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ON FORBIDDEN CONFIGURATION OF
0-DISTRIBUTIVE LATTICES

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Dedicated to my Mother

Abstract. In this paper we obtain the forbidden configuration for 0-distributive lattices.

Keywords: forbidden configuration, 0-distributive lattice

MSC 2010: 06D05, 06D15

INTRODUCTION

Grillet and Varlet [2] introduced the concept of 0-distributive lattices as a generalization of distributive lattices.

A lattice L with 0 is called *0-distributive* if, for every triplet $\langle a, b, c \rangle$ of elements of L , $a \wedge b = a \wedge c = 0$ implies $a \wedge (b \vee c) = 0$. Dually, one can define the 1-distributive lattice.

Grillet (see Varlet [4]) has given the forbidden configuration of modular 0-distributive lattices as follows:

Theorem 1. *A modular lattice L with 0 is 0-distributive if and only if it contains no sublattice isomorphic to the lattice of Figure 1(a) or to the lattice of Figure 1(b) (next page).*

In Stern [3], it is mentioned that, till now no such forbidden configuration is known for 0-distributivity.

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In this paper we obtain forbidden configuration for 0-distributive lattices. The idea of the proof is taken from the paper of Davey, Poguntke and Rival [1].

A subset A of a lattice L is an antichain if x is incomparable with y for each pair of distinct elements x, y in A . For example, if a triplet $\langle a, b, c \rangle$ of elements of L violates 0-distributivity, then $\{a, b, c\}$ must be a three-element antichain.

For antichains A and B in L we define $A \ll B$ if, for every $a \in A$ there is $b \in B$ such that $a \leq b$; this defines a partial order on the set of antichains in L .

Throughout this paper, let L be a lattice of finite length with 0 which does not satisfy 0-distributivity and let $\{a, b, c\}$ be a maximal antichain in L such that $\langle a, b, c \rangle$ violates 0-distributivity. Since L is of finite length we may, without loss of generality, assume that every proper sub-interval of L satisfies 0-distributivity and $a \vee b \vee c = 1$.

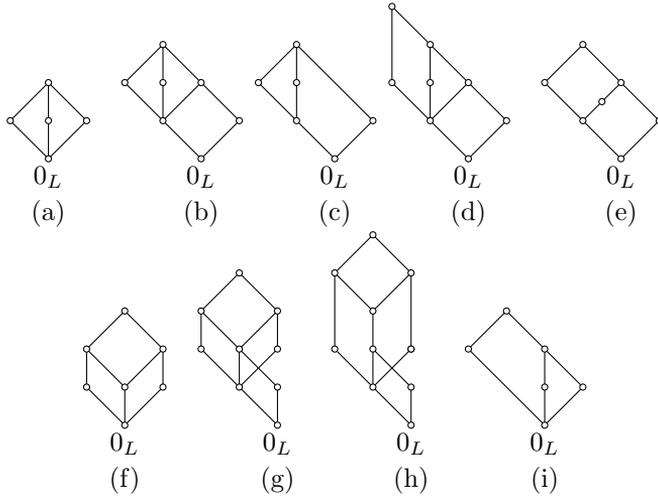


Figure 1

Lemma 1. (i) If $b < s < 1$, then $b \wedge c = s \wedge c$. (ii) If $c < t < 1$, then $b \wedge c = b \wedge t$.

Proof. Assume $b \wedge c < s \wedge c$. Hence $b < [b \vee (s \wedge c)]$. Clearly, $a \wedge [b \vee (s \wedge c)] \geq 0$. We claim that $a \wedge [b \vee (s \wedge c)] = 0$. If $a \wedge [b \vee (s \wedge c)] > 0$, then $\langle a \wedge [b \vee (s \wedge c)], b, s \wedge c \rangle$ is a triplet which violates 0-distributivity in the interval $[0, s]$, a contradiction to the assumption that every proper sub-interval satisfies 0-distributivity.

Therefore $a \wedge [b \vee (s \wedge c)] = 0$. But then $\langle a, [b \vee (s \wedge c)], c \rangle$ violates 0-distributivity, a contradiction to maximality of $\{a, b, c\}$. Thus, $b \wedge c = s \wedge c$. Similarly (ii) can be proved. \square

Lemma 2. (i) If $b \wedge c < s < b$, then $s \vee c = 1$. (ii) If $b \wedge c < t < c$, then $b \vee t = 1$.

