, then f is Pettis integrable on E.*

P r o o f. With no loss of generality we may assume that E contains a and b. We let G denote the linear extension of F from E to $[a,b]$. By part (f) of Theorem 3.1, G is both AC and scalarly differentiable on $[a,b]$ to a function g such that $g|_E = f$. By the previous theorem, the function g is Pettis integrable on $[a,b]$. Hence, f is Pettis integrable on E which is what we desired. □

We give below an example showing that the scalar differentiability hypothesis of Theorem 5.1 cannot be eliminated.

Example 5.1 ([1]). Let $\{x_{ij}\}_{i,j=1}^{\infty}$ denote a doubly infinite complete orthonormal system in L^2 and let Σ be the σ-algebra of Lebesgue measurable sets in $[0,1]$. To each t in $[0,1]$ we assign $g_i(t) = x_{ij} \cdot \chi_{[(j-1)/2^i, j/2^i)}(t)$ $(j = 1, \ldots, 2^i)$ and, for each positive integer n, we let $f_n(t)$ denote the sum $g_1(t) + \ldots + g_n(t)$. Finally, for each $E \in \Sigma$ we write $\nu_n(E)$ to represent the vector $(\mathrm{P}) \int_E f_n$. Pettis [12, p. 303] observed that there exists a countably additive and absolutely continuous function ν from Σ to L^2 such that

$$\lim_n \sup_{E\in\Sigma} \|\nu_n(E) - \nu(E)\| = 0.$$

On the other hand, Birkhoff [1, p. 376] actually proved that there exists no Pettis integrable function f from $[0,1]$ to L^2 satisfying

$$\lim_n \sup_{E\in\Sigma} \Big\|\nu_n(E) - (\mathrm{P}) \int_E f\Big\| = 0.$$

It is evident that $F(t)=\nu([0,t])$ is AC on $[0,1]$. Now it follows from Theorem 5.1 that F has no scalar derivative on $[0,1]$.

We define four extensions of the Pettis integral.

Definition 5.1. Let $f\colon [a,b]\to X$.

(a) The function f is *Henstock-Kurzweil-Pettis integrable* or *Denjoy-Pettis integrable* on $[a,b]$ if there exists a function $F\colon [a,b]\to X$ such that $F(a)=0$ and F is scalarly ACG^* or ACG and weakly continuous on $[a,b]$ and f is respectively a scalar derivative or an approximate scalar derivative of F on $[a,b]$.

(b) The function f is *Henstock-Kurzweil-Pettis** *integrable* or *Denjoy-Pettis** *integrable* on $[a,b]$ if there exists a function $F\colon [a,b]\to X$ such that $F(a)=0$ and F is ACG^* or ACG and continuous on $[a,b]$ and f is a scalar derivative or an approximate scalar derivative of F on $[a,b]$, respectively.

A straightforward argument can be given to show that function F in the above definition is unique. Throughout such a function will be referred to as the *indefinite integral* of the function f. Given I, we write $\int_I f$ to denote the vector $\Delta F(I)$. Further, we say that the function f is integrable on a set $E\subset[a,b]$ if the function $f\chi_E$ is integrable on $[a,b]$.

It should be noted that Theorem 5.1 yields the Henstock-Kurzweil-Pettis* integrability of a Pettis integrable function. The Henstock-Kurzweil-Pettis and Denjoy-Pettis integrals have received a considerable study—see [2], [6], [5] and the references therein. Gámez and Mendoza [5] refined Gordon's Example 4.1, showing that there exists a Dunford integrable function $f\colon [a,b]\to c_0$ with the indefinite Dunford integral F such that $\Delta F(I)\in c_0$ for each interval I in $[a,b]$ and f is not Pettis integrable on any subinterval of $[a,b]$. On the other hand, it can easily be seen that the Denjoy-Pettis* integrability suffices to insure the Pettis integrability on a portion of an arbitrary perfect set. More precisely, we have the following result.

Theorem 5.2. *Suppose that $f\colon [a,b]\to X$ is Denjoy-Pettis* integrable on $[a,b]$. Then each perfect set E in $[a,b]$ contains a portion P on which f is Pettis integrable. Moreover, if $\{(a_n,b_n)\}$ is an enumeration of the intervals in $[a,b]$ contiguous to $\overline{P}$, then $\sum\limits_n (\mathrm{DP}^*)\int_{a_n}^{b_n} f$ is unconditionally convergent and*

$$(\mathrm{DP}^*)\int_a^b f=(\mathrm{P})\int_P f+\sum_n(\mathrm{DP}^*)\int_{a_n}^{b_n} f.$$

P r o o f. Let F be the indefinite Denjoy-Pettis* integral of f. By part (c) of Theorem 3.2 there exists a portion P of E such that F is AC on P. Hence, F is AC on $\overline{P}$ and Corollary 5.1 yields the Pettis integrability of f on $\overline{P}$. Let $I_n=$

$[a_n, b_n]$. Reference to the proof of Theorem 5.1 makes it obvious that $\sum_n (\mathrm{DP}^*) \int_{I_n} f$ is unconditionally convergent. Finally, by Theorem 9 of [5], the desired equality is valid and the theorem is proved. □

The next theorem gives a condition that ensures the Denjoy-Pettis* integrability of a Denjoy-Pettis integrable function.

Theorem 5.3. *Suppose that X does not contain an isomorphic copy of c_0 and let $f\colon [a,b] \to X$ be Denjoy-Pettis integrable on $[a,b]$. If $F(t) = (\mathrm{DP}) \int_a^t f$ is continuous on $[a,b]$, then f is Denjoy-Pettis* integrable on $[a,b]$ and $F(t) = (\mathrm{DP}^*) \int_a^t f$ for all t in $[a,b]$.*

Proof. It suffices to show that the function F is ACG on $[a,b]$. By part (a) of Theorem 3.1, F is scalarly VBG on $[a,b]$. Since F is weakly continuous on $[a,b]$, it follows from Theorem 3.3 that F is VBG on $[a,b]$. Now Corollary 4.3 applies to F. □

Recall that a real Banach space X is said to have the *Schur property* (or to be a *Schur space*, in short) if each weakly null sequence in X converges in norm.

Theorem 5.4. *The following two assertions are equivalent:*

(i) *X is a Schur space;*

(ii) *if $f\colon [0,1] \to X$ is Denjoy-Pettis integrable on $[0,1]$, then f is Denjoy-Pettis* integrable on $[0,1]$.*

Proof. (i) $\Rightarrow$ (ii). Let $f\colon [0,1] \to X$ be Denjoy-Pettis integrable on $[0,1]$ and let $F(t) = \int_0^t f$ for all t in $[0,1]$. Since F is weakly continuous on $[0,1]$ and X is a Schur space, F is continuous on $[0,1]$. Now Theorem 5.3 applies to f.

(ii) $\Rightarrow$ (i). On the contrary, assume that X fails the Schur property, then there is a sequence $\{x_n\}$ in X such that for all x^* in X^* we have $\lim_n x^*(x_n) = 0$ and $\|x_n\| \geqslant 1$ for all n. Let C denote the Cantor ternary set and let $\{(a_k^{(i)}, b_k^{(i)})\}$, $k = 1, \ldots$, $i = 1, \ldots, 2^{k-1}$ be the natural enumeration of the intervals in $[0,1]$ contiguous to C. For a fixed positive integer k, we let F_k denote the real-valued function defined on $[0,1]$ that equals 0 on the set $\{0, 1, a_k^{(1)}, b_k^{(1)}, \ldots, a_k^{(2^{k-1})}, b_k^{(2^{k-1})}\}$, equals 1 on the set $\{\frac{1}{2}(a_k^{(1)} + b_k^{(1)}), \ldots, \frac{1}{2}(a_k^{(2^{k-1})} + b_k^{(2^{k-1})})\}$, and is linear on the intervals between these points. Then define $F(t)$ by $\sum_{k=1}^{\infty} F_k(t) x_k$ for all t in $[0,1]$. It is obvious that F is ACG on $[0,1]$ and discontinuous on C. We claim that F is weakly continuous on $[0,1]$. Fix an arbitrary balanced weak neighborhood $\mathscr{O}$ of 0. Then there exists a positive integer K such that $x_k \in \mathscr{O}$ for each $k > K$. Since $0 \leqslant F_k(t) \leqslant 1$ for all t in $[0,1]$ and for all k, it follows that $\sum_{k=K+1}^{\infty} F_k(t)x_k \in \mathscr{O}$ for all t in $[0,1]$. By Lemma 1 of [15], F is weakly continuous on $[0,1]$. On the other hand, it is clear that $F' = f$ almost everywhere on $[0,1]$. So, by (ii), f is Denjoy-Pettis* integrable on $[0,1]$. Thus, the function F is an indefinite Denjoy-Pettis* integral while it is discontinuous on C. This is the desired contradiction. □

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