Aleksander Misiak; Eugeniusz Stasiak

$G$-space of isotropic directions and $G$-spaces of $\varphi$-scalars with $G = O(n, 1, \mathbb{R})$


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Abstract. There exist exactly four homomorphisms $\varphi$ from the pseudo-orthogonal group of index one $G = O(n, 1, \mathbb{R})$ into the group of real numbers $\mathbb{R}_0$. Thus we have four $G$-spaces of $\varphi$-scalars $(\mathbb{R}, G, h_\varphi)$ in the geometry of the group $G$. The group $G$ operates also on the sphere $S^{n-2}$ forming a $G$-space of isotropic directions $(S^{n-2}, G, \ast)$. In this note, we have solved the functional equation $F(A \ast q_1, A \ast q_2, \ldots, A \ast q_m) = \varphi(A) \cdot F(q_1, q_2, \ldots, q_m)$ for given independent points $q_1, q_2, \ldots, q_m \in S^{n-2}$ with $1 \leq m \leq n$ and an arbitrary matrix $A \in G$ considering each of all four homomorphisms. Thereby we have determined all equivariant mappings $F: (S^{n-2})^m \to \mathbb{R}$.

Keywords: $G$-space, equivariant map, pseudo-Euclidean geometry

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1. Introduction

For $n \geq 2$ consider the matrix $E_1 = \text{diag}(+1, \ldots, +1, -1) \in GL(n, \mathbb{R})$.

Definition 1. A pseudo-orthogonal group of index one is a subgroup of the group $GL(n, \mathbb{R})$ satisfying the condition

$$G = O(n, 1, \mathbb{R}) = \{ A : A \in GL(n, \mathbb{R}) \land A^T \cdot E_1 \cdot A = E_1 \}.$$

It is known that there exist exactly four homomorphisms $\varphi$ from the group $G$ into the group $\mathbb{R}_0$. Denoting $A = [A^i_j]_{1}^{n} \in G$ we can specify these homomorphisms, namely $1(A) = 1, \varepsilon(A) = \det A = \text{sign}(\det A), \eta(A) = \text{sign}(A^n_n)$ and $\varepsilon(A) \cdot \eta(A)$.

Definition 2. A $G$-space is the triple $(M, G, f)$, where $f$ is an operation of the group $G$ on the set $M$. 

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Definition 3. By a $G$-space of $\varphi$-scalars we understand the triple $(\mathbb{R}, G, h_{\varphi})$, where the mappings $\varphi: G \rightarrow \mathbb{R}_0$ and $h_{\varphi}: \mathbb{R} \times G \rightarrow \mathbb{R}$ fulfil the conditions

a) $\bigwedge_{A,B \in G} \varphi(AB) = \varphi(A) \cdot \varphi(B)$,

b) $\bigwedge_{x \in \mathbb{R}} \bigwedge_{A \in G} h_{\varphi}(x, A) = \varphi(A) \cdot x$.

Let two $G$-spaces $(M_\alpha, G, f_\alpha)$ and $(M_\beta, G, f_\beta)$ be given.

Definition 4. A mapping $F_{\alpha\beta}: M_\alpha \rightarrow M_\beta$ is called equivariant if the condition

\[
\bigwedge_{x \in M_\alpha} \bigwedge_{A \in G} F_{\alpha\beta}(f_\alpha(x, A)) = f_\beta(F_{\alpha\beta}(x), A)
\]

is fulfilled.

The class of $G$-spaces with equivariant maps as morphisms constitutes a category which is called a pseudo-Euclidean geometry of index one. In particular, there exist in this geometry the $G$-space of contravariant vectors

\[
(\mathbb{R}^n, G, f), \quad \text{where } \bigwedge_{u \in \mathbb{R}^n} \bigwedge_{A \in G} f(u, A) = A \cdot u,
\]

and four $G$-spaces of objects with one component and linear transformation rule

\[
(\mathbb{R}, G, h), \quad \text{where } \bigwedge_{x \in \mathbb{R}} \bigwedge_{A \in G} h(x, A) = \begin{cases} 
1 \cdot x & \text{for } 1 \text{-scalars,} \\
\varepsilon(A) \cdot x & \text{for } \varepsilon \text{-scalars,} \\
\eta(A) \cdot x & \text{for } \eta \text{-scalars,} \\
\varepsilon(A) \cdot \eta(A) \cdot x & \text{for } \varepsilon\eta \text{-scalars.}
\end{cases}
\]

All equivariant maps from the product of linearly independent contravariant vectors into $G$-spaces of $\varphi$-scalars were determined in [4], [5] and [6]. In particular, the equivariant in the $G$-space of 1-scalars of a pair of vectors $u$ and $v$ is the invariant $p(u, v) = u^T \cdot E_1 \cdot v$. In fact, for an arbitrary matrix $A \in G$ we have $p(Au, Av) = (Au)^T \cdot E_1 \cdot (Av) = u^T \cdot (A^T \cdot E_1 \cdot A) \cdot v = u^T \cdot E_1 \cdot v = p(u, v)$. The invariant $p$ enables us to determine an invariant subset of isotropic vectors, namely the transitive, isotropic cone $V = \{u: u \in \mathbb{R}^n \& p(u, u) = 0 \& u \neq 0\}$. Let us introduce in addition the sphere $S^{n-2}$ included in the hyperplane $q^n = 1$ and immersed in the space $\mathbb{R}^n$, namely

\[
S^{n-2} = \left\{ q: q = [q^1, q^2, \ldots, q^{n-1}, 1]^T, \text{ where } \sum_{i=1}^{n-1} (q^i)^2 = 1 = q^n \right\}.
\]
Let $q \in S^{n-2}$ and $A \in G$. For brevity let us denote $W(q, A) = \sum_{i=1}^{n} A_{i}^{n} q_{i}$. Let us recall (see [5]) that

$$\bigwedge_{q \in S^{n-2}} \bigwedge_{A \in G} \text{sign} W(q, A) = \text{sign}(A_{n}) = \eta(A). \quad (4)$$

Because of $u^{n} \neq 0$ we can write every isotropic vector $u \in V$ in the form

$$u = [u^{1}, u^{2}, \ldots, u^{n}]^{T} = u^{n} \cdot \left[\frac{u^{1}}{u^{n}}, \ldots, \frac{u^{n-1}}{u^{n}}, 1\right]^{T} = u^{n} \cdot [q^{1}, q^{2}, \ldots, q^{n-1}, 1]^{T} = u^{n} \cdot q,$$

where $q \in S^{n-2}$. Let us call $u^{n} = u^{n}(u)$ the parameter and $q = q(u)$ the direction of the isotropic vector $u$. For an arbitrary matrix $A \in G$ we have $A \cdot u \in V$ and applying the transformation rule for the vector $(2)$ we get

$$A \cdot u = \left[\sum_{i=1}^{n} A_{i}^{1} u^{i}, \ldots, \sum_{i=1}^{n} A_{i}^{n} u^{i}\right]^{T} = \left(\sum_{i=1}^{n} A_{i}^{n} u^{i}\right) \cdot \left[\sum_{i=1}^{n} A_{i}^{1} u^{i}, \ldots, \sum_{i=1}^{n} A_{i}^{n-1} u^{i}, 1\right]^{T}$$

$$= (u^{n} \cdot W(q, A)) \cdot \left(\frac{1}{W(q, A)} \cdot A \cdot q\right).$$

So, we have obtained the transformation rules for the parameter and the direction of the isotropic vector $u$:

$$u^{n}(A \cdot u) = u^{n}(u) \cdot W(q, A) \text{ and } q(A \cdot u) = \frac{1}{W(q, A)} \cdot A \cdot q(u) = A \ast q. \quad (5)$$

Let us observe that $B \ast (A \ast q) = (B \cdot A) \ast q$ holds for $A, B \in G$ and $E \ast q = q$ for the unit matrix $E$. In what follows the group $G$ operates on the sphere $S^{n-2}$.

**Definition 5.** The $G$-space

$$\bigwedge_{q \in S^{n-2}} \bigwedge_{A \in G} * (q, A) = A \ast q = \frac{A \cdot q}{W(q, A)}, \quad (6)$$

is called a $G$-space of isotropic directions.

**Definition 6.** The system of directions $q_{i} = q(u)_{i} \in S^{n-2}$ for $i = 1, 2, \ldots, m$ is called independent if the system of vectors $u_{1}, u_{2}, \ldots, u_{m} \in V$ is linearly independent.
In this paper we determine all equivariant mappings from the product of isotropic directions into \( \varphi \)-scalars. More accurately, having in mind (1), (3) and (6) we solve the functional equations

\[
\begin{align*}
(7) & \quad F(A * q_1, A * q_2, \ldots, A * q_m) = 1 \cdot F(q_1, q_2, \ldots, q_m), \\
(8) & \quad F(A * q_1, A * q_2, \ldots, A * q_m) = \varepsilon(A) \cdot F(q_1, q_2, \ldots, q_m), \\
(9) & \quad F(A * q_1, A * q_2, \ldots, A * q_m) = \eta(A) \cdot F(q_1, q_2, \ldots, q_m), \\
(10) & \quad F(A * q_1, A * q_2, \ldots, A * q_m) = \varepsilon(A) \cdot \eta(A) \cdot F(q_1, q_2, \ldots, q_m)
\end{align*}
\]

for an arbitrary matrix \( A \in G \) and the given system of independent points \( q_1, q_2, \ldots, q_m \in S^{n-2} \) with \( 1 \leq m \leq n \).

2. Certain particular solutions

For the pair of points \( q_i, q_j \in S^{n-2} \) let us denote \( 1 - \sum_{k=1}^{n-1} q_i^k q_j^k = Q(q_i, q_j) = Q_{ij} \) for brevity. The Euclidean distance between these points

\[
\|q_i, q_j\| = \sqrt{\sum_{k=1}^{n-1} (q_j^k - q_i^k)^2} = \sqrt{2 \cdot \left( 1 - \sum_{k=1}^{n-1} q_i^k q_j^k \right)} = \sqrt{2 \cdot Q(q_i, q_j)} = \sqrt{2 \cdot Q_{ij}}
\]

is not an invariant under the operation of the group \( G \). Let the isotropic vectors \( u, u \) correspond to the directions \( q_i, q_j \), respectively. Since we have \( p(A u_i, A u_j) = p(u_i, u_j) \) for an arbitrary matrix \( A \in G \), according to (5) we get

\[
(11) \quad Q(A * q_i, A * q_j) = \frac{Q(q_i, q_j)}{W(q_i, A) \cdot W(q_j, A)},
\]

which means

\[
\|A * q_i, A * q_j\| = \frac{\|q_i, q_j\|}{\sqrt{W(q_i, A) \cdot W(q_j, A)}}.
\]

For different points \( q_1, q_2, q_3, q_4 \in S^{n-2} \), which is possible if \( n > 2 \), we can construct easily two simple but nontrivial invariants

\[
\frac{Q_{13}Q_{24}}{Q_{12}Q_{34}}, \frac{Q_{14}Q_{23}}{Q_{12}Q_{34}} \quad \text{or equivalently} \quad \frac{\|q_1, q_3\| \cdot \|q_2, q_4\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}, \frac{\|q_1, q_4\| \cdot \|q_2, q_3\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}
\]

which can be interpreted in a quadrilateral or tetrahedron with vertices \( q_1, q_2, q_3, q_4 \).

In addition we have

\[
\det(A u_1, A u_2, \ldots, A u_n) = \varepsilon(A) \cdot \det(u_1, u_2, \ldots, u_n),
\]

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An analogous result is obtained for the two remaining equations using 

\[ \det(A \ast q_1, A \ast q_2, \ldots, A \ast q_n) = \frac{\varepsilon(A) \cdot \det(q_1, q_2, \ldots, q_n)}{W(q_1, A) \cdot W(q_2, A) \cdots \cdot W(q_n, A)} \]

Now (12) together with (4) yields

\[ \text{sign } \det(A \ast q_1, \ldots, A \ast q_n) = \begin{cases} \varepsilon(A) \cdot \text{sign } \det(q_1, \ldots, q_n) & \text{for even } n, \\ \varepsilon(A) \cdot \eta(A) \cdot \text{sign } \det(q_1, \ldots, q_n) & \text{for odd } n. \end{cases} \]

**Lemma 1.** For arbitrary possible \( m = 1, 2, \ldots \) and an arbitrary matrix \( A \in G \) the functional equation

\[ F(A \ast q_1, \ldots, A \ast q_m) = \begin{cases} \eta(A) \cdot F(q_1, \ldots, q_m) & \text{if } n = 2, 3, 4, \ldots, \\ \varepsilon(A) \cdot F(q_1, \ldots, q_m) & \text{if } n = 3, 5, 7, \ldots, \\ \varepsilon(A) \cdot \eta(A) \cdot F(q_1, \ldots, q_m) & \text{if } n = 2, 4, 6, \ldots \end{cases} \]

has only the trivial solution \( F(q_1, q_2, \ldots, q_m) = 0 \).

**Proof.** If \( A \in G \) then obviously \( (-A) \in G \) and \( A \ast q = (-A) \ast q \). Inserting \( A \) and then \( (-A) \) into the first equation and having in mind \( \eta(-A) = -\eta(A) \) we get simultaneously

\[ F(q_1, \ldots, q_m) = \eta(A) \cdot F(A \ast q_1, \ldots, A \ast q_m) = -\eta(A) \cdot F(A \ast q_1, \ldots, A \ast q_m). \]

An analogous result is obtained for the two remaining equations using \( \varepsilon(-A) = -\varepsilon(A) \) in the case of \( n \) odd and \( \varepsilon(-A) \cdot \eta(-A) = -\varepsilon(A) \cdot \eta(A) \) in the case of \( n \) even.

We have to consider the cases \( n = 2 \) and \( n = 3 \). If \( n = 2 \) then the sphere \( S^0 \) has only two different points \( q_1 = [q_1^1, 1]^T \) and \( q_2 = [q_2^1, 1]^T = [-q_1^1, 1]^T \) where \( (q_1^1)^2 = 1 \). An arbitrary pseudo-orthogonal matrix is of the form

\[ A(\varepsilon, \eta, x) = \begin{bmatrix} \varepsilon \cdot \eta \cdot \cosh x & \varepsilon \cdot \eta \cdot \sinh x \\ \eta \cdot \sinh x & \eta \cdot \cosh x \end{bmatrix}, \]

where \( \varepsilon^2 = 1, \eta^2 = 1, x \in \mathbb{R} \). Since we have \( A \ast q_1 = [\varepsilon q_1^1, 1]^T \), so putting the matrix \( A(q_1^1, \eta, x) \) into functional equations (7) and (8) we get solutions

1-scalars) \( F(q_1) = c \) and \( F(q_1, q_2) = c, \)

\( \varepsilon \)-scalars) \( F(q_1) = c \cdot q_1^1 \) and \( F(q_1, q_2) = 2c \cdot q_1^1 = -2c \cdot q_2^1 = c \cdot \begin{pmatrix} q_1^1 & 1 \\ q_2^1 & 1 \end{pmatrix}, \)

where \( c \) denotes a constant.
In the case \( n = 3 \) the circle \( S^1 \) is an uncountable set. For the given different points \( q_1, q_2, q_3 \in S^1 \) there exists a matrix \( A \in G \) such that \( \varepsilon(A) = 1, \eta(A) = \text{sign} \det(q_1, q_2, q_3) \) and \( A \ast q_1 = [0, 1, 1]^T, A \ast q_2 = [0, -1, 1]^T \) and \( A \ast q_3 = [1, 0, 1]^T \). Inserting this matrix into equations (7) and (10) we get solutions

1-scalars) \( F(q_1) = c \) and \( F(q_1, q_2) = c \) and \( F(q_1, q_2, q_3) = c \),

\( \varepsilon \eta \)-scalars) \( F(q_1) = 0 \) and \( F(q_1, q_2) = 0 \) and \( F(q_1, q_2, q_3) = c \cdot \text{sign} \det(q_1, q_2, q_3) \),

where \( c \) again denotes an arbitrary constant.

Just in the case \( m = 4 \) and \( q_4 \notin \{q_1, q_2, q_3\} \) we get two non-trivial invariants and general solutions of the equations:

1-scalars) \( F(q_1, q_2, q_3, q_4) = \Theta(q_{13}Q_{24}/Q_{12}Q_{34}, Q_{14}Q_{23}/Q_{12}Q_{34}) = \Theta(x_4, y_4), \)

\( \varepsilon \eta \)-scalars) \( F(q_1, q_2, q_3, q_4) = \Theta(x_4, y_4) \cdot \text{sign} \det(q_1, q_2, q_3) \), where \( \Theta \) is an arbitrary function of two variables.