Proof. Suppose $s \vee c < 1$. Clearly $c < s \vee c < 1$. By Lemma 1, $b \wedge c = b \wedge (s \vee c)$. But then $b \wedge c = b \wedge (s \vee c) \geq s > b \wedge c$, a contradiction. Hence $s \vee c = 1$. Similarly (ii) can be proved. \square

Theorem 2. A finite lattice L with 0 is 0-distributive if and only if it contains no sublattice isomorphic to one of the lattices of Figure 1(a), (b), (c), (d), (e), (f), (g), (h), (i).

Proof. Suppose $a \wedge b = a \wedge c = 0$ but $a \wedge (b \vee c) \neq 0$. Let p be an atom of L such that $p \leq a \wedge (b \vee c)$. Then $p \leq b \vee c$ implies $b \vee c = (b \vee p) \vee (c \vee p)$.

We have the following three main cases:

$$[A] \quad p \vee b = p \vee c;$$

$$[B] \quad p \vee b \leq p \vee c;$$

$$[C] \quad p \vee b \parallel p \vee c.$$

Case [A]: Suppose $b \vee p = c \vee p$. Then $b \vee c = b \vee p = c \vee p$. This case has the following two subcases:

$$[A_1] \quad (b \wedge c) \vee p = b \vee p;$$

$$[A_2] \quad (b \wedge c) \vee p < b \vee p.$$

Subcase [A₁]: If $(b \wedge c) \vee p = b \vee p$, then $b \wedge c \neq 0$, otherwise $p = b$, a contradiction to the fact that $p \not\leq b$. Therefore, $L_1 = \{0, b, c, b \wedge c, p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(c).

Subcase [A₂]: Let $(b \wedge c) \vee p < b \vee p$. This subcase has the following three subcases:

$$[A_{21}] \quad [(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[A_{22}] \quad [(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[A_{23}] \quad [(b \wedge c) \vee p] \wedge b > b \wedge c < [(b \wedge c) \vee p] \wedge c.$$

Subcase [A₂₁]: If $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$, then $L_2 = \{0, p, b, c, b \wedge c, (b \wedge c) \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(b) when $b \wedge c > 0$ and to the lattice of Figure 1(a) if $b \wedge c = 0$.

Subcase [A₂₂]: Suppose $b \wedge c < [(b \wedge c) \vee p] \wedge b$ and $b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds.

Clearly, $b \wedge c < [(b \wedge c) \vee p] \wedge b \leq b$. If $[(b \wedge c) \vee p] \wedge b = b$, then $[(b \wedge c) \vee p] = b \vee p$, a contradiction to $[(b \wedge c) \vee p] < b \vee p$ (Subcase [A₂]). Thus, $b \wedge c < [(b \wedge c) \vee p] \wedge b < b$. By Lemma 2, $\{[(b \wedge c) \vee p] \wedge b\} \vee c = 1$. Then $L_3 = \{0, b \wedge c, p, (b \wedge c) \vee p, [(b \wedge c) \vee p] \wedge b, c, 1\}$ forms a sublattice isomorphic to the lattice of Figure 1(e). Note that in this case $b \wedge c \neq 0$, otherwise $0 = b \wedge c < [(b \wedge c) \vee p] \wedge b = p \wedge b = 0$.

From the symmetry of b, c , the subcases $b \wedge c < [(b \wedge c) \vee p] \wedge c$ and $b \wedge c = [(b \wedge c) \vee p] \wedge b$ follow.

Subcase [A₂₃]: Suppose $b \wedge c < [(b \wedge c) \vee p] \wedge b$ and $b \wedge c < [(b \wedge c) \vee p] \wedge c$ hold.

Put $[(b \wedge c) \vee p] \wedge b = x$ and $[(b \wedge c) \vee p] \wedge c = y$. As shown in case [A₂₂], we have $b \wedge c < [(b \wedge c) \vee p] \wedge b < b$ and $b \wedge c < [(b \wedge c) \vee p] \wedge c < c$. By Lemma 2, $\{[(b \wedge c) \vee p] \wedge b\} \vee c = 1$ and $\{[(b \wedge c) \vee p] \wedge c\} \vee b = 1$, that is, $x \vee c = y \vee b = 1$. Clearly, $p \wedge x = p \wedge y = 0$ and every proper sub-interval $[0, (b \wedge c) \vee p]$ of L is 0-distributive, hence we have $p \wedge (x \vee y) = 0$. Clearly, $x \vee y \leq (b \wedge c) \vee p$. If $x \vee y < (b \wedge c) \vee p$, then $L_4 = \{0, p, c, y, x \vee y, [(b \wedge c) \vee p], 1\}$ forms a sublattice isomorphic to the lattice of Figure 1(e) and if $x \vee y = (b \wedge c) \vee p$, then $L_4 = \{0, p, x, y, x \vee y, x \wedge y = b \wedge c\}$ forms a sublattice isomorphic to the lattice of Figure 1(c).

Case [B]: Without loss of generality, suppose $b \vee p < c \vee p$. Then $b \vee c = c \vee p$. Note that $(b \wedge c) \vee p \neq b \vee c$, otherwise $b \vee p = c \vee p$, a contradiction to $b \vee p < c \vee p$. This case has the following two subcases:

$$[B_1] \quad (b \wedge c) \vee p = b \vee p;$$

$$[B_2] \quad (b \wedge c) \vee p < b \vee p.$$

Subcase [B₁]: Suppose $(b \wedge c) \vee p = b \vee p$ holds. Clearly, $b < b \vee p < 1$, and by Lemma 1, $b \wedge c = (b \vee p) \wedge c$. Then $L_5 = \{0, b, c, b \wedge c, p, b \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(e) when $b \wedge c > 0$ and is isomorphic to the lattice of Figure 1(i) when $b \wedge c = 0$.

Subcase [B₂]: Suppose $(b \wedge c) \vee p < b \vee p$. We have the following three subcases in this case.

$$[B_{21}] \quad [(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[B_{22}] \quad [(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[B_{23}] \quad [(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c;$$

$$[B_{24}] \quad [(b \wedge c) \vee p] \wedge b > b \wedge c < [(b \wedge c) \vee p] \wedge c.$$

Subcase [B₂₁]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. It is clear that $b < b \vee p < 1$, hence by Lemma 1, $b \wedge c = (b \vee p) \wedge c$. Thus, $L_6 = \{0, b, c, b \wedge c, p, (b \wedge c) \vee p, b \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(d) when $b \wedge c > 0$ and is isomorphic to the lattice of Figure 1(i) when $b \wedge c = 0$.

Subcase [B₂₂]: Suppose $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. Further, we claim that $[(b \wedge c) \vee p] \wedge b \neq b$. If possible, then $b \leq [(b \wedge c) \vee p]$. Taking join with p , we get $b \vee p \leq (b \wedge c) \vee p$, a contradiction to $(b \wedge c) \vee p < b \vee p$. Hence $(b \wedge c) < [(b \wedge c) \vee p] \wedge b < b$. By Lemma 2, $\{[(b \wedge c) \vee p] \wedge b\} \vee c = 1$. Thus, $L_7 = \{0, c, b \wedge c, p, [(b \wedge c) \vee p] \wedge b, (b \wedge c) \vee p, 1\}$ forms a sublattice isomorphic to the lattice of Figure 1(e).

Subcase [B₂₃]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c$ holds. Along similar lines as in Subcase [B₂₂], one can show that $\{[(b \wedge c) \vee p] \wedge c\} \vee b = 1$, which implies that $b \vee p = 1$, a contradiction to $b \vee p < 1$. Hence this case can not occur. Similarly, Subcase [B₂₄] also.

Case [C]: Suppose $b \vee p \parallel c \vee p$. Then $b \vee p, c \vee p < b \vee c$. Note that $(b \wedge c) \vee p \leq (b \vee p) \wedge (c \vee p)$ is always true. Hence, we have the following two subcases:

$$[C_1] \quad (b \wedge c) \vee p = (b \vee p) \wedge (c \vee p);$$

$$[C_2] \quad (b \wedge c) \vee p < (b \vee p) \wedge (c \vee p).$$

Subcase [C₁]: Suppose $(b \wedge c) \vee p = (b \vee p) \wedge (c \vee p)$ holds. For this subcase, we have the following three subcases:

$$[C_{11}] \quad [(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[C_{12}] \quad [(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[C_{13}] \quad [(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c.$$

Subcase [C₁₁]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. This together with $(b \wedge c) \vee p = (b \vee p) \wedge (c \vee p)$ gives $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c = b \wedge (c \vee p) = c \wedge (b \vee p)$. Thus, $L_8 = \{0, p, b, c, (b \wedge c) \vee p, b \vee p, c \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(g) when $b \wedge c \neq 0$ and L_8 forms a sublattice isomorphic to the lattice of Figure 1(f) when $b \wedge c = 0$.

