Srinivasa Rao Kola; Pratima Panigrahi
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NEARLY ANTIPODAL CHROMATIC NUMBER \(ac'(P_n)\) OF THE PATH \(P_n\)

Srinivasa Rao Kola, Pratima Panigrahi, Kharagpur

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Abstract. Chartrand et al. (2004) have given an upper bound for the nearly antipodal chromatic number \(ac'(P_n)\) as \((n-2)/2 + 2\) for \(n \geq 9\) and have found the exact value of \(ac'(P_n)\) for \(n = 5, 6, 7, 8\). Here we determine the exact values of \(ac'(P_n)\) for \(n \geq 8\). They are \(2p^2 - 6p + 8\) for \(n = 2p\) and \(2p^2 - 4p + 6\) for \(n = 2p + 1\). The exact value of the radio antipodal number \(ac(P_n)\) for the path \(P_n\) of order \(n\) has been determined by Khennoufa and Togni in 2005 as \(2p^2 - 2p + 3\) for \(n = 2p + 1\) and \(2p^2 - 4p + 5\) for \(n = 2p\). Although the value of \(ac(P_n)\) determined there is correct, we found a mistake in the proof of the lower bound when \(n = 2p\) (Theorem 6). However, we give an easy observation which proves this lower bound.

Keywords: radio \(k\)-coloring, span, radio \(k\)-chromatic number

MSC 2010: 05C78, 05C12, 05C15

1. Introduction

Let \(G\) be a connected graph. For any positive integer \(k\), a radio \(k\)-coloring of \(G\) is an assignment \(f\) of colors (positive integers) to the vertices of \(G\) such that \(|f(u) - f(v)| \geq 1 + k - d(u,v)\) for every two distinct vertices \(u, v\) of \(G\). The span \(rc_k(f)\) of a radio \(k\)-coloring \(f\) of \(G\) is the maximum color assigned to a vertex of \(G\). The radio \(k\)-chromatic number \(rc_k(G)\) of \(G\) is the minimum span of all radio \(k\)-colorings of \(G\). If \(rc_k(f) = rc_k(G)\) for some radio \(k\)-coloring \(f\) of \(G\) then we say that \(f\) is a minimal radio \(k\)-coloring. The radio 1-chromatic number \(rc_1(G)\) is then the chromatic number \(\chi(G)\). If diam\(G = d\), the radio \(d\)-coloring of \(G\) is referred to as the radio coloring of \(G\), and the radio \(d\)-chromatic number \(rc_d(G)\) is called the radio number of \(G\) that was introduced in [1]. The radio \((d-1)\)-coloring of \(G\) is referred to as the radio antipodal coloring of \(G\) and the radio \((d-1)\)-chromatic number \(ac(G)\) is called the radio antipodal chromatic number or simply the antipodal number of \(G\).
The radio \((d - 2)\)-coloring is known as the nearly antipodal coloring of \(G\) and the radio \((d - 2)\) coloring number is called the nearly antipodal chromatic number \(ac'(G)\) of \(G\). The radio \(k\)-coloring of a graph was defined by Chartrand et al. in [3], [4]. The radio antipodal coloring for graphs was first studied by Chartrand et al. [2], [3].

Khchikech et al. [7] have given the exact values of the radio \(k\)-chromatic number of \(P_n\) for \(k \geq n\), which are \((n - 1)k - \frac{1}{2}n(n - 2) + 1\) if \(n\) is even and \((n - 1)k - \frac{1}{2}(n - 1)^2 + 2\) if \(n\) is odd. Liu and Zhu [6] determined the exact value of the radio \((n - 1)\)-coloring number (radio number) as given below.

**Theorem 1.1** ([6]). For any integer \(n \geq 3\),

\[
rc_{(n-1)}(P_n) = \begin{cases} 
2p^2 + 3 & \text{if } n = 2p + 1, \\
2p^2 - 2p + 2 & \text{if } n = 2p.
\end{cases}
\]

Next we define the symbols \(\varepsilon_j, 2 \leq j \leq n\), which are used in the sequel.

**Definition 1.2.** For any radio \(k\)-coloring \(f\) of a path \(P_n\) and an ordering \(x_1, x_2, \ldots, x_n\) of vertices of \(P_n\) with \(f(x_i) \leq f(x_{i+1})\), \(1 \leq i \leq n - 1\), we define \(\varepsilon_j = (f(x_j) - f(x_{j-1})) - (1 + k - d(x_j, x_{j-1}))\), \(2 \leq j \leq n\). It is clear from the definition of a radio \(k\)-coloring that \(\varepsilon_j \geq 0, 2 \leq j \leq n\).

Khennoufa and Togni [5] determined the exact value of the radio antipodal number for the path \(P_n\) as given below.

**Theorem 1.3** ([5]). For any integer \(n \geq 5\),

\[
ac(P_n) = \begin{cases} 
2p^2 - 2p + 3 & \text{if } n = 2p + 1, \\
2p^2 - 4p + 5 & \text{if } n = 2p.
\end{cases}
\]

Although the value of \(ac(P_n)\) determined in [5] is correct, we found a mistake in the proof of the lower bound when \(n = 2p\) (Theorem 6 of [5]). In Theorem 6 of [5], the inequality \(rc_{2p-1}(P_{2p}) \leq ac(P_{2p}) + (2p - 1) - \varepsilon_3 - \varepsilon_{2p-1}\) is incorrect, because from Lemma 2 of [5] one gets \(rc_{2p-1}(f') \leq rc_{2p-2}(f) + (2p - 1) - \varepsilon_3 - \varepsilon_{2p-1}\), where \(f\) is a radio \(k\)-coloring of \(P_n\) and \(f'\) is a radio \(k'\)-coloring of \(P_n, k' \geq k\), with \(rc_{k'}(f') \leq rc_k(f) + (n - 1)(k' - k) - \sum_{i \in I} \min(k' - k, \varepsilon_i), I = \{i_1, i_2, \ldots, i_s\} \subset \{2, 3, \ldots, n\}\) such that \(i_{j+1} > i_j + 1, 1 \leq j \leq s - 1\). Since \(rc_{2p-1}(P_{2p}) \leq rc_{2p-1}(f')\), we get \(rc_{2p-1}(P_{2p}) \leq rc_{2p-2}(f) + (2p - 1) - \varepsilon_3 - \varepsilon_{2p-1}\). We have \(ac(P_{2p}) \leq rc_{2p-2}(f)\). The number \(rc_{2p-2}(f)\) cannot be replaced by \(ac(P_{2p})\) unless \(f\) is a minimal coloring. If we assume (or know) that \(f\) is a minimal coloring then there is no need of proving this theorem.

The observation below establishes the lower bound of \(ac(P_{2p})\) (Theorem 6 of [5]).
Observation 1.4. For \( n = 2p \), the lower bound of the antipodal number of \( P_n \) may be proved as follows. We know that \( rc_k(P_n) \geq rc_k(P_{n-1}) \) for any positive integer \( k \). So \( ac(P_{2p}) = rc_{n-2}(P_{n=2p}) \geq rc_{n-2}(P_{n-1=2(p-1)+1}) = 2(p-1)^2 + 3 = 2p^2 - 4p + 5 \) due to Liu and Zhu [6].

Chartrand et al. [4] have given an upper bound for the radio \( k \)-chromatic number when \( 1 \leq k \leq n - 3 \) as follows.

**Theorem 1.5 ([4]).** For \( 1 \leq k \leq n - 3 \),

\[
rc_k(P_n) \leq \begin{cases} 
\frac{1}{2}(k^2 + 2k + 1) & \text{if } k \text{ is odd,} \\
\frac{1}{2}(k^2 + 2k + 2) & \text{if } k \text{ is even.}
\end{cases}
\]

Chartrand et al. [4] have also given a lower bound for the same number. However, from Theorem 1.3, we get the following improved lower bound for \( rc_k(P_n), 1 \leq k \leq n - 3 \).

**Theorem 1.6 ([7]).** For \( 1 \leq k \leq n - 3 \),

\[
rc_k(P_n) \geq \begin{cases} 
\frac{1}{2}(k^2 + 5) & \text{if } k \text{ is odd,} \\
\frac{1}{2}(k^2 + 6) & \text{if } k \text{ is even.}
\end{cases}
\]

Chartrand et al. [4] have further improved the upper bound for the particular case \( k = n - 3 \) (i.e. for \( ac'(P_n) \)) as \( \binom{n-2}{2} + 2 \) for \( n \geq 9 \) and have given the exact values of \( ac'(P_n) \) for \( n = 5, 6, 7, 8 \) as \( 5, 7, 11, 16 \), respectively. In this paper we give the exact value of the nearly antipodal chromatic number of \( P_n \) for \( n \geq 8 \). And consequently, the lower bound of the radio \( k \)-chromatic number of \( P_n \) for \( 5 \leq k \leq n - 4 \) is also improved.

2. Upper bound

Now we give an upper bound for \( ac'(P_n) \) by defining radio \((n-3)\)-colorings of \( P_n \).

**Theorem 2.1.** For any integer \( n \geq 8 \),

\[
ac'(P_n) \leq \begin{cases} 
2p^2 - 6p + 8 & \text{if } n = 2p, \\
2p^2 - 4p + 6 & \text{if } n = 2p + 1.
\end{cases}
\]

**Proof.** Let \( P_n \) be the path \( a_1 a_2 \ldots a_n \).
Case I. In this case we take \( n = 2p \). Define a map \( f \) as
\[
\begin{align*}
  f(a_1) &= p - 1, \\
  f(a_{p-1-i}) &= 2p - 1 + i(2p - 3) \quad \text{for } 0 \leq i \leq p - 3, \\
  f(a_p) &= 2p^2 - 6p + 8, \\
  f(a_{p+1}) &= 1, \\
  f(a_{2p-1-j}) &= p + 1 + j(2p - 3) \quad \text{for } 0 \leq j \leq p - 3, \\
  f(a_{2p}) &= 2p^2 - 7p + 10.
\end{align*}
\]
Observe that \( a_p \) gets the maximum color, i.e. \( 2p^2 - 6p + 8 \). Here the distance condition is verified only for vertices of the form \( a_{p-1-i} \) and \( a_{2p-1-j} \), \( 0 \leq i, j \leq p - 3 \) as the other conditions can be checked easily.

For \( 0 \leq i, j \leq p - 3 \), \(|f(a_{2p-1-j}) - f(a_{p-1-i})| = |p + 1 + j(2p - 3) - (2p - 1 + i(2p - 3))| = |(j - i)(2p - 3) - (p - 2)| \geq p - 2 + (j - i) = 1 + (2p - 3) - (2p - 1 - j - (p - 1 - i)) \) if \( j \leq i \). For \( j - i = 1 \), \(|(j-i)(2p-3)-(p-2)| = p-1 = p-2+(j-i) = 1+(n-3)-(2p-1-j-(p-1-i)) \).

For \( j - i \geq 2 \), \(|(j-i)(2p-3)-(p-2)| \geq 3p-4 > 2p-5 \geq p-2 + (j - i) = 1 + (n - 3) - (2p - 1 - j - (p - 1 - i)) \).

Therefore the above mapping \( f \) is a radio \((n-3)\)-coloring and \( ac'(P_n) \leq 2p^2 - 6p + 8 \) for \( n = 2p \).

Case II. In this case we take \( n = 2p + 1 \). Define a map \( g \) as
\[
\begin{align*}
  g(a_1) &= p + 1, \\
  g(a_2) &= 2p^2 - 5p + 7, \\
  g(a_{3+i}) &= 3p + 1 + i(2p - 1), \ 0 \leq i \leq p - 4, \\
  g(a_{p+j}) &= 1 + j(2p - 1), \ 0 \leq j \leq 1, \\
  g(a_{p+2}) &= 2p^2 - 4p + 6, \\
  g(a_{p+3+l}) &= 4p + l(2p - 1), \ 0 \leq l \leq p - 4, \\
  g(a_{2p+m}) &= p + m(2p - 1), \ 0 \leq m \leq 1.
\end{align*}
\]
Observe that \( a_{p+2} \) gets the maximum color, i.e. \( 2p^2 - 4p + 6 \). Here the distance condition is verified only for vertices of the form \( a_{3+i} \) and \( a_{p+3+l} \), \( 0 \leq i, l \leq p - 4 \) as the other cases can be checked easily. For \( 0 \leq i, l \leq p - 4 \), \(|g(a_{p+3+l}) - g(a_{3+i})| = |4p + l(2p - 1) - (3p + 1 + i(2p - 1))| = |(l - i)(2p - 1) + p - 1| \geq p - 1 - (l - i) = 1 + (2p - 2) - (p + 3 + l - (3 + i)) = 1 + (n - 3) - (p + 3 + l - (3 + i)) \) if \( l \geq i \). For \( i - l = 1 \), \(|(l-i)(2p-1)+p-1|=p-1-(l-i)=1+(n-3)-(p+3+l-(3+i)) \). For \( i - l \geq 2 \), \(|(l-i)(2p-1)+p-1| \geq 3p-4 > 2p-5 \geq p-1 - (l - i) = 1 + (n - 3) - (p + 3 + l - (3 + i)) \).

Therefore the above mapping \( g \) is a radio \((n-3)\)-coloring and \( ac'(P_n) \leq 2p^2 - 4p + 6 \) for \( n = 2p + 1 \).
Example 2.2. Here we illustrate Theorem 2.1 by giving an example of nearly antipodal colorings of $P_n$ for $n = 8, 11, 12$.

\[
\begin{array}{cccccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\
  3 & 12 & 7 & 16 & 1 & 10 & 5 & 14 \\
  a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\
  6 & 32 & 16 & 25 & 1 & 10 & 36 & 29 \\
  a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\
  5 & 38 & 29 & 20 & 11 & 44 & 1 & 34 \\
\end{array}
\]

3. Lower bound

Here we first give a result of Liu and Zhu [6] and also a detailed proof of this which will be useful in deriving some more results in the sequel.

Theorem 3.1 [6]. For any radio $k$-coloring $f$ of a path $P_n$: $a_1 a_2 a_3 \ldots a_{n-1} a_n$,

\[
\sum_{i=2}^{n} d(x_i, x_{i-1}) \leq \begin{cases} 
2p^2 - 1 & \text{if } n = 2p, \\
2p^2 + 2p - 1 & \text{if } n = 2p + 1,
\end{cases}
\]

where $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ is an ordering of vertices of $P_n$ such that $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n - 1$.

Proof. Let $x_i = a_{\sigma(i)}$, $1 \leq i \leq n$. Then $\sigma$ is a permutation of $\{1, 2, 3, \ldots, n\}$. Note that $d(x_i, x_{i-1})$ is equal to either $\sigma(i) - \sigma(i - 1)$ or $\sigma(i - 1) - \sigma(i)$, whichever is positive. Replacing each term $d(x_i, x_{i-1})$ in $\sum_{i=2}^{n} d(x_i, x_{i-1})$ by the corresponding $\sigma(i) - \sigma(i - 1)$ or $\sigma(i - 1) - \sigma(i)$, whichever is positive, we obtain a summation whose entries are $\pm j$ for $j \in \{1, 2, 3, \ldots, n\}$. Altogether there are $2(n-1)$ terms of the form $\pm j$ in the summation $\sum_{i=2}^{n} d(x_i, x_{i-1})$, half of them positive and half negative. To maximize the summation $\sum_{i=2}^{n} d(x_i, x_{i-1})$, one needs to minimize the absolute values for negative terms while maximizing the values of positive terms. It is easy to verify that the following are the only possibilities achieving the maximum summation.

For $n = 2p$: In the summation $\sum_{i=2}^{n} d(x_i, x_{i-1})$ each of $\{1, 2, 3, \ldots, p - 1\}$ occurs twice with a negative sign, each of $\{p+2, p+3, \ldots, 2p\}$ occurs twice with a positive sign, $p$ occurs once as negative and $p + 1$ occurs once as positive. Since $p$ and $p + 1$ occur only once in the summation, we get $\{f(x_1), f(x_{2p})\} = \{f(a_p), f(a_{p+1})\}$.

For $n = 2p + 1$:
(i) Each of the numbers \( \{ p + 2, p + 3, p + 4, \ldots, 2p + 1 \} \) occurs twice with a positive sign, each of \( \{ 1, 2, 3, \ldots, p - 1 \} \) occurs twice with a negative sign, and each of \( p \) and \( p + 1 \) occurs once as negative. Since both \( p \) and \( p + 1 \) occur only once in the summation, we get \( \{ f(x_1), f(x_{2p}) \} = \{ f(a_p), f(a_{p+1}) \} \).

(ii) Each of the numbers \( \{ p + 3, p + 4, \ldots, 2p + 1 \} \) occurs twice with a positive sign, each of \( \{ 1, 2, 3, \ldots, p \} \) occurs twice with a negative sign, and each of \( p + 1 \) and \( p + 2 \) occurs once as positive. Since both \( p \) and \( p + 2 \) occur only once in the summation, we get \( \{ f(x_1), f(x_{2p}) \} = \{ f(a_{p+1}), f(a_{p+2}) \} \).

Next we give the possibilities of having the distance summation \( 2p^2 - 2 \) and \( 2p^2 - 3 \) when \( n = 2p \) is any radio \( k \)-coloring of \( P_n \).

**Lemma 3.2.** For any radio \( k \)-coloring of \( P_n \), the following are the only possibilities for the sum \( \sum_{i=2}^{n} d(x_i, x_{i-1}) \) to be \( 2p^2 - 2 \) when \( n = 2p \).

(i) In the summation each of \( \{ 1, 2, 3, \ldots, p - 1 \} \) occurs twice with a negative sign, each of \( \{ p + 1, p + 3, \ldots, 2p \} \) occurs twice with a positive sign, \( p \) occurs once as negative and \( p + 2 \) occurs once as positive. Since both \( p \) and \( p + 2 \) occur only once in the summation, we get \( \{ f(x_1), f(x_{2p}) \} = \{ f(a_p), f(a_{p+2}) \} \).

(ii) In the summation each of \( \{ 1, 2, 3, \ldots, p - 2, p \} \) occurs twice with a negative sign, each of \( \{ p + 2, p + 3, \ldots, 2p \} \) occurs twice with a positive sign, \( p - 1 \) