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THE (SIGNLESS) LAPLACIAN SPECTRAL RADIUS OF  
UNICYCLIC AND BICYCLIC GRAPHS WITH  $n$  VERTICES AND  
 $k$  PENDANT VERTICES

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*Abstract.* In this paper, the effects on the signless Laplacian spectral radius of a graph are studied when some operations, such as edge moving, edge subdividing, are applied to the graph. Moreover, the largest signless Laplacian spectral radius among the all unicyclic graphs with  $n$  vertices and  $k$  pendant vertices is identified. Furthermore, we determine the graphs with the largest Laplacian spectral radii among the all unicyclic graphs and bicyclic graphs with  $n$  vertices and  $k$  pendant vertices, respectively.

*Keywords:* Laplacian matrix, signless Laplacian matrix, spectral radius

*MSC 2010:* 05C50, 05C75

## 1. INTRODUCTION

Throughout the paper,  $G = (V, E)$  is a connected undirected simple graph with  $V = \{v_1, v_2, \dots, v_n\}$  and  $E = \{e_1, e_2, \dots, e_m\}$ . Especially, if  $m = n$  or  $m = n + 1$ , then  $G$  is called a unicyclic or bicyclic graph, respectively. The notation  $N(v)$  is used to denote the neighbors of vertex  $v$ . The degree of vertex  $v$ , written by  $d(v)$ , is  $d(v) = |N(v)|$ . Specially, we use  $\Delta(G)$  to indicate the maximum degree of  $G$ . If  $d(v) = 1$ , then  $v$  is called a pendant vertex of  $G$ . Let the adjacency matrix, degree matrix of  $G$  be  $A(G) = [a_{ij}]$ ,  $D(G) = \text{diag}\{d(v_1), d(v_2), \dots, d(v_n)\}$ , respectively. The *Laplacian matrix* of  $G$  is  $L(G) = D(G) - A(G)$  and the *signless Laplacian matrix* of  $G$  is  $Q(G) = D(G) + A(G)$ . Denote the spectral radii of  $A(G)$ ,  $L(G)$  and  $Q(G)$

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by  $\varrho(G)$ ,  $\lambda(G)$ , and  $\mu(G)$ , respectively. For the relation between  $\lambda(G)$  and  $\mu(G)$ , it is well known that

**Proposition 1.1** ([15], [14]).  $\lambda(G) \leq \mu(G)$ , the equality holds if and only if  $G$  is bipartite.

If  $G$  is connected, by the Perron-Frobenius Theorem of non-negative matrices,  $\mu(G)$  has multiplicity one and there exists a unique positive unit eigenvector corresponding to  $\mu(G)$ . We refer to such an eigenvector as the *Perron vector* of  $\mu(G)$ .

Our terminology and notation are standard except as indicated. For terminology and notation not defined here, we refer the readers to [1], [2], [4]–[6], [10], [12], [13], [17], [18] and the references therein.

It is well known that graph spectrum has great important application in many fields. Several graph spectra, i.e., spectra of  $A(G)$ ,  $L(G)$  and  $Q(G)$ , have been defined in [3]. The spectra of  $A(G)$ ,  $L(G)$  are well studied (for instance see [4], [6], [8], [12], [13]), but the spectrum of  $Q(G)$  seems to be less well known. It is not until recent years, some researchers found that the spectrum of  $Q(G)$  has a strong connection with the structure of the graph (see [7], [10]). Thus, more and more mathematicians became interested in it and devoted themselves to the study [2], [5], [7], [10].

The problem concerning graphs with maximal or minimal spectral radius over a given class of graphs proposed in [1] has been studied extensively. In this direction, Wu et al. [17] determined the unique tree with the largest spectral radius in the class of trees with  $n$  vertices and  $k$  pendant vertices, and Guo [9] identified the graphs with the largest spectral radius in the class of unicyclic and bicyclic graphs with  $n$  vertices and  $k$  pendant vertices, respectively. Very recently, Geng et al. [6] obtained the unique tricyclic graph with the largest spectral radius in the class of tricyclic graphs with  $n$  vertices and  $k$  pendant vertices. In this paper, we shall consider the similar problem for signless Laplacian spectral radius and Laplacian spectral radius. We determine the unique graph with the largest signless Laplacian spectral radius among all unicyclic graphs with  $n$  vertices and  $k$  pendant vertices, and the graphs with the largest Laplacian spectral radii among all unicyclic graphs and bicyclic graphs with  $n$  vertices and  $k$  pendant vertices, respectively.

The paper is organized as follows. In the second section, we obtain some properties for the signless Laplacian spectral radius of a graph when some operations, such as edge moving, edge subdividing, are applied to the graph. In the third section, we determine the graphs with the largest signless Laplacian spectral radius and the largest Laplacian spectral radius among all unicyclic graphs having  $n$  vertices and  $k$  pendant vertices, respectively. In the fourth section, we identify the graph with

the largest Laplacian spectral radius among all bicyclic graphs having  $n$  vertices and  $k$  pendant vertices.

## 2. SOME PROPERTIES OF THE SIGNLESS LAPLACIAN SPECTRAL RADIUS

Let  $P_n$  and  $C_n$  be the path and cycle on  $n$  vertices, respectively. Let  $G - u$  or  $G - uv$  denote the graph that obtained from  $G$  by deleting the vertex  $u \in V(G)$  or the edge  $uv \in E(G)$ . Similarly,  $G + uv$  is a graph that arises from  $G$  by adding an edge  $uv \notin E(G)$ , where  $u, v \in V(G)$ .

Let  $X(G)$  be the line graph of  $G$ . It is well known that (for example, see [13], p. 23):

$$(1) \quad \mu(G) = 2 + \varrho(X(G)).$$

In the study of spectral theory, the effects on the spectrum are observed when some operations, such as edge moving, edge subdividing, are applied to the graph. For example, the following lemmas are stated for the spectral radius of the adjacency matrix.

**Lemma 2.1** ([12]). *Let  $uv$  be an edge of a graph  $G$  satisfying  $d(u) \geq 2$  and  $d(v) \geq 2$ , and suppose that two new paths  $P: uu_1u_2 \dots u_k$  and  $Q: vv_1v_2 \dots v_m$  of length  $k$  and  $m$  ( $k \geq m \geq 1$ ) are attached to  $G$ , respectively, to form  $M_{k,m}$ , where  $u_1, u_2, \dots, u_k$  and  $v_1, v_2, \dots, v_m$  are distinct new vertices. Then, we have  $\varrho(M_{k,m}) > \varrho(M_{k+1,m-1})$ .*

Suppose  $v$  is a vertex of a connected graph  $G$  with at least two vertices. Let  $G_{k,l}$  ( $l \geq k \geq 1$ ) be the graph obtained from  $G$  by attaching two new paths  $P: v(= v_0)v_1v_2 \dots v_k$  and  $Q: v(= u_0)u_1u_2 \dots u_l$  of length  $k$  and  $l$ , respectively, at  $v$ , where  $v_1, v_2, \dots, v_k$  and  $u_1, u_2, \dots, u_l$  are distinct new vertices. Let  $G_{k-1,l+1} = G_{k,l} - v_{k-1}v_k + u_lv_k$ . It has been proved that

**Lemma 2.2** ([12]). *Let  $G$  be a connected graph on  $n \geq 2$  vertices. If  $l \geq k \geq 1$ , then*

$$\varrho(G_{k,l}) > \varrho(G_{k-1,l+1}).$$

By  $G \subset G'$ , we mean that  $G$  is a subgraph of  $G'$  and  $G \not\cong G'$ . It is well known that (for example, see [13], p. 17–18):

**Lemma 2.3.** *If  $G \subset G'$  and  $G'$  is a connected graph, then  $\varrho(G) < \varrho(G')$ .*

By Lemma 2.3, it immediately follows

**Proposition 2.1.** *If  $G \subset G'$  and  $G'$  is a connected graph, then  $\mu(G) < \mu(G')$ .*

*Proof.* Since  $G \subset G'$  and  $G'$  is a connected graph, then  $X(G) \subset X(G')$  and  $X(G')$  is a connected graph. This implies that  $\varrho(X(G)) < \varrho(X(G'))$ . Bearing in mind the equality (1), then the result follows.  $\square$

**Lemma 2.4** ([4]). *Suppose  $M_{n \times n}$  is a symmetric, nonnegative matrix,  $y$  is an  $n$ -tuple positive vector and  $\mu'$  is a positive real number. If  $My \leq \mu'y$  and  $My \neq \mu'y$ , then  $\varrho_1(M) < \mu'$ , where  $\varrho_1(M)$  is the largest eigenvalue of  $M$ .*

With the help of the above lemmas, we can obtain the similar results on  $\mu(G)$  for the general connected graphs.

### 1. Edge moving operation

**Theorem 2.1.** *Let  $G$  be a connected graph on  $n \geq 2$  vertices. If  $l \geq k \geq 1$ , then*

$$\mu(G_{k,l}) > \mu(G_{k-1,l+1}).$$

*Proof.* We consider the next two cases.

*Case 1.*  $k = 1$ . Without loss of generality, suppose  $e_1 = vv_1$ ,  $e_2 = vu_1$ ,  $e_3 = u_1u_2, \dots, e_{l+1} = u_{l-1}u_l$ ,  $e_t = u_lv_1$ . Then  $G_{1,l}, G_{0,l+1}, X(G_{1,l}), X(G_{0,l+1})$  are the graphs as shown in Fig. 1. Let  $G_1 = X(G_{1,l}) \setminus \{wv_{e_1} : w \neq v_{e_2}\}$ , then  $G_1 \subset X(G_{1,l})$ , thus  $\varrho(G_1) < \varrho(X(G_{1,l}))$  follows from Lemma 2.3.

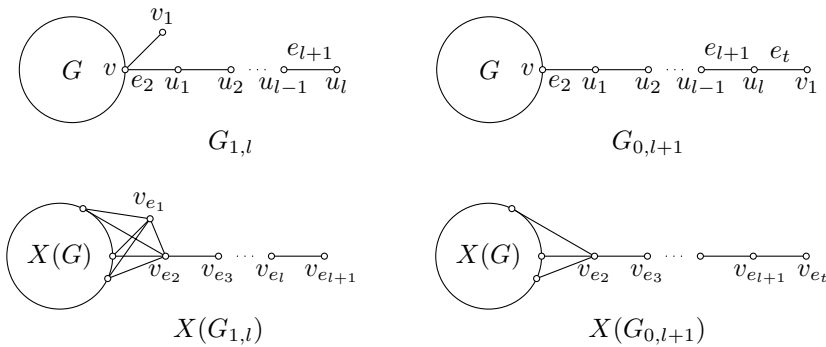


Fig. 1

*Subcase 1.1.*  $l = 1$ . It is easy to see that  $G_1 \cong X(G_{0,l+1})$ , this implies that  $\varrho(X(G_{0,l+1})) = \varrho(G_1) < \varrho(X(G_{1,l}))$ . Thus, the result follows from equality (1).

*Subcase 1.2.*  $l \geq 2$ . By Lemma 2.2, we have  $\varrho(X(G_{0,l+1})) < \varrho(G_1) < \varrho(X(G_{1,l}))$ . Thus, the result follows from equality (1).

*Case 2.*  $k \geq 2$ . Without loss of generality, suppose  $e_1 = vv_1$ ,  $e_2 = v_1v_2, \dots$ ,  $e_k = v_{k-1}v_k$  and  $e_{k+1} = vu_1$ ,  $e_{k+2} = u_1u_2, \dots$ ,  $e_{k+l} = u_{l-1}u_l$ ,  $e_t = u_lv_k$ . Then  $G_{k,l}$ ,  $G_{k-1,l+1}$ ,  $X(G_{k,l})$ ,  $X(G_{k-1,l+1})$  are the graphs as shown in Fig. 2. By Lemma 2.1, it follows that  $\varrho(X(G_{k-1,l+1})) < \varrho(X(G_{k,l}))$ . Bearing in mind the equality (1), the result follows.

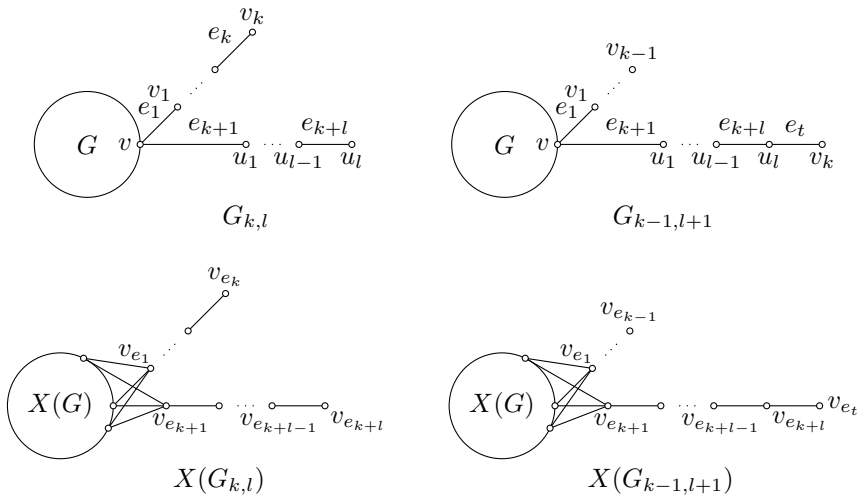


Fig. 2

By combining the above discussion, the assertion follows.  $\square$

By Proposition 1.1 and Theorem 2.1, we have

**Corollary 2.1** ([8]). *Let  $G$  be a connected bipartite graph on  $n \geq 2$  vertices. If  $l \geq k \geq 1$ , then  $\lambda(G_{k,l}) > \lambda(G_{k-1,l+1})$ .*

**Lemma 2.5** ([18]). *Let  $G = (V(G), E(G))$  be a connected simple graph with  $uv_i \in E(G)$  and  $wv_i \notin E(G)$  for  $i = 1, \dots, k$ . Let  $G' = (V'(G), E'(G))$  be a new graph obtained from  $G$  by deleting edges  $uv_i$  and adding edges  $wv_i$  for  $i = 1, \dots, k$ . Let  $x = (x_1, x_2, \dots, x_n)^T$  be a Perron vector of  $\mu(G)$ . If  $x_w \geq x_u$ , then  $\mu(G) < \mu(G')$ .*