Subcase [C₁₂]: Suppose $[(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. We claim that $b \wedge (c \vee p) \neq b$. Otherwise $b \leq c \vee p$, which gives $b \vee c \leq c \vee p$, a contradiction to $c \vee p < b \vee c$. Hence $b \wedge c < b \wedge (c \vee p) < b$. By Lemma 2, $[b \wedge (c \vee p)] \vee c = 1$, which implies $c \vee p = 1$, a contradiction. Thus, this case can not occur. Similarly, one can show that the Subcase [C₁₃] can not occur.

Subcase [C₂]: Suppose $(b \wedge c) \vee p < (b \vee p) \wedge (c \vee p) < 1$ holds. For this subcase, we have the following three subcases.

$$[C_{21}] \quad [(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[C_{22}] \quad [(b \wedge c) \vee p] \wedge b > b \wedge c = [(b \wedge c) \vee p] \wedge c;$$

$$[C_{23}] \quad [(b \wedge c) \vee p] \wedge b = b \wedge c < [(b \wedge c) \vee p] \wedge c.$$

Subcase [C₂₁]: Suppose $[(b \wedge c) \vee p] \wedge b = b \wedge c = [(b \wedge c) \vee p] \wedge c$ holds. This subcase has again the following three subcases.

$$[C_{211}] \quad (b \vee p) \wedge c = b \wedge c = (c \vee p) \wedge b;$$

$$[C_{212}] \quad (b \vee p) \wedge c = b \wedge c < (c \vee p) \wedge b;$$

$$[C_{213}] \quad (b \vee p) \wedge c > b \wedge c < (c \vee p) \wedge b.$$

Subcase [C₂₁₁]: Suppose $(b \vee p) \wedge c = b \wedge c = (c \vee p) \wedge b$ holds. Then $L_9 = \{0, p, b, c, b \wedge c, (b \wedge c) \vee p, (b \vee p) \wedge (c \vee p), b \vee p, c \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(h) when $b \wedge c > 0$ and $L_9 = \{0, b, c, (b \vee p) \wedge (c \vee p), b \vee p, c \vee p, b \vee c\}$ forms a sublattice isomorphic to the lattice of Figure 1(f) when $b \wedge c = 0$.

From Subcase C₁₂, we can show that the Subcases [C₂₁₂], [C₂₁₃], [C₂₂] and [C₂₃] can not occur.

The converse is obvious.

This completes the proof. □

Remark. Note that the lattice of Figure 2 violates 0-distributivity, but has no sublattice isomorphic to Figure 1 (a), (b), (c), (d), (e), (f), (g), (e), (h) or (i). Hence, Theorem 2 fails in a lattice of infinite chains.

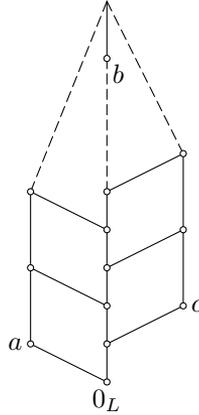


Figure 2

From the proof of Theorem 2 and Lemma 2, the next theorem can be easily proved.

Theorem 3. A finite modular lattice L with 0 is 0-distributive if and only if it contains no sublattice isomorphic to the lattice of Figure 3(a) or to the lattice of Figure 3(b) (where covers indicated by double lines are preserved).

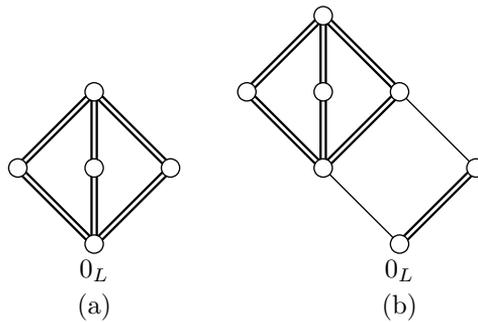


Figure 3

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