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Modal operators on bounded residuated l-monoids

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1. Introduction

Residuated lattice ordered monoids (Rl-monoids) form a large class of algebras containing the class of all lattice ordered groups (l-groups) as well as classes of algebras of several propositional logics. Recall that commutative Rl-monoids were introduced in [26] as a common generalization of Abelian l-groups and Heyting algebras (i.e. algebras of the intuitionistic propositional logic). At the same time, the algebras of fuzzy logics, such as MV-algebras [4] (see also [5]) (which are also categorically equivalent to Wajsberg algebras [9]), i.e., algebras of the Lukasiewicz infinite valued logic, and BL-algebras [13], i.e., algebras of Hájek’s basic fuzzy logic, can be recognized as special cases of bounded commutative Rl-monoids.

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More generally, RI-monoids without the requirement of the commutativity of the semigroup binary operation were introduced in [16] and further developed in [17]. Analogously as in the commutative cases, the algebras of non-commutative generalizations of fuzzy logics can be also viewed as special cases of bounded RI-monoids. Namely, GMV-algebras [21] or pseudo MV-algebras [11] (which are also categorically equivalent to pseudo Wajsberg algebras [3]), i.e., algebras of the non-commutative Lukasiewicz logic [19], and pseudo BL-algebras [6], i.e., algebras of Hájek’s non-commutative basic fuzzy logic [14], form proper classes of the class of bounded RI-monoids.

Modal operators (special cases of closure operators) on Heyting algebras were introduced and studied by Macnab in [20]. Analogously, modal operators on MV-algebras were introduced in [15]. A generalization of those operators to all bounded commutative RI-monoids were studied by the authors in [23].

In this paper we define and study modal operators on bounded RI-monoids which need not be commutative. Many of results are obtained for the variety of good normal RI-monoids that contains, among others, both the variety of good pseudo BL-algebras and that of Heyting algebras.

2. Bounded RI-monoids

A bounded RI-monoid is an algebra $M = (M; \odot, \lor, \land, \rightarrow, \sim, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid (need not be commutative).
(ii) $(M; \lor, \land, 0, 1)$ is a bounded lattice.
(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \sim z$ for any $x, y \in M$.
(iv) $(x \rightarrow y) \odot x = x \land y = y \odot (y \sim x)$.

Recall that the lattice $(M; \lor, \land)$ is distributive and that bounded RI-monoids form a variety of algebras of the indicated type. Moreover, the bounded RI-monoids can be recognized as bounded integral generalized BL-algebras in the sense of [1] and [2] and hence it is possible to prove (see [7]) that the operation “$\odot$” distributes over the lattice operations “$\lor$” and “$\land$”.

In what follows, by an RI-monoid we will mean a bounded RI-monoid.

If the operation “$\odot$” in an RI-monoid $M$ is commutative then $M$ is called a commutative RI-monoid.

For any RI-monoid $M$ we define two unary operations (negations) “$-$” and “$\sim$” on $M$ such that $x^- := x \rightarrow 0$ and $x^- := x \sim 0$ for every $x \in M$.

Recall that the algebras of the mentioned propositional logics are characterized in the class of RI-monoids as follows:
An \(Rl\)-monoid \(M\) is

a) a pseudo BL-algebra ([18]) if and only if \(M\) satisfies the identities of pre-linearity
\((x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \sim y) \vee (y \sim x)\);

b) a GMV-algebra (pseudo MV-algebra) ([22]) if and only if \(M\) fulfils the identities
\(x \sim y = x \sim y\);

c) a Heyting algebra ([26]) if and only if the operations “\(\odot\)” and “\(\land\)” coincide
on \(M\).

Basic properties of bounded \(Rl\)-monoids have been given in many articles (e.g. [25]), here we present some of them which will be used in this paper.

**Lemma 1.** In any bounded \(Rl\)-monoid \(M\) we have for any \(x, y \in M\):

1. \(x \leq y \iff x \rightarrow y = 1 \iff x \sim y = 1\).
2. \(x \leq y \implies z \rightarrow x \leq z \rightarrow y, \ z \sim x \leq z \sim y\).
3. \(x \leq y \implies y \rightarrow z \leq x \rightarrow z, \ y \sim z \leq x \sim z\).
4. \(1 \sim = 1 = 1 \sim, \ 0 \sim = 0 = 0 \sim\).
5. \(x \leq x \sim, \ x \leq x \sim\).
6. \(x \sim x, \ x \sim x = x\).
7. \(x \odot x = 0 = x \odot x\).
8. \(x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z, \ x \sim (y \sim z) = (y \odot x) \sim z\).
9. \(x \odot y = x \rightarrow y, \ (x \odot y) = y \sim x\).
10. \(x \lor y = x \lor y, \ (x \lor y) = x \lor y\).

An \(Rl\)-monoid \(M\) is called **good** if \(M\) satisfies the identity \(x \sim = x \sim\). (The notion of a good \(Rl\)-monoid is a generalization of that of a good pseudo BL-algebra introduced in [12].)

Obviously, every GMV-algebra or every commutative \(Rl\)-monoid is good. Moreover (see also [8]), if \(M_1\) and \(M_2\) are non-trivial GMV-algebras then their ordinal sum is a good \(Rl\)-monoid which need not be either a GMV-algebra or a pseudo BL-algebra.

Let \(M\) be a good \(Rl\)-monoid. According to [8], we define a binary operation “\(\oplus\)” on \(M\) as follows:

\[
\forall x, y \in M; x \oplus y := (y \sim x) \sim.
\]

By [25], every good \(Rl\)-monoid fulfils the identity \((x \odot y) \sim = (x \sim \odot y \sim) \sim\).

**Lemma 2.** If \(M\) is a good \(Rl\)-monoid and \(x, y \in M\) then

\[
(x \oplus y) \sim = x \sim \oplus y \sim = x \sim \oplus y = x \oplus y \sim = x \oplus y.
\]
Proof. 

\[(x \oplus y) \sim = (y \sim \odot x \sim)^\sim = x \oplus y, \]

\[x \sim \oplus y \sim = (y \sim \odot x \sim)^\sim = \sim x \oplus y, \]

\[= (y \sim \odot x \sim)^\sim = x \oplus y \sim = (y \sim x \sim)^\sim = x \sim \oplus y. \]

\[\square\]

Proposition 3 ([8]). Let \(M\) be a good \(Rl\)-monoid. Then for all \(x, y, z \in M\):

(i) \(x \oplus y = (y \sim \odot x \sim)^\sim\);

(ii) \(x \oplus (y \oplus z) = (x \oplus y) \oplus z\);

(iii) \(x, y \leq x \oplus y\);

(iv) \(x \oplus 0 = x \sim = 0 \oplus x\);

(v) \(x \oplus 1 = 1 = 1 \oplus x\);

(vi) \(x \oplus y = x \sim y \sim = y \sim \to x \sim\).

3. Modal operators—definition and properties

Definition. Let \(M\) be an \(Rl\)-monoid. A mapping \(f \colon M \to M\) is called a modal operator on \(M\) if, for any \(x, y \in M\),

1. \(x \leq f(x)\);
2. \(f(f(x)) = f(x)\);
3. \(f(x \odot y) = f(x) \odot f(y)\).

Moreover, if an \(Rl\)-monoid \(M\) is good and for any \(x, y \in M\),

4. \(f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus y)\),

then a modal operator \(f\) is called strong.

Proposition 4. a) If \(f\) is a modal operator on an \(Rl\)-monoid \(M\) and \(x, y \in M\) then

(i) \(x \leq y \Rightarrow f(x) \leq f(y)\);

(ii) \(f(x \to y) \leq f(x) \to f(y) = f(f(x) \to f(y)) = x \to f(y) = f(x \to f(y))\),

(iii) \(f(x) \leq (x \sim f(0)) \sim f(0), f(x) \leq (x \sim f(0)) \to f(0)\);

(iv) \(x^{-} \odot f(x) \leq f(0), f(x) \odot x^{-} \leq f(0)\);

(v) \(f(x \vee y) \leq f(x \vee f(y)) = f(f(x) \vee f(y))\).

b) If \(M\) is a good \(Rl\)-monoid and \(f\) is a strong modal operator on \(M\) then

(iii) \(x \oplus f(0) \geq f(x^{-}) \geq f(x), f(0) \oplus x \geq f(x^{-}) \geq f(x)\).

Proof. The proof of (i), (ii), (iv) and (v) runs as in [23].
(iii) By (i) and (ii), we have
\[
(f(x) \rightarrow f(0)) \odot f(x) = f(x) \land f(0) = f(0) \\
\implies f(x) \leq (f(x) \rightarrow f(0)) \bowtie f(0) = (x \rightarrow f(0)) \bowtie f(0).
\]
The latter inequality can be proved similarly.
(vi) By Proposition 3(vi), Lemma 1(2)(5) and properties (ii), (i) above,
\[
x \oplus f(0) = x^- \bowtie f(0)^- \bowtie \geq x^- \bowtie f(0) = f(x^- \bowtie f(0)) \geq f(x^- \bowtie 0) \\
= f(x^-) \geq f(x).
\]
The proof of the remaining inequalities is analogous. □

**Proposition 5.** Let \( M \) be a good \( Rl \)-monoid. If \( f \) is a strong modal operator on \( M \) and \( x, y \in M \) then
(i) \( f(x \oplus y) = f(f(x) \oplus f(y)) \);
(ii) \( x \oplus f(0) = f(x^-) = f(0) \oplus x \).

**Proof.** (i) By the definition of a strong modal operator it is obvious.
(ii) We have \( f(x \oplus f(0)) = f(x \oplus 0) = f(x^-) \implies f(x^-) = f(x \oplus f(0)) \geq x \oplus f(0) \geq f(x^-) \).

**Theorem 6.** Let \( M \) be an \( Rl \)-monoid and \( f: M \rightarrow M \) be a mapping. Then \( f \) is a modal operator on \( M \) if and only if for any \( x, y \in M \) it is satisfied:
(a) \( x \rightarrow f(y) = f(x) \rightarrow f(y) \);
(b) \( x \bowtie f(y) = f(x) \bowtie f(y) \);
(c) \( f(x) \odot f(y) \geq f(x \odot y) \).

**Proof.** Let a mapping \( f \) fulfil conditions (a)–(c).
Properties 1 and 2 from the definition of a modal operator follow in the same way as in the commutative case (see [23]).
We prove the property 3 from definition:
\[
x \odot y \leq f(x \odot y) \implies x \leq y \rightarrow f(x \odot y) = f(y) \rightarrow f(x \odot y) \implies x \odot f(y) \leq f(x \odot y) \implies f(y) \leq x \rightarrow f(x \odot y) = f(x \rightarrow f(x \odot y) \implies f(x) \odot f(y) \leq f(x \odot y) \implies f(x) \odot f(y) = f(x \odot y) \implies f(x) \odot f(y) = f(x \odot y).
\]

**Remark 7.** M. Galatos and C. Tsinakis introduced in [10] the notion of a **nucleus** of a residuated lattice \( L \) as a closure operator \( \gamma \) on \( L \) satisfying \( \gamma(a) \gamma(b) \leq \gamma(ab) \). From this point of view, a modal operator \( f \) on an \( Rl \)-monoid \( M \) is a nucleus of \( M \) satisfying \( f(x) \odot f(y) \geq f(x \odot y) \).

As a consequence we obtain:
Proposition 8. Let $M$ be an $Rl$-monoid and $f: M \rightarrow M$ be a mapping. Then $f$ is a nucleus of $M$ if and only if $f$ satisfies (a) and (b) of Theorem 6.

Proof. Considering the previous explication it remains to prove the isotony of $f$.

We first note $y \rightarrow f(y) = f(y) \rightarrow f(y) = 1$. Further, $x \leq y \implies 1 = y \rightarrow f(y) \leq x \rightarrow f(y) = f(x) \rightarrow f(y) \implies f(x) \leq f(y)$. □

Remark 9. In [23, Corollary 8], which is the analogy of Proposition 8 for the commutative case, we required the isotony of $f$ besides. It is evident now that this requirement is superfluous.

An $Rl$-monoid is called normal if it satisfies the identities

$$(x \circ y)^{-} = x^{-} \circ y^{-} \quad (x \circ y)^{-} = x^{-} \circ y^{-}.$$

For example, every good pseudo BL-algebra or every Heyting algebra is normal [25].

Let $M$ be a good $Rl$-monoid and $a \in M$. We denote by $\varphi_a: M \rightarrow M$ the mapping such that $\varphi_a(x) = a \oplus x$ for every $x \in M$. By Lemma 2, we have $\varphi_a$ coincides with $\varphi_{a^{-}}$.