## 2. Edge subdividing operation

Let  $G$  be a connected graph, and  $uv \in E(G)$ . The graph  $G_{u,v}^*$  is obtained from  $G$  by subdividing the edge  $uv$ , i.e., adding a new vertex  $w$  and edges  $wu, wv$  in  $G - uv$ . An *internal path*, say  $v_1v_2 \dots v_{s+1}$  ( $s \geq 1$ ), is a path joining  $v_1$  and  $v_{s+1}$  (which need not be distinct) such that  $v_1$  and  $v_{s+1}$  have degree greater than 2, while all other

vertices  $v_2, \dots, v_s$  are of degree 2. A *pendant path* of a graph is a path with one of its end vertices having degree one and all the internal vertices having degree two. Clearly, a pendant path of length one is a pendant edge.

**Theorem 2.2.** *Let  $uv$  be an edge of the connected graph  $G$ .*

- (1) *If  $uv$  belongs to a pendant path of  $G$ , then  $\mu(G_{u,v}^*) > \mu(G)$ .*
- (2) *If  $uv$  belongs to an internal path of  $G$ , then  $\mu(G_{u,v}^*) < \mu(G)$ .*

*Proof.* (1) Since  $G \subset G_{u,v}^*$ , then  $\mu(G_{u,v}^*) > \mu(G)$  follows from Proposition 2.1.

(2) For convenience, we assume  $v_1v_2 \dots v_a$  ( $a \geq 2$ ) is an internal path of  $G$  and  $x = (x_1, x_2, \dots, x_n)^T$  is the Perron vector of  $\mu(G)$ , where  $x_i$  ( $> 0$ ) corresponds to the vertex  $v_i$  ( $1 \leq i \leq n$ ). Without loss of generality, suppose that  $x_1 \leq x_a$ , and  $x_t = \min\{x_t, x_{t+1}, \dots, x_a\}$  such that  $x_t < x_i$  for  $1 \leq i \leq t-1$ . We divide the proof into the next two cases.

*Case 1.*  $t = 1$ . Let  $G' = G - v_1v_2 + v_1w + wv_2$ , where  $w \notin V(G)$ . It is easy to see that  $G_{u,v}^* \cong G'$ . Let  $y = (y_1, y_w, y_2, \dots, y_n)^T$ , where  $y_w = y_1 = x_1$  and  $y_i = x_i$  for  $2 \leq i \leq n$ . This implies that  $y$  is an  $(n+1)$ -tuple positive vector. Let  $s = \sum_{v_j \in N(v_1)} x_j - x_2$ , where  $N(v_1)$  is the set of neighbors of  $v_1$  in  $G$ . Then,

$$\begin{aligned} (Q(G')y)_1 &= d(v_1)x_1 + s + y_w = s + (d(v_1) + 1)x_1, \\ (\mu(G)y)_1 &= \mu(G)y_1 = \mu(G)x_1 = d(v_1)x_1 + s + x_2. \end{aligned}$$

Since  $x_1 \leq x_2$ , then  $(Q(G')y)_1 \leq (\mu(G)y)_1$ . Moreover, we have

$$\begin{aligned} (Q(G')y)_w &= 2y_w + x_1 + x_2 = 3x_1 + x_2, \\ (\mu(G)y)_w &= \mu(G)y_w = \mu(G)x_1 = d(v_1)x_1 + s + x_2. \end{aligned}$$

Since  $d(v_1) \geq 3$ ,  $s > 0$ , thus  $(Q(G')y)_w < (\mu(G)y)_w$ .

For the other vertex  $v_j$  ( $j \neq 1, w$ ), we have  $(Q(G')y)_j = (\mu(G)y)_j$ . Combining the above discussion, we can conclude that  $Q(G')y \leq \mu(G)y$  and  $Q(G')y \neq \mu(G)y$ , thus  $\mu(G_{u,v}^*) = \mu(G') < \mu(G)$  follows from Lemma 2.4.

*Case 2.*  $1 < t < a$ . Let  $G' = G - v_{t-1}v_t + v_{t-1}w + wv_t$ , where  $w \notin V(G)$ . It is easy to see that  $G^*(u, v) \cong G'$ . Let  $y = (y_1, \dots, y_{t-1}, y_w, y_t, \dots, y_n)^T$ , where  $y_w = x_t$  and  $y_i = x_i$  for  $1 \leq i \leq n$ . This implies that  $y$  is an  $(n+1)$ -tuple positive vector. Then

$$\begin{aligned} (Q(G')y)_w &= 2y_w + x_{t-1} + x_t = x_{t-1} + 3x_t, \\ (\mu(G)y)_w &= \mu(G)y_w = \mu(G)x_t = 2x_t + x_{t-1} + x_{t+1}. \end{aligned}$$

Since  $x_t \leq x_{t+1}$ , thus  $(Q(G')y)_w \leq (\mu(G)y)_w$ . Moreover, we have

$$\begin{aligned}(Q(G')y)_t &= 2x_t + y_w + x_{t+1} = 3x_t + x_{t+1}, \\ (\mu(G)y)_t &= \mu(G)y_t = \mu(G)x_t = 2x_t + x_{t-1} + x_{t+1}.\end{aligned}$$

Since  $x_t < x_{t-1}$ , thus  $(Q(G')y)_t < (\mu(G)y)_t$ .

For the other vertex  $v_j$  ( $j \neq t, w$ ), we have  $(Q(G')y)_j = (\mu(G)y)_j$ . Combining the above discussion, we can conclude that  $Q(G')y \leq \mu(G)y$  and  $Q(G')y \neq \mu(G)y$ , thus  $\mu(G_{u,v}^*) = \mu(G') < \mu(G)$  follows from Lemma 2.4.

By combining the above arguments, we have  $\mu(G_{u,v}^*) < \mu(G)$ . This completes the proof.  $\square$

**Corollary 2.2.** *Suppose  $uv$  is an edge of the connected bipartite graph  $G$ .*

- (1) *If  $uv$  belongs to a pendant path of  $G$ , then  $\lambda(G_{u,v}^*) > \lambda(G)$ .*
- (2) *If  $uv$  belongs to an internal path of  $G$  and  $G_{u,v}^*$  is also a bipartite graph, then  $\lambda(G_{u,v}^*) < \lambda(G)$ .*

*Proof.* We only prove (1), because (2) can be proved similarly. It is easy to see that  $G_{u,v}^*$  is also bipartite as  $G$  is bipartite, then  $\lambda(G_{u,v}^*) = \mu(G_{u,v}^*) > \mu(G) = \lambda(G)$  follows from Proposition 1.1 and Theorem 2.2. Thus (1) holds.  $\square$

### 3. THE LARGEST $\mu(G)$ (RESP. $\lambda(G)$ ) IN THE CLASS OF UNICYCLIC GRAPHS WITH $n$ VERTICES AND $k$ PENDANT VERTICES

Let  $G$  be a connected graph and let  $T$  be a tree such that  $T$  is attached to a vertex  $v$  of  $G$ . The vertex  $v$  is called the *root* of  $T$ . Throughout this paper, we assume that  $T$  does not include the root. Given  $u, v \in V(G)$ , the symbol  $d(u, v)$  is used to denote the *distance* between  $u$  and  $v$ , i.e., the length of (number of edges in) the shortest path that connects  $u$  and  $v$  in  $G$ . Paths  $P_{l_1}, \dots, P_{l_k}$  are said to *have almost equal lengths* if  $l_1, \dots, l_k$  satisfy  $|l_i - l_j| \leq 1$  for  $1 \leq i \leq j \leq k$ .

For integers  $n, k$ , let  $\mathbb{U}_n(k)$  denote the class of connected unicyclic graphs with  $n$  vertices and  $k$  pendant vertices, and let  $\mathbb{U}_n(t, k)$  be the class of connected unicyclic graphs on  $n$  vertices and  $k$  pendant vertices with the unique cycle of length  $t$ . The notation  $W_n(t, k)$  denotes the unicyclic graph on  $n$  vertices obtained from a cycle, say  $C_t$ , by attaching  $k$  paths of almost equal lengths to one vertex of  $C_t$ . Obviously,  $W_n(t, k) \in \mathbb{U}_n(t, k) \subseteq \mathbb{U}_n(k)$ .



**Lemma 3.1.** *Suppose  $t$  and  $k$  are integers with  $t \geq 3$  and  $1 \leq k \leq n - t$ . If  $G \in \mathbb{U}_n(t, k)$ , then  $\mu(G) \leq \mu(W_n(t, k))$ , with equality holding if and only if  $G \cong W_n(t, k)$ .*

*Proof.* Choose  $G \in \mathbb{U}_n(t, k)$  such that the signless Laplacian spectral radius of  $G$  is as large as possible. Denote the vertex set of  $G$  by  $\{v_1, \dots, v_n\}$  and the Perron vector of  $\mu(G)$  by  $x = (x_1, \dots, x_n)^T$ , where  $x_i (> 0)$  corresponds to the vertex  $v_i$  ( $1 \leq i \leq n$ ).

We first prove that  $G$  is a graph obtained by attaching some tree to only one vertex of  $C_t$ . On the contrary, assume that there exist trees  $T_1, T_2$  attached to  $v_1, v_2$  of  $C_t$ , respectively. Without loss of generality, suppose  $x_1 \leq x_2$ . Note that there must be some vertex  $u \in V(T_1) \cap N(v_1)$  such that  $u \notin N(v_2)$ , let

$$G_1 = G - v_1u + v_2u,$$

then  $G_1 \in \mathbb{U}_n(t, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

Thus,  $G$  is a graph obtained by attaching some tree  $T$  to one vertex, say  $v_1$ , of  $C_t$ . We now prove that each vertex  $v$  of  $T$  has degree  $d(v) \leq 2$ . On the contrary, assume there exists one vertex  $v_i \in V(T)$  such that  $d(v_i) \geq 3$  and  $d(v_i, v_1)$  is as small as possible.

If  $x_1 \geq x_i$ , since  $d(v_i) \geq 3$ , then there must exist one vertex  $u \in N(v_i)$  such that  $d(v_1, u) > d(v_1, v_i)$ . Clearly,  $u \notin N(v_1)$ . Let

$$G_1 = G - uv_i + uv_1,$$

then  $G_1 \in \mathbb{U}_n(t, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

If  $x_1 < x_i$ , we consider the next two cases.

*Case 1.*  $d(v_i, v_1) = 1$ . Assume  $C_t = v_1v_2v_3 \dots v_tv_1$ . Clearly,  $v_2 \notin N(v_i)$ . Let

$$G_1 = G - v_1v_2 + v_iv_2, \quad G_2 = G_1 - v_iv_2 - v_2v_3 + v_iv_3, \quad G_3 = G_2 - v_2, \quad G_4 = G_2 + v_2v_s,$$

where  $v_s$  is a pendant vertex of  $G$ .

By Lemma 2.5,  $\mu(G) < \mu(G_1)$ . Since  $k \geq 1$ , then  $d(v_1) \geq 3$ . Thus,  $v_1v_tv_{t-1} \dots v_3v_i$  or  $v_iv_1v_tv_{t-1} \dots v_3v_i$  is in an internal path of  $G_3$ . By Theorem 2.2,  $\mu(G_1) < \mu(G_3) < \mu(G_4)$ . Thus, we can conclude that  $\mu(G) < \mu(G_4)$ . But  $G_4 \in \mathbb{U}_n(t, k)$ , a contradiction to the choice of  $G$ .

*Case 2.*  $d(v_i, v_1) \geq 2$ . Suppose  $P = v_1v_2 \dots v_lv_i$  is the unique path of length  $l$  from  $v_1$  to  $v_i$ , then  $l \geq 2$  by  $d(v_i, v_1) \geq 2$ . Let

$$G_1 = G - v_iv_l - v_lv_{l-1} + v_iv_{l-1}, \quad G_2 = G_1 - v_l, \quad G_3 = G_1 + v_lv_s,$$

where  $v_s$  is a pendant vertex of  $G$ .

Clearly,  $G_3 \in \mathbb{U}_n(t, k)$ . By the choice of  $v_i$ ,  $v_1v_2 \dots v_{l-1}v_i$  is an internal path of  $G_2$ . By Theorem 2.2, we have  $\mu(G) < \mu(G_2) < \mu(G_3)$ , a contradiction.

Thus,  $G$  is a graph obtained by attaching  $k$  paths to the vertex  $v_1$  of  $C_t$ . Finally, we prove that  $G \cong W_n(t, k)$ , i.e., the  $k$  paths have almost equal lengths. On the contrary, assume that there exist two paths, say  $P_{l_1}$  and  $P_{l_2}$ , such that  $l_1 - l_2 \geq 2$  and  $l_2 \geq 2$ . Denote  $P_{l_1} = u_1 \dots u_{l_1}$  and  $P_{l_2} = w_1 \dots w_{l_2}$ , where  $u_1 = v_1 = w_1$ . Let

$$G_1 = G - u_{l_1-1}u_{l_1} + w_{l_2}u_{l_1},$$

then  $G_1 \in \mathbb{U}_n(t, k)$ . By Theorem 2.1,  $\mu(G) < \mu(G_1)$ , a contradiction to the choice of  $G$ .

By combining the above arguments, we have  $G \cong W_n(t, k)$ . This completes the proof.  $\square$

**Lemma 3.2.** *Suppose  $t$  and  $k$  are integers with  $t \geq 4$  and  $1 \leq k \leq n - t$ . Then,*

$$\mu(W_n(t, k)) < \mu(W_n(t - 1, k)).$$

*Proof.* By the definition,  $W_n(t, k)$  is the graph obtained by attaching  $k$  paths of almost equal lengths to  $v_1$  of  $C_t$ . Assume  $C_t = v_1v_2v_3 \dots v_tv_1$ . Let

$$G_1 = W_n(t, k) - v_1v_2 - v_2v_3 + v_1v_3, \quad G_2 = G_1 - v_2, \quad G_3 = G_1 + v_2v_s,$$

where  $v_s$  is a pendant vertex of  $W_n(t, k)$ .