3. General solution of equation (7)

For \( n = 4, 5, 6, \ldots \) let \( n \) independent points \( q_i = [q_i^1, q_i^2, \ldots, q_i^{n-1}, 1]^T \in S^{n-2} \) be given, where \( i = 1, 2, \ldots, n \) and let \( Q(s) = \det[Q_{ij}] \) for \( s = 2, 3, \ldots, n \). Let us remark that \( [\det(q_1, q_2, \ldots, q_n)]^2 = (-1)^{n+1}Q(n) \) and \( (-1)^{s+1}Q(s) > 0 \). We are going to construct a matrix \( C = C(q_1, q_2, \ldots, q_n) = [C_i^j]_{i,j} \in G \) which will enable us to solve equation (7). We start with the last three rows. For \( i = 1, 2, \ldots, n - 1 \) let

\[
C_{i}^{n-2} = \frac{Q_{23}q_i^1 + Q_{13}q_i^2 - Q_{12}q_i^3}{(-1)^n \sqrt{Q(3)}}, \quad C_{i}^{n-1} = \frac{Q_{13}q_i^3 - Q_{23}q_i^1}{(-1)^n \sqrt{Q(3)}}, \quad C_{i}^{n} = \frac{Q_{23}q_i^1 + Q_{13}q_i^2}{(-1)^n \sqrt{Q(3)}}
\]

We have formulas for the \((n - 2)\)-nd and \((n - 1)\)-st components of an arbitrary point \( C \ast q_r \), namely

\[
\begin{align*}
(C \ast q_r)^{n-2} &= \frac{Q_{13}Q_{2r} + Q_{23}Q_{1r} - Q_{12}Q_{3r}}{Q_{13}Q_{2r} + Q_{23}Q_{1r}}, \\
(C \ast q_r)^{n-1} &= \frac{Q_{13}Q_{2r} - Q_{23}Q_{1r}}{Q_{13}Q_{2r} + Q_{23}Q_{1r}}.
\end{align*}
\]

These components in accordance with (11) are 1-scalars. In particular, for \( r = 1, 2, 3 \) we get

\[
(C \ast q_1) = [0, \ldots, 0, 1, 1]^T, \quad (C \ast q_2) = [0, \ldots, 0, -1, 1]^T, \quad (C \ast q_3) = [0, \ldots, 0, 1, 0, 1]^T.
\]
Let the elements of the first row $C^1_i$ of the matrix $C$ be coefficients of $z_i$ in the Laplace expansion in terms of elements of the last row of the determinant

$$C^1 = \frac{\text{sign} \det(q_1, \ldots, q_n)}{\sqrt{(-1)^n Q(n-1)}} \begin{vmatrix} q_1^n & q_1^{n-1} & \ldots & q_1^1 & 1 \\ q_2^n & q_2^{n-1} & \ldots & q_2^1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n-1}^n & q_{n-1}^{n-1} & \ldots & q_{n-1}^1 & 1 \\ z_1 & z_2 & \ldots & z_{n-1} & z_n \end{vmatrix}$$

Then we have $(C * q_r)^1 = 0$ for $r = 1, 2, \ldots, n - 1$. Analogously, the coefficients of $z_i$ in the Laplace expansion in terms of elements of the last row of the determinant $C^2$ are the elements $C^2_i$ of the second row of the matrix $C$. Now, $(C * q_r)^2 = 0$ for $r = 1, 2, \ldots, n - 2$. Proceeding in the same way we can determine $(k - 1)$ rows of the matrix $C$ and then the $k$-th row using the determinant

$$C^k = \frac{1}{\sqrt{(-1)^{n-k+1} Q(n-k)}} \begin{vmatrix} q_1^n & q_1^{n-1} & \ldots & q_1^1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n-k}^n & q_{n-k}^{n-1} & \ldots & q_{n-k}^1 & 1 \\ C_1^1 & C_1^2 & \ldots & C_1^{n-1} & -C_n^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k}^{k-1} & C_{n-k}^{k-2} & \ldots & C_{n-k}^{k-1} & -C_n^{k-1} \\ z_1 & z_2 & \ldots & z_{n-1} & z_n \end{vmatrix}$$

We get $(C * q_r)^k = 0$ only for $r = 1, 2, \ldots, n - k$. In this way we construct the rows number $k = 2, 3, \ldots, n - 3$ and $(n - 2)$ again. We describe the $k$-th coordinate of the point $C * q_r$ by the formula

$$\begin{equation}
(C * q_r)^k = \frac{\sqrt{Q(3)} \cdot W_r^k}{(Q_{13}Q_{2r} + Q_{23}Q_{1r}) \sqrt{-Q(n-k)Q(n-k+1)}}
\end{equation}$$

where

$$W_r^k = \begin{vmatrix}
0 & Q_{12} & Q_{13} & \ldots & Q_{1,n-k-1} & Q_{1,n-k} & Q_{1,n-k+1} \\
Q_{21} & 0 & Q_{23} & \ldots & Q_{2,n-k-1} & Q_{2,n-k} & Q_{2,n-k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q_{n-k,1} & Q_{n-k,2} & Q_{n-k,3} & \ldots & Q_{n-k,n-k-1} & 0 & Q_{n-k,n-k+1} \\
Q_{r1} & Q_{r2} & Q_{r3} & \ldots & Q_{r,n-k-1} & Q_{r,n-k} & Q_{r,n-k+1}
\end{vmatrix}$$

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which holds true for \( k = 1, 2, \ldots, n - 2 \) and arbitrary \( r \). Considering the formulas (14) and (16) we see that \( C * q_r \) depends on \( q_1, q_2, \ldots, q_r \) only, in spite of \( C = C(q_1, q_2, \ldots, q_n) \). It allows us to select the lacking points of the sphere and construct the matrix \( C \) in the case \( m < n \). Formula (11) implies that \( (C * q_r)^k \) is an invariant. Considering the case when \( n = 2, n = 3 \) and (15) we have

**Lemma 2.** In the case \( 1 \leq m < 4 \), equation (7) has only the trivial solution \( F(q_1) = c \) for \( n \geq 2 \), \( F(q_1, q_2) = c \) for \( n \geq 2 \) and \( F(q_1, q_2, q_3) = c \) for \( n > 2 \), where \( c \) is an arbitrary constant.

Considering the case \( n = 3 \) and formulas (14) and (15) and using for \( m = n = 4 \) simply formula (16) we obtain

**Lemma 3.** The general solution of equation (7) in the case \( n > 2 \) and \( m = 4 \) is of the form

\[
F(q_1, q_2, q_3, q_4) = \Theta\left(\frac{\|q_1, q_3\| \cdot \|q_2, q_4\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}, \frac{\|q_1, q_4\| \cdot \|q_2, q_3\|}{\|q_1, q_2\| \cdot \|q_3, q_4\|}\right)
\]

where \( \Theta \) is an arbitrary function of two variables.

We can conclude with

**Lemma 4.** The general solution of equation (7) for arbitrary \( 4 \leq m \leq n \) is of the form

\[
F(q_1, q_2, \ldots, q_m) = \Theta((C * q_r)^k)
\]

where \( r \) runs from \( 4 \) to \( m \) and for every fixed \( r \) the index \( k \) changes from \( (n + 1 - r) \) to \( (n - 1) \) and \( \Theta \) is an arbitrary function of \( \frac{m}{2}(m - 3)(m + 2) \) variables.

Despite omitting in Lemma 4 the trivial 1-scalars \(-1, 0, +1\), we have relations \( C * q_r \in S^{n-2} \) as a result of the fact that \( (m - 3) \) arguments of the function \( \Theta \) are dependent on the others. Analysing formula (16) one can suppose that other kinds of invariants exist, in addition to the arguments of the function \( \Theta \) in Lemma 3. Because it is easy to find the correct number \( \frac{m}{2}m(m - 3) \) of simple and independent 1-scalars, we have

**Theorem 1.** The general solution of the functional equation

\[
F(A * q_1, A * q_2, \ldots, A * q_m) = F(q_1, q_2, \ldots, q_m)
\]