Denote by

$I(M) = \{a \in M: a \circ a = a\}$

the set of all multiplicative idempotents in an $Rl$-monoid $M$. It is obvious that $0, 1 \in I(M)$. By [17, Lemma 2.8.3], $a \circ x = a \wedge x$ holds for any $a \in I(M)$, $x \in M$. Further we can prove, if $M$ is a normal $Rl$-monoid and $a \in I(M)$ then also $a^{-} \in I(M)$. Recall that by [8], $(M; \oplus)$ is a semigroup. Further, if $a^{-} \in I(M)$ then $a \oplus a = (a^{-} \circ a^{-})^{-} = a^{-}$.

Theorem 10. If $M$ is a good and normal $Rl$-monoid and $a \in M$ then $\varphi_a$ is a strong modal operator on $M$ if and only if $a^{-}, a^{-} \in I(M)$.

Proof. a) Let $a, x, y \in M$, $a^{-}, a^{-} \in I(M)$.

1. $\varphi_a(x) = a \oplus x = (x^{-} \circ a^{-})^{-} \geq x^{-} \geq x$.
2. $\varphi_a(\varphi_a(x)) = a \oplus (a \oplus x) = (a \oplus a) \oplus x = a^{-} \oplus x = a \oplus x = \varphi_a(x)$.
3. We first prove that $a \oplus x = (a \vee x)^{-}$.

$a \oplus x = (x^{-} \circ a^{-})^{-} = (x^{-} \wedge a^{-})^{-} = (x \vee a)^{-}$, by Lemma 1(10).

Now, we prove condition 3 from the definition of a modal operator.
By Lemma 2, the normality of $M$ and the distributivity \( \odot \) over \( \lor \), we have
\[
\varphi_a(x) \odot \varphi_a(y) = (a \oplus x) \odot (a \oplus y) = (a^- \oplus x) \odot (a^- \oplus y)
\]
\[
= (a^- \lor x)^- \odot (a^- \lor y)^- = ((a^- \lor x) \odot (a^- \lor y))^-
\]
\[
= ((a^- \odot a^-) \lor (x \odot a^-) \lor (a^- \odot y) \lor (x \odot y))^-
\]
\[
= (a^- \lor (x \lor y))^- = a^- \lor (x \lor y) = a \lor (x \lor y) = \varphi_a(x \lor y).
\]

4. By \([8]\), \((M; \oplus)\) is a semigroup.

For this reason and by the above and Lemma 2,
\[
\varphi_a(x \oplus \varphi_a(y)) = a \oplus (x \oplus (a \oplus y)) = ((a \oplus a) \oplus x) \oplus y = (a^- \oplus x) \oplus y
\]
\[
= (a \oplus x) \oplus y = a \oplus (x \oplus y) = \varphi_a(x \oplus y).
\]

b) Let \( \varphi_a \) be a strong modal operator on \( M \). Then \( \varphi_a(x \lor y) = \varphi_a(x) \odot \varphi_a(y) \), hence \( a \oplus (x \lor y) = (a \oplus x) \odot (a \oplus y) \) for any \( x, y \in M \). For \( x = y = 0 \), we obtain
\[
a \oplus 0 = (a \oplus 0) \odot (a \oplus 0), \text{ thus } a^- = a^- \lor a^- = a^- \lor a^- = a^- \lor a^- = a^- \lor a^- \text{ and } a^- \in I(M).
\]

Further, \( \varphi_a(x \lor y) = \varphi_a(x \lor \varphi_a(y)) \), hence \( a \oplus (x \lor y) = a \oplus (x \oplus (a \oplus y)) \) for any \( x, y \in M \). For \( x = y = 0 \), we have \( a^- = a \oplus 0 = a \oplus (0 \oplus 0) = a \oplus (0 \oplus (a \oplus 0)) = (a \oplus 0) \odot a^- = a^- \lor a^- \), thus \( a^- = (a^- \lor a^-)^- \lor a^- \). From this it follows that \( a^- = (a^- \lor a^-)^- \lor a^- \lor a^- = a^- \lor a^- = a^- \lor a^- \) and so \( a^- \in I(M) \).

\textbf{Theorem 11.} Let \( M \) be a good normal Rl-monoid and \( f \) be a modal operator on \( M \) such that \( f(x) = f(x^-) \) for all \( x \in M \). Then \( f \) is strong if and only if \( f = \varphi_{f(0)} \) and \( f(0)^- \in I(M) \).

\textbf{Proof.} Let \( f \) be a modal operator on \( M \) satisfying \( f(x) = f(x^-) \) for every \( x \in M \).

If \( f \) is strong then by Proposition 5, \( f(x) = f(x^-) = x \oplus f(0) \) for any \( x \in M \). Hence \( f = \varphi_{f(0)} \) and moreover, by Theorem 10, \( f(0)^-, f(0)^- \in I(M) \).

If \( f \) is any modal operator then \( f(0^-)^- = f(0 \lor 0^-)^- = (f(0) \lor f(0))^- = f(0)^- = f(f(0)^-), \text{ thus } f(0)^- \in I(M) \).

Therefore conversely, if \( f = \varphi_{f(0)} \) and \( f(0)^- \in I(M) \), then by Theorem 10, we conclude that \( f \) is strong. \( \square \)

\textbf{Corollary 12.} If \( M \) is a GMV-algebra (pseudo MV-algebra, equivalently) and \( f \) is a modal operator on \( M \), then \( f \) is strong if and only if \( f = \varphi_{f(0)} \).

\textbf{Proof.} It is sufficient to show that \( f(0)^- \in I(M) \) for any modal operator \( f \).

It is known (see \([11]\) and \([22]\)) that the set \( I(M) \) coincides with the set \( B(M) \) of all elements having complements in the lattice \( (M; \lor, \land, 0, 1) \) in any GMV-algebra \( M \), and if \( x \in B(M) \) and \( x' \) is its complement, then \( x' = x^- = x^- \in B(M) = I(M) \).

Since \( f(0) \in I(M) \) in our case, we obtain \( f(0)^- \in I(M) \) as well. \( \square \)
Let $M$ be an $Rl$-monoid and $a \in I(M)$. Let us consider mappings $\psi^1_a : M \to M$ and $\psi^2_a : M \to M$ such that $\psi^1_a(x) := a \to x$ and $\psi^2_a(x) := a \sim x$ for every $x \in M$.

**Proposition 13.** If $M$ is an $Rl$-monoid and $a \in I(M)$ then for any $x, y \in M$

$$x \to \psi^1_a(y) = \psi^1_a(x) \to \psi^1_a(y),$$

$$x \sim \psi^2_a(y) = \psi^2_a(x) \sim \psi^2_a(y).$$

**Proof.** By the definition of $\psi^1_a$ and Lemma 1(8), $x \to \psi^1_a(y) = x \to (a \to y) = (x \circ a) \to y = (a \circ x) \to y$ and $\psi^1_a(x) \to \psi^1_a(y) = (a \to x) \to (a \to y) = ((a \to x) \circ a) \to y = (a \land x) \to y = (a \circ x) \to y$.

The other identity would be proved analogously. \qed

**Corollary 14.** Let $M$ be an $Rl$-monoid and $a \in I(M)$. Then $\psi^1_a$ is a modal operator on $M$ if and only if for any $x, y \in M$

$$x \sim \psi^1_a(y) = \psi^1_a(x) \sim \psi^1_a(y),$$

$$\psi^1_a(x) \circ \psi^1_a(y) \geq \psi^1_a(x \circ y).$$

Let $M$ be an $Rl$-monoid and $f$ be a modal operator on $M$. Then $\text{Fix}(f) = \{x \in M : f(x) = x\}$ will denote the set of all fixed elements of the operator $f$. By the definition of a modal operator it is obvious that $\text{Fix}(f) = \text{Im}(f)$.

**Theorem 15.** If $f$ is a modal operator on an $Rl$-monoid $M$ then $\text{Fix}(f)$ is closed under the operations $\land, \circ, \to$ and $\sim$, and $\text{Fix}(f) = (\text{Fix}(f); \circ, \lor_F, \land, \to, \sim, f(0), 1)$, where $x \lor_F y = f(x \lor y)$ for any $x, y \in \text{Fix}(f)$, is an $Rl$-monoid.

**Proof.** (i) Since $f$ is a closure operator on the lattice $(M; \lor, \land)$, it holds $x \land y \in \text{Fix}(f)$ for any $x, y \in \text{Fix}(f)$, and so $(\text{Fix}(f); \lor_F, \land)$ is a lattice.

(ii) $(\text{Fix}(f); \lor_F, \land, f(0), 1)$ is a bounded lattice.

(iii) Let $x, y \in \text{Fix}(f)$. Then $f(x \circ y) = f(x) \circ f(y) = x \circ y$, thus $x \circ y \in \text{Fix}(f)$.

(iv) If $y, z \in \text{Fix}(f)$ then by Proposition 5 we have $y \to z = f(y) \to f(z) = f(f(y) \to f(z)) = f(y \to z)$, hence $y \to z \in \text{Fix}(f)$. For any $y, z \in \text{Fix}(f)$, $y \sim z \in \text{Fix}(f)$ analogously.

Therefore, if $x, y, z \in \text{Fix}(f)$ then $x \circ y, y \to z, x \sim z \in \text{Fix}(f)$ and for this reason $x \circ y \leq z$ holds in $\text{Fix}(f)$ if and only if $x \leq y \to z$ and it is equivalent to $y \leq x \sim z$.

(v) By foregoing, $\text{Fix}(f)$ also satisfies the identities $(x \to y) \circ x = x \land y = y \circ (y \sim x)$.
4. Modal operators on intervals

Let $M$ be an Rl-monoid. For $a \in I(M)$, let

$$I(a) := [0, a] = \{ x \in M : 0 \leq x \leq a \}. $$

**Theorem 16.** Let $M$ be an Rl-monoid and $a \in I(M)$. For any $x, y \in I(a)$ we set $x \circ_a y = x \circ y$, $x \rightarrow_a y := (x \rightarrow y) \land a$ and $x \sim_a y := (x \sim y) \land a$. Then $I(a) = (I(a); \circ_a, \lor, \land, \rightarrow_a, \sim_a, 0, a)$ is an Rl-monoid.

**Proof.** (i) If $x, y \in I(a)$ then $x \circ y \in I(a)$ and $x \circ a = a \circ x = x \land a = x$, hence $(I(a); \circ_a, a)$ is a monoid.

(ii) Obviously, $(I(a); \lor, \land, 0, a)$ is a bounded lattice.

(iii) Let $x, y \in I(a)$. It holds that $x \rightarrow y$ is the greatest element $z \in M$ such that $z \circ x \leq y$. Therefore $(x \rightarrow y) \land a$ is the greatest element in $I(a)$ with this property. Analogously, for $y, z \in I(a)$, $z \sim y$ is the greatest element $x \in M$ such that $z \circ_a x \leq y$. Hence, $(z \sim y) \land a$ is the greatest element in $I(a)$ with this property. That means, $z \circ_a x \leq y$ if and only if $z \leq (x \rightarrow y) \land a = x \rightarrow_a y$, and if and only if $x \leq (z \sim y) \land a = z \sim_a y$ for any $x, z \in I(a)$.

(iv) For any $x, y \in I(a)$ we have $(x \rightarrow_a y) \circ_a x = ((x \rightarrow y) \land a) \circ x = (x \rightarrow y) \circ a \circ x = (x \rightarrow y) \circ x \circ a = (x \land y) \land a = x \land y$. We obtain $y \circ_a (y \sim_a x) = x \land y$ analogously. □

Let $M$ be an Rl-monoid, $a \in I(M)$ and $x \in I(a)$. We denote by $x^{-a}$ and $x^\sim a$ the negations of an element $x$ in $I(a)$.

**Proposition 17.** a) If $M$ is an Rl-monoid, $a \in I(M)$ and $x \in I(a)$, then

$$x^{-a} = x^{-} \land a, \quad x^\sim a = x^{\sim} \land a. $$

b) Moreover, if $M$ is good and satisfying the identities

$$(v \land w)^{-} = v^{-} \lor w^{-}, \quad (v \land w)^{\sim} = v^{\sim} \lor w^{\sim},$$

then the Rl-monoid $I(a)$ is good, too. If we denote by $x \oplus_a y$ the sum of elements $x, y \in I(a)$ in the Rl-monoid $I(a)$ then it holds

$$x \oplus_a y = (x \oplus y) \land a.$$

**Proof.** a) $x^{-a} = x \rightarrow_a 0 = (x \rightarrow 0) \land a = x^{-} \land a$, $x^\sim a = x \sim_a 0 = (x \sim 0) \land a = x^{\sim} \land a$. □
b) Let $M$ be good and satisfy $(\ast)$. Then

\[
x^{-a\sim a} = (x^{-a})\sim a = (x^- \land a)\sim a = (x^- \lor a^\sim) \land a = (x^- \land a) \lor (a^\sim \land a)
\]

\[
= (x^- \land a) \lor 0 = x^- \land a = x^- \land a = (x^- \land a) \lor (a^- \land a)
\]

hence $I(a)$ is also a good Rl-monoid and therefore we can define $x \oplus_{a} y$ for any $x, y \in I(a)$.