Since  $k \geq 1$ , then  $d(v_1) \geq 3$ . Thus,  $v_1v_3v_4 \dots v_tv_1$  is an internal path of  $G_2$ . By Theorem 2.2,  $\mu(W_n(t, k)) < \mu(G_2) < \mu(G_3)$ . Moreover, note that  $G_3 \in \mathbb{U}_n(t - 1, k)$ , thus we can conclude that  $\mu(W_n(t, k)) < \mu(G_3) \leq \mu(W_n(t - 1, k))$  by Lemma 3.1.  $\square$

For  $G \in \mathbb{U}_n(k)$ , it has been proved (see [9]) that  $\varrho(G) \leq \varrho(W_n(3, k))$ , with equality holding if and only if  $G \cong W_n(3, k)$ . The next theorem shows the similar result to  $\mu(G)$  for  $G \in \mathbb{U}_n(k)$

**Theorem 3.1.** *Suppose  $k$  is an integer with  $1 \leq k \leq n - 3$ . If  $G \in \mathbb{U}_n(k)$ , then*

$$\mu(G) \leq \mu(W_n(3, k)),$$

*and the equality holds if and only if  $G \cong W_n(3, k)$ .*

*Proof.* Since  $k \geq 1$  and  $G \in \mathbb{U}_n(k)$ , then there exists an integer  $t (\geq 3)$  such that  $G \in \mathbb{U}_n(t, k)$ . By Lemmas 3.1-3.2, it follows that  $\mu(G) \leq \mu(W_n(t, k)) \leq \mu(W_n(3, k))$ , with equality holding if and only if  $G \cong W_n(3, k)$ . This completes the proof of the assertion.  $\square$

In the following, we shall determine the unique graph with the largest Laplacian spectral radius in the class of unicyclic graphs with  $n$  vertices and  $k$  pendant vertices. By Proposition 1.1 and Lemma 3.1, it follows

**Lemma 3.3.** *Suppose  $t$  is a positive even number and  $k$  is an integer with  $1 \leq k \leq n - t$ . If  $G \in \mathcal{U}_n(t, k)$ , then  $\lambda(G) \leq \lambda(W_n(t, k))$ , with equality holding if and only if  $G \cong W_n(t, k)$ .*

**Lemma 3.4.** *Suppose  $t (\geq 5)$  is a positive odd number and  $k$  is an integer with  $1 \leq k \leq n - t$ . If  $G \in \mathcal{U}_n(t, k)$ , then  $\lambda(G) < \lambda(W_n(t - 1, k))$ .*

*Proof.* Since  $t$  is odd, by Proposition 1.1 and Lemmas 3.1–3.2 we have  $\lambda(G) < \mu(G) \leq \mu(W_n(t, k)) < \mu(W_n(t - 1, k)) = \lambda(W_n(t - 1, k))$ . Thus, the result follows.  $\square$

**Lemma 3.5.** *Suppose  $t$  and  $k$  are integers with  $t \geq 4$  and  $1 \leq k \leq n - t$ . If  $G \in \mathcal{U}_n(t, k)$ , then  $\lambda(G) \leq \lambda(W_n(4, k))$ , with equality holding if and only if  $G \cong W_n(4, k)$ .*

*Proof.* We divide the proof into the following two cases.

*Case 1.*  $t$  is even. We may assume that  $t \geq 6$ . By Proposition 1.1, Lemmas 3.2 and 3.3, we have  $\lambda(G) \leq \lambda(W_n(t, k)) = \mu(W_n(t, k)) < \mu(W_n(t - 1, k)) < \mu(W_n(t - 2, k)) = \lambda(W_n(t - 2, k))$ . Since  $t$  is even, by repeating the above process, we can conclude that  $\lambda(G) < \lambda(W_n(4, k))$  for  $t \geq 6$ .

*Case 2.*  $t$  is odd. Since  $t \geq 5$ , then  $\lambda(G) < \lambda(W_n(t - 1, k))$  follows from Lemma 3.4. Combining with Case 1, we have  $\lambda(G) < \lambda(W_n(t - 1, k)) \leq \lambda(W_n(4, k))$ .

By combining the above arguments, the result follows.  $\square$

**Lemma 3.6** ([8]). *Let  $v$  be a vertex of a connected graph  $G$  and suppose that  $v_1, \dots, v_s$  are pendant vertices of  $G$  which are adjacent to  $v$ . Let  $G^*$  be the graph obtained from  $G$  by adding any  $b$  ( $1 \leq b \leq \frac{1}{2}s(s - 1)$ ) edges between  $v_1, \dots, v_s$ . Then,  $\lambda(G) = \lambda(G^*)$ .*

The next lemma gives an upper bound for  $\lambda(G)$ , which does not exceed  $n$ .

**Lemma 3.7** ([16]).  $\lambda(G) \leq \max\{|N(u) \cup N(v)| : u, v \in V(G)\}$ .

The next lemma gives a lower bound for  $\lambda(G)$ .

**Lemma 3.8** ([14]). *If  $G$  is a graph with at least one edge, then  $\lambda(G) \geq \Delta(G) + 1$ , where equality holds if and only if  $\Delta(G) = n - 1$ .*

**Theorem 3.2.** *Suppose  $k$  is an integer with  $1 \leq k \leq n - 4$ . If  $G \in \mathbb{U}_n(k)$ , then*

$$\lambda(G) \leq \lambda(W_n(4, k)),$$

where the equality holds if and only if  $G \cong W_n(4, k)$ .

*Proof.* Since  $k \geq 1$  and  $G \in \mathbb{U}_n(k)$ , then there exists an integer  $t (\geq 3)$  such that  $G \in \mathbb{U}_n(t, k)$ . If  $t \geq 4$ , the result follows from Lemma 3.5. Next we shall consider the case of  $t = 3$ .

By the definition,  $\mathbb{U}_n(3, k)$  denotes the class of connected unicyclic graphs on  $n$  vertices having  $k$  pendant vertices and a cycle  $C_3 = v_1 v_2 v_3 v_1$ . Choose  $G \in \mathbb{U}_n(3, k)$  such that the Laplacian spectral radius of  $G$  is as large as possible.

We first proved that  $G$  is a graph by attaching some tree to only one vertex of  $C_3$ . On the contrary, assume that there exist trees  $T_1, T_2$  attached to  $v_1, v_2$  of  $C_3$ , respectively. Note that  $1 \leq k \leq n - 4$ . By Lemmas 3.7–3.8 we have

$$\lambda(G) \leq \max\{|N(u) \cup N(v)| : u, v \in V(G)\} \leq k + 3 = \Delta(W_n(3, k)) + 1 < \lambda(W_n(3, k)).$$

But  $W_n(3, k) \in \mathbb{U}_n(3, k)$ , it is a contradiction to the choice of  $G$ .

Thus,  $G$  is a graph obtained by attaching some tree to one vertex, say  $v_1$ , of  $C_3$ . Let  $T = G - v_2 v_3$ , then  $T$  is a tree. By Lemma 3.6,  $\lambda(G) = \lambda(T)$ . Next we shall prove that  $\lambda(T) < \lambda(W_n(4, k))$ .

Choose a pendant vertex, say  $u$ , of  $V(T)$  such that  $d(v_1, u)$  is as large as possible in  $T$ . Let  $G_1 = T + uv_2$ . Since  $T \subset G_1$ , then  $\lambda(T) = \mu(T) < \mu(G_1)$  follows from Proposition 2.1.

Note that  $G_1$  contains a cycle, say  $C_a$ , clearly  $a \geq 4$  because  $1 \leq k \leq n - 4$ , thus  $G_1 \in \mathbb{U}_n(a, k)$ . By Lemmas 3.1–3.2,  $\mu(G_1) \leq \mu(W_n(a, k)) \leq \mu(W_n(4, k))$ .

Combining the above arguments, we can conclude that

$$\lambda(G) = \lambda(T) = \mu(T) < \mu(G_1) \leq \mu(W_n(4, k)) = \lambda(W_n(4, k)).$$

This completes the proof. □

4. THE LARGEST  $\lambda(G)$  IN THE CLASS OF BICYCLIC GRAPHS WITH  $n$  VERTICES  
AND  $k$  PENDANT VERTICES

Let  $G$  be a bicyclic graph. The *base* of  $G$ , denoted by  $\hat{G}$ , is the (unique) minimal connected bicyclic subgraph of  $G$ . It is easy to see that  $\hat{G}$  is the unique bicyclic subgraph of  $G$  containing no pendant vertices, while  $G$  can be obtained from  $\hat{G}$  by attaching trees to some vertices of  $\hat{G}$ .

Let  $C_p$  and  $C_q$  be two vertex-disjoint cycles. Suppose that  $u \in V(C_p)$  and  $v \in V(C_q)$ . In [9], Guo introduced the graph  $B(p, l, q)$  (Fig. 3), which is arisen from  $C_p$  and  $C_q$  by joining  $u$  and  $v$  by a path  $(u =)v_1v_2 \dots v_l(= v)$  of length  $l - 1$ , where  $l = 1$  means identifying  $u$  and  $v$ .

Let  $P_{p+1}$ ,  $P_{q+1}$  and  $P_{l+1}$  be three vertex-disjoint paths, where  $p, l, q \geq 1$  and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resulting graph (Fig. 3), denoted by  $P(p, l, q)$ , is also reported in [9].

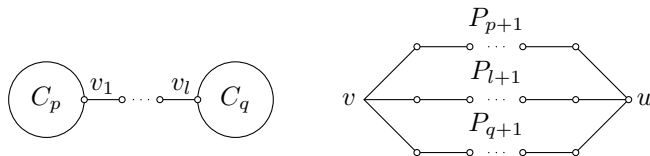


Fig. 3. The graphs  $B(p, l, q)$  and  $P(p, l, q)$ .

For integers  $n, k$ , let  $\mathbb{B}(n, k)$  be the class of connected bicyclic graphs with  $n$  vertices and  $k$  pendant vertices. Now we define the following two kinds of bicyclic graphs with  $n$  vertices and  $k$  pendant vertices:

$$\mathbb{B}_1(n, k) = \{G \in \mathbb{B}(n, k) : \hat{G} = B(p, l, q)\},$$

$$\mathbb{B}_2(n, k) = \{G \in \mathbb{B}(n, k) : \hat{G} = P(p, l, q)\}.$$

The girth of  $G$  is the length of a shortest cycle in  $G$  and its length is denoted by  $g(G)$ . For convenience, we introduced more notation as follows.

$$\mathcal{B}^1(n, k) = \{G \in \mathbb{B}_1(n, k) : g(G) \geq 4\}, \quad \mathcal{B}^2(n, k) = \{G \in \mathbb{B}_1(n, k) : g(G) = 3\},$$

$$\mathcal{B}^3(n, k) = \{G \in \mathbb{B}_2(n, k) : g(G) \geq 4\}, \quad \mathcal{B}^4(n, k) = \{G \in \mathbb{B}_2(n, k) : g(G) = 3\}.$$

It is easy to see that

$$\mathbb{B}(n, k) = \mathbb{B}_1(n, k) \cup \mathbb{B}_2(n, k),$$

$$\mathbb{B}_1(n, k) = \mathcal{B}^1(n, k) \cup \mathcal{B}^2(n, k),$$

$$\mathbb{B}_2(n, k) = \mathcal{B}^3(n, k) \cup \mathcal{B}^4(n, k).$$

Let  $W_1$  be the graph on  $n$  vertices obtained from  $B(4, 1, 4)$  by attaching  $k$  paths of almost equal lengths to the vertex of degree 4. Let  $W_2$  and  $W_3$  be the graphs on  $n$  vertices arisen from  $P(3, 1, 3)$  by attaching  $k$  paths of almost equal lengths to one vertex of degree 3 and one vertex of degree 2, respectively. Let  $W_4$  and  $W_5$  be the graphs on  $n$  vertices obtained from  $P(2, 2, 2)$  by attaching  $k$  paths of almost equal lengths to one vertex of degree 3 and one vertex of degree 2, respectively.

Let  $m(v)$  denote the average of the degrees of the vertices adjacent to  $v$ , i.e.,  $m(v) = \sum_{u \in N(v)} d(u)/d(v)$ .

**Lemma 4.1** ([11]).

$$\lambda(G) \leq \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} : uv \in E(G) \right\}.$$

**Lemma 4.2.** *If  $1 \leq k \leq n - 7$ , then  $\lambda(W_i) < \lambda(W_1)$  holds for  $2 \leq i \leq 5$ .*

*Proof.* By Lemmas 3.8 and 4.1, we have

$$\begin{aligned} \lambda(W_2) &\leq \max \left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} : uv \in E(W_2) \right\} \\ &= \max \left\{ \frac{k^2 + 9k + 32}{k + 6}, \frac{k^2 + 9k + 25}{k + 5}, \frac{k^2 + 9k + 19}{k + 4} \right\} \\ &\leq k + 5 = \Delta(W_1) + 1 < \lambda(W_1). \end{aligned}$$

By Lemmas 3.7–3.8, we have

$$\lambda(W_3) \leq \max\{|N(u) \cup N(v)| : u, v \in V(W_3)\} = k + 5 = \Delta(W_1) + 1 < \lambda(W_1).$$

It can be proved similarly as  $\lambda(W_3) < \lambda(W_1)$  that  $\lambda(W_4) < \lambda(W_1)$  and  $\lambda(W_5) < \lambda(W_1)$ .

By combining the above discussion, the assertions follow.  $\square$

**Lemma 4.3.** *If  $1 \leq k \leq n - 7$  and  $G \in \mathcal{B}^1(n, k)$ , then  $\mu(G) \leq \mu(W_1)$ , with equality holding if and only if  $G \cong W_1$ .*

*Proof.* Choose  $G \in \mathcal{B}^1(n, k)$  such that  $\mu(G)$  is as large as possible. Denote the vertex set of  $G$  by  $\{v_1, \dots, v_n\}$  and the Perron vector of  $\mu(G)$  by  $x = (x_1, \dots, x_n)^T$ , where  $x_i (> 0)$  corresponds to the vertex  $v_i$  ( $1 \leq i \leq n$ ).