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for given independent points \( q_1, q_2, \ldots, q_m \in S^{n-2} \) and an arbitrary matrix \( A \in G \) is of the form

\[
F(q_1, q_2, \ldots, q_m) = \begin{cases} 
  c & \text{if } m = 1, 2, 3, \\
  \Theta \left( \frac{Q_{13}Q_{24}}{Q_{12}Q_{34}}, \frac{Q_{14}Q_{23}}{Q_{12}Q_{34}} \right) & \text{if } m = 4, \\
  \Theta \left( \frac{Q_{13}Q_{2i}}{Q_{12}Q_{3i}}, \frac{Q_{23}Q_{1i}}{Q_{12}Q_{3i}}, \frac{Q_{1i}Q_{2j}}{Q_{12}Q_{ij}} \right) & \text{if } 4 < m \leq n,
\end{cases}
\]

where \( 4 \leq j < i = 4, 5, \ldots, m \), \( c \) is an arbitrary constant and \( \Theta \) is an arbitrary function of \( \frac{1}{2}m(m - 3) \) variables.

4. General solutions to equations (8) and (10)

**Theorem 2.** The general solution of the functional equation

\[
F(A * q_1, A * q_2, \ldots, A * q_m) = \varepsilon(A) \cdot F(q_1, q_2, \ldots, q_m)
\]

for given independent points \( q_1, q_2, \ldots, q_m \in S^{n-2} \) and an arbitrary matrix \( A \in G \) is of the form

\[
F(q_1, q_2, \ldots, q_m) = \begin{cases} 
  c \cdot q_1 & \text{if } n = 2 \text{ and } m = 1, \\
  0 & \text{if } n \text{ is odd}, \\
  0 & \text{if } n > 2 \text{ and } m < n, \\
  \Psi \cdot \text{sign det}(q_1, q_2, \ldots, q_n) & \text{if } n \text{ is even and } m = n,
\end{cases}
\]

where \( c \) is an arbitrary constant and \( \Psi \) is the general solution of equation (7).

**Proof.** We have already proved the first two cases. Now, let \( m < n \) and \( n > 2 \). Then the matrix \( C \) in the case \( n \geq 4 \) (or \( A \) in the case \( n = 3 \)) satisfies \((C * q_r)^1 = 0\) for \( r = 1, 2, \ldots, m \). Let \( \overline{C} \) denote a matrix obtained from the matrix \( C \) by multiplying its elements of the first row by \(-1\). From the relations \( \varepsilon(\overline{C}) = -\varepsilon(C) \) and \((C * q_r) = (\overline{C} * q_r)\) we get simultaneously

\[
F(q_1, q_2, \ldots, q_m) = \varepsilon(C)F(C * q_1, C * q_2, \ldots, C * q_m) = \varepsilon(\overline{C})F(\overline{C} * q_1, \overline{C} * q_2, \ldots, \overline{C} * q_m) = -\varepsilon(C)F(C * q_1, C * q_2, \ldots, C * q_m).
\]

Let \( F(q_1, q_2, \ldots, q_n) \) be the general solution of equation (8) in the case \( m = n \) and \( n \) even. Then the quotient \( F(q_1, q_2, \ldots, q_n) : \text{sign det}(q_1, q_2, \ldots, q_n) \) is the general solution of equation (7), which proves the assertion of the theorem in the last case.
Analogously we can prove

**Theorem 3.** The general solution of the functional equation

\[ F(A \ast q_1, A \ast q_2, \ldots, A \ast q_m) = \varepsilon(A) \cdot \eta(A) \cdot F(q_1, q_2, \ldots, q_m) \]

for given independent points \( q_1, q_2, \ldots, q_m \in S^{n-2} \) and an arbitrary matrix \( A \in G \) is of the form

\[
F(q_1, q_2, \ldots, q_m) = \begin{cases} 
0 & \text{if } n \text{ is even or } m < n, \\
\Psi \cdot \text{sign det}(q_1, q_2, \ldots, q_n) & \text{if } n \text{ is odd and } m = n,
\end{cases}
\]

where \( \Psi \) is the general solution of equation (7).

**References**


Authors’ address: Aleksander Misiak, Eugeniusz Stasiak, Instytut Matematyki, Politechnika Szczecińska, Al. Piastów 17, 70-310 Szczecin, Poland, e-mail: misiak@ps.pl.