Then it holds, using Lemma 1(9),

\[
x \oplus_{a} y = (y^{-a} \circ x^{-a})^\sim a = (y^{-a} \circ x^{-a})^\sim a = (y^- \circ a \circ x^- \circ a)^\sim a
\]

\[
= (y^- \circ x^- \circ a)^\sim a = (a \rightsquigarrow (y^- \circ x^-))^\sim a = a \circ (a \rightsquigarrow (y^- \circ x^-))^\sim
\]

\[
= a \land (y^- \circ x^-)^\sim = (x \oplus y) \land a.
\]

\[
\]

Remark 18. For example, every pseudo BL-algebra satisfies the identities $(\ast)$ (see [25]).

Let $M$ be an Rl-monoid, $a \in I(M)$ and let $f$ be a modal operator on $M$. Let us consider a mapping $f^a: I(a) \rightarrow I(a)$ such that $f^a(x) := f(x) \land a(= f(x) \circ a)$, for every $x \in I(a)$.

**Theorem 19.**

a) Let $M$ be an Rl-monoid, $a \in I(M)$ and $f$ be a modal operator on $M$. Then $f^a$ is a modal operator on the Rl-monoid $I(a)$.

b) If $M$ is good and it satisfies the identities $(\ast)$, and $f$ is strong, then $f^a$ is also a strong modal operator on $I(a)$.

**Proof.**

a) Consider $x, y \in I(a)$.

1. $x \leq a$ and $x \leq f(x)$, hence $x \leq a \land f(x) = f^a(x)$.

2. $f^a(f^a(x)) = f^a(f^a(x)) = f(f(x) \land a) \land a = f(f(x) \circ a) \land a = (f(f(x)) \circ f(a)) \land a = f(x) \land f(a) \land a = f(x) \land a = f^a(x)$.

3. $f^a(x \circ y) = f(x \circ y) \land a = f(x) \lor f(y) \circ a \circ a = (f(x) \land a) \circ (f(y) \land a) = f^a(x) \circ f^a(y)$.

b) Let $f$ be strong. Then

\[
f^a(x \oplus_{a} f^a(y)) = f^a(x \oplus_{a} f^a(y)) = f^a((x \oplus (f(y) \land a)) \land a)
\]

\[
= f((x \oplus (f(y) \land a)) \land a) \land a
\]

\[
= f(x \oplus (f(y) \land a)) \land a = f(x \oplus f(f(y) \land a)) \land a
\]

\[
= f(x \oplus ((f(f(y))) \land f(a))) \land a = f(x \oplus (f(y) \land f(a))) \land a
\]

\[
= f(x \oplus f(y \land a)) \land a = f(x \oplus f(y)) \land a = f(x \oplus y) \land a
\]

\[
= f^a(x \oplus y).
\]
5. The set of idempotent elements

**Proposition 20.** If $M$ is an $Rl$-monoid then $I(M)$ is a subalgebra of its reduct $(M; \odot, \lor, \land, 0, 1)$.

**Proof.** Suppose $M$ is an $Rl$-monoid and $x, y \in I(M)$. Then

$$(x \odot y) \odot (x \odot y) = x \odot (y \odot x) \odot y = x \odot (x \odot y) \odot y = (x \odot x) \odot (y \odot y) = x \odot y,$$

thus $x \odot y = x \land y \in I(M)$. Further,

$$(x \lor y) \odot (x \lor y) = (x \odot x) \lor (y \odot x) \lor (x \odot y) \lor (y \odot y) = x \lor y \lor (x \odot y) = x \lor y,$$

hence also $x \lor y \in I(M)$.

Obviously, $0, 1 \in I(M)$.

Let $f$ be a modal operator on an $Rl$-monoid $M$ and $\hat{f} = f|I(M)$. Assume $x \in I(M)$. Then $f(x) = f(x \odot x) = f(x) \odot f(x)$, so $f(x) \in I(M)$. Therefore, we can consider $\hat{f}$ as the mapping of $I(M)$ into $I(M)$.

**Theorem 21.** Let $M$ be an $Rl$-monoid and let $f$ be a modal operator on $M$. Then $\hat{f}: I(M) \rightarrow I(M)$ fulfills conditions 1, 2, 3 from the definition of a modal operator.

**Proof.** Theorem is the immediate consequence of previous considerations. □

**Theorem 22.** Let $M$ be a good and normal $Rl$-monoid and let $x^- \in I(M)$ for any $x \in I(M)$. Then $I(M)$ is closed under the operation “$\oplus$”. Furthermore, if $f$ is a strong modal operator on $M$ then $\hat{f}$ satisfies the condition 4 from the definition of a strong modal operator, too.

**Proof.** Let $x, y \in I(M)$. By the proof of Theorem 10, part 3, it holds $x \oplus y = (x \lor y)^-$. Further,

$$(x \oplus y) \odot (x \oplus y) = (x \lor y)^- \odot (x \lor y)^- = ((x \lor y) \odot (x \lor y))^-
= ((x \odot x) \lor (x \lor y) \lor (y \lor x) \lor (y \lor y))^- = (x \lor y)^- = x \oplus y.$$

Therefore, $x \oplus y \in I(M)$.

Now it is obvious that $\hat{f}$ satisfies also the condition 4 from the definition of a strong modal operator. □
Let us remind that an \( Rl \)-monoid is called representable if and only if it is isomorphic with a subdirect product of linearly ordered \( Rl \)-monoids. It is obvious that every (bounded) linearly ordered \( Rl \)-monoid is a pseudo BL-algebra. Therefore, representable \( Rl \)-monoids are pseudo BL-algebras as well; and by [18], they form a proper subclass of the class of all pseudo BL-algebras. (Let us recall that, by [21], the class of representable commutative \( Rl \)-monoids and the class of all BL-algebras coincide.)

**Theorem 23.** Let \( M \) be a representable pseudo BL-algebra. Then \( I(M) \) is a subalgebra in \( M \) which is a Heyting algebra. If \( f \) is a modal operator on \( M \) then \( \hat{f} \) is a modal operator on \( I(M) \). Moreover, if \( M \) is good and \( x^\sim \in I(M) \) for every \( x \in I(M) \) and \( f \) is a strong modal operator on \( M \), then \( \hat{f} \) is a strong modal operator on \( I(M) \).

**Proof.** Let a representable pseudo BL-algebra \( M \) be isomorphic with a subdirect product of pseudo BL-chains \( M_\alpha, \alpha \in A \). Let \( a = (a_\alpha; \alpha \in A) \in M \). Then \( a \in I(M) \) if and only if \( a_\alpha \in I(M_\alpha) \) for every \( \alpha \in A \). Suppose \( x = (x_\alpha; \alpha \in A), y = (y_\alpha; \alpha \in A) \in I(M) \). Then \( x_\alpha \rightarrow y_\alpha = 1 \) for \( y_\alpha \geq x_\alpha \) and \( x_\alpha \rightarrow y_\alpha = y_\alpha \) for \( x_\alpha > y_\alpha \). Hence \( (x_\alpha \rightarrow y_\alpha; \alpha \in A) \in I(M) \) and \( (x_\alpha \rightarrow y_\alpha; \alpha \in A) = x \rightarrow y \). Similarly, \( (x_\alpha \rightarrow y_\alpha; \alpha \in A) \in I(M) \) and \( (x_\alpha \rightarrow y_\alpha; \alpha \in A) = x \rightarrow y \). By [17], \( I(M) \) is a Heyting algebra.

Therefore, if \( f \) is a modal operator on \( M \) then \( \hat{f} \) is a modal operator on \( I(M) \). Further by [25], every good pseudo BL-algebra is a normal \( Rl \)-monoid. For that reason, if \( M \) is good and \( x^\sim \in I(M) \) for every \( x \in I(M) \), and if a modal operator on \( M \) is strong, then \( \hat{f} \) is a strong modal operator on the Heyting algebra \( I(M) \).

\[ \square \]

**References**


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