Suppose  $\hat{G} = B(p, l, q)$ , and  $v_1 \dots v_l$  is the unique path from  $v_1 \in V(C_p)$  to  $v_l \in V(C_q)$ . We claim that  $l = 1$ . Assume, on the contrary, that  $l > 1$ . Without loss of

generality, suppose that  $x_1 \geq x_l$ . Clearly, there exists some vertex  $u \in N(v_l) \cap V(C_q)$ , and  $u \notin N(v_1)$ . Let

$$G_1 = G - v_l u + v_1 u,$$

then  $G_1 \in \mathcal{B}^1(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction. Hence,  $l = 1$ .

We now prove that  $G$  is the graph that arises from  $B(p, 1, q)$  by attaching a tree to the vertex of degree 4, say  $v_1$ , in  $B(p, 1, q)$ . Assume that there exists a vertex  $v_i$  of  $B(p, 1, q)$  such that  $v_i \neq v_1$  and there exists a tree  $T$  attached to  $v_i$ . By symmetry, we may assume that  $v_i \in V(C_p)$ .

If  $x_1 \geq x_i$ , choose  $u \in N(v_i) \cap V(T)$ , clearly  $u \notin N(v_1)$ . Let

$$G_1 = G - v_i u + v_1 u,$$

then  $G_1 \in \mathcal{B}^1(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

If  $x_1 < x_i$ , suppose  $\{u, v\} = N(v_1) \cap V(C_q)$ , clearly  $u, v \notin N(v_i)$ . Let

$$G_1 = G - v_1 u - v_1 v + v_i u + v_i v,$$

then  $G_1 \in \mathcal{B}^1(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

Thus,  $G$  is a graph obtained by attaching one tree, say  $T$ , to the vertex  $v_1$  of  $B(p, 1, q)$ . We now prove that each vertex of  $T$  has degree  $d(v) \leq 2$ . On the contrary, assume there exists one vertex  $v_i \in V(T)$  such that  $d(v_i) \geq 3$ .

If  $x_1 \geq x_i$ , since  $d(v_i) \geq 3$ , then there must exist some vertex  $u \in N(v_i)$  such that  $d(v_1, u) > d(v_1, v_i)$ . Clearly,  $u \notin N(v_1)$ . Let

$$G_1 = G - u v_i + v_1 u$$

then  $G_1 \in \mathcal{B}^1(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

If  $x_1 < x_i$ , suppose  $\{u, v\} = N(v_1) \cap V(C_q)$ , clearly  $u, v \notin N(v_i)$ . Let

$$G_1 = G - v_1 u - v_1 v + v_i u + v_i v,$$

then  $G_1 \in \mathcal{B}^1(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

Thus,  $G$  is a graph obtained by attaching  $k$  paths to the vertex  $v_1$  of  $B(p, 1, q)$ . Next we shall prove that  $p = q = 4$ . On the contrary, we assume that  $p > 4$  and  $C_p = v_1 v_2 \dots v_p v_1$ . Let

$$G_1 = G - v_1 v_2 - v_2 v_3 + v_1 v_3, \quad G_2 = G_1 - v_2, \quad G_3 = G_1 + v_2 v_s,$$

where  $v_s$  is a pendant vertex of  $G$ .

Note that  $v_1v_3 \dots v_pv_1$  is an internal path of  $G_2$ , then  $\mu(G) < \mu(G_2)$  follows from Theorem 2.2. Moreover, since  $G_2 \subset G_3$ , by Proposition 2.1 it follows that  $\mu(G_2) < \mu(G_3)$ . Thus, we can conclude that  $\mu(G) < \mu(G_3)$ . But  $G_3 \in \mathcal{B}^1(n, k)$ , a contradiction. Thus,  $p = 4$ . By the same reason,  $q = 4$ .

Finally, we prove that  $G \cong W_1$ , i.e., the  $k$  paths have almost equal lengths. On the contrary, if there exist two paths, say  $P_{l_1}$  and  $P_{l_2}$ , such that  $l_1 - l_2 \geq 2$  and  $l_2 \geq 2$ . Denote  $P_{l_1} = u_1 \dots u_{l_1}$  and  $P_{l_2} = w_1 \dots w_{l_2}$ , where  $u_1 = v_1 = w_1$ . Let

$$G_1 = G - u_{l_1-1}u_{l_1} + w_{l_2}u_{l_1},$$

then  $G_1 \in \mathcal{B}^1(n, k)$ . By Theorem 2.1,  $\mu(G) < \mu(G_1)$ , a contradiction.

By combining the above arguments, we have  $G \cong W_1$ . This completes the proof.  $\square$

**Corollary 4.1.** *If  $1 \leq k \leq n - 7$  and  $G \in \mathcal{B}^1(n, k)$ , then  $\lambda(G) \leq \lambda(W_1)$ , with equality holding if and only if  $G \cong W_1$ .*

*Proof.* By Proposition 1.1 and Lemma 4.3, we have

$$\lambda(G) \leq \mu(G) \leq \mu(W_1) = \lambda(W_1).$$

Thus, the conclusion follows from Lemma 4.3.

**Lemma 4.4.** *If  $1 \leq k \leq n - 7$  and  $G \in \mathcal{B}^2(n, k)$ , then  $\lambda(G) < \lambda(W_1)$ .*

*Proof.* Choose  $G \in \mathcal{B}^2(n, k)$  such that  $\lambda(G)$  is as large as possible. Without loss of generality, we assume that  $p \geq q$  in the proof of this lemma. By the definition,  $\hat{G} = B(p, l, 3)$ . Suppose that  $C_q (= C_3) = v_1v_2v_3v_1$ , where  $d(v_1) \geq 3$  in  $B(p, l, 3)$ . Two cases occur as follows.

*Case 1.* If there exists a vertex  $w$  ( $w = v_2$  or  $v_3$ ) of degree 2 in  $C_3$  of  $B(p, l, 3)$  such that there exists a tree  $T$  attached to  $w$ , by Lemmas 3.7–3.8 it follows that

$$\lambda(G) \leq \max\{|N(u) \cup N(v)| : u, v \in V(G)\} \leq k + 5 = \Delta(W_1) + 1 < \lambda(W_1).$$

*Case 2.* There exists no tree attached to  $v_2$  or/and  $v_3$  in  $C_q (= C_3)$  of  $B(p, l, 3)$ . Let

$$G_1 = G - v_2v_3.$$

Then,  $\lambda(G) = \lambda(G_1)$  follows from Lemma 3.6. Note that  $G_1 \in \mathcal{U}_n(k + 2)$  and  $k + 2 < n - 4$ , by Theorem 3.2 we have  $\lambda(G_1) \leq \lambda(W_n(4, k + 2))$ . Choose two different pendant vertices, say  $u$  and  $v$ , of  $V(W_n(4, k + 2))$  such that  $d(v_i, u)$  and



$d(v_i, v)$  are as large as possible in  $W_n(4, k+2)$ , where  $v_i$  is the unique vertex of degree greater than 4 in  $W_n(4, k+2)$ . Let

$$G_2 = W_n(4, k+2) + uv.$$

Since  $W_n(4, k+2) \subset G_2$ , by Proposition 2.1 we have  $\mu(W_n(4, k+2)) < \mu(G_2)$ . Note that  $G_2 \in \mathcal{B}^1(n, k)$ , then  $\mu(G_2) \leq \mu(W_1)$  by Lemma 4.3.

Combining the above discussion and Proposition 1.1, we can conclude that

$$\lambda(G) = \lambda(G_1) \leq \lambda(W_n(4, k+2)) = \mu(W_n(4, k+2)) < \mu(G_2) \leq \mu(W_1) = \lambda(W_1).$$

By the above arguments, we have  $\lambda(G) < \lambda(W_1)$ . This completes the proof.  $\square$

**Lemma 4.5.** *If  $1 \leq k \leq n-7$  and  $G \in \mathcal{B}^3(n, k)$ , then  $\mu(G) \leq \max\{\mu(W_2), \mu(W_3), \mu(W_4), \mu(W_5)\}$ .*

*Proof.* Choose  $G \in \mathcal{B}^3(n, k)$  such that  $\mu(G)$  is as large as possible. Denote the vertex set of  $G$  by  $\{v_1, \dots, v_n\}$  and the Perron vector of  $\mu(G)$  by  $x = (x_1, \dots, x_n)^T$ , where  $x_i (> 0)$  corresponds to the vertex  $v_i$  ( $1 \leq i \leq n$ ). Without loss of generality, we assume that  $l = \min\{p, l, q\}$  in the proof of this lemma.

We first prove that  $G$  is the graph obtained from  $P(p, l, q)$  by attaching some tree to only one vertex of  $P(p, l, q)$ . On the contrary, assume there exist trees  $T_i$  and  $T_j$  attached to  $v_i$  and  $v_j$  of  $P(p, l, q)$ , respectively. By symmetry, we may assume that  $x_i \geq x_j$ . Choose  $u \in N(v_j) \cap V(T_j)$ , clearly  $u \notin N(v_i)$ . Let

$$G_1 = G - v_j u + v_i u.$$

Then,  $G_1 \in \mathcal{B}^3(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

Thus,  $G$  is the graph arisen from  $P(p, l, q)$  by attaching some tree, say  $T$ , to unique vertex, say  $v_1$ , of  $P(p, l, q)$ . We now prove that every vertex  $v$  of  $T$  has degree  $d(v) \leq 2$ . On the contrary, assume that there exists  $v_j \in V(T)$  such that  $d(v_j) \geq 3$ .

If  $x_j \leq x_1$ , since  $d(v_j) \geq 3$ , then there must exist some vertex  $u \in N(v_j)$  such that  $d(v_1, u) > d(v_1, v_j)$ . Clearly,  $u \notin N(v_1)$ . Let

$$G_1 = G - uv_j + uv_1,$$

then  $G_1 \in \mathcal{B}^3(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

If  $x_j > x_1$ , suppose  $u \in N(v_1) \cap V(P(p, l, q))$ , clearly  $u \notin N(v_j)$ . Let

$$G_1 = G - v_1 u + v_j u,$$

then  $G_1 \in \mathcal{B}^3(n, k)$ . By Lemma 2.5,  $\mu(G) < \mu(G_1)$ , a contradiction.

Thus,  $G$  is a graph obtained by attaching  $k$  paths to some vertex  $v_1$  of  $P(p, l, q)$ . We divide the proof into the next two cases.

*Case 1.*  $d(v_1) = 3$  in  $P(p, l, q)$ .

*Subcase 1.1.*  $l = 1$ . We shall prove that  $p = q = 3$ . On the contrary, assume that  $p \geq 4$ . Suppose  $P_{p+1} = v_1 v_2 \dots v_{p+1}$ , then  $d(v_1) \geq 4$  and  $d(v_{p+1}) = 3$ . Let

$$G_1 = G - v_1 v_2 - v_2 v_3 + v_1 v_3, \quad G_2 = G_1 - v_2, \quad G_3 = G_1 + v_2 v_s,$$

where  $v_s$  is a pendant vertex of  $G$ .

Note that  $v_1 v_3 \dots v_{p+1}$  is an internal path of  $G_2$  and  $G_2 \subset G_3$ , then  $\mu(G) < \mu(G_2) < \mu(G_3)$  follows from Proposition 2.1 and Theorem 2.2, a contradiction to the choice of  $G$ . Thus,  $p = 3$ . By the same reason,  $q = 3$ . Thus,  $G$  is a graph obtained by attaching  $k$  paths to one vertex of degree 3 of  $P(3, 1, 3)$ . By Theorem 2.1, we have  $G \cong W_2$ .

*Subcase 1.2.*  $l \geq 2$ . By the same method as Subcase 1.1, we can prove that  $G \cong W_4$ .

*Case 2.*  $d(v_1) = 2$  in  $P(p, l, q)$ .

*Subcase 2.1.*  $l = 1$ . By the same method as Subcase 1.1, we can prove that  $G \cong W_3$ .

*Subcase 2.2.*  $l \geq 2$ . By the same method as Subcase 1.1, we can prove that  $G \cong W_5$ .

By the above arguments, this completes the proof. □

**Corollary 4.2.** *If  $1 \leq k \leq n - 7$  and  $G \in \mathcal{B}^3(n, k)$ , then  $\lambda(G) < \lambda(W_1)$ .*

*Proof.* By Proposition 1.1,

$$\max\{\mu(W_2), \mu(W_3), \mu(W_4), \mu(W_5)\} = \max\{\lambda(W_2), \lambda(W_3), \lambda(W_4), \lambda(W_5)\}.$$

Combining with Proposition 1.1, Lemmas 4.2 and 4.5, we have

$$\lambda(G) \leq \mu(G) \leq \max\{\lambda(W_2), \lambda(W_3), \lambda(W_4), \lambda(W_5)\} < \lambda(W_1).$$

Thus, the conclusion follows. □

**Lemma 4.6.** *If  $1 \leq k \leq n - 7$  and  $G \in \mathcal{B}^4(n, k)$ , then  $\lambda(G) < \lambda(W_1)$ .*

*Proof.* If  $G \in \mathcal{B}^4(n, k)$ , by Lemmas 3.7–3.8 it follows that

$$\lambda(G) \leq \max\{|N(u) \cup N(v)| : u, v \in V(G)\} \leq k + 5 = \Delta(W_1) + 1 < \lambda(W_1).$$

This completes the proof. □

By Corollaries 4.1–4.2, Lemmas 4.4 and 4.6, we can conclude

**Theorem 4.1.** *If  $1 \leq k \leq n - 7$  and  $G \in \mathbb{B}(n, k)$ , then*

$$\lambda(G) \leq \lambda(W_1),$$

where the equality holds if and only if  $G \cong W_1$ .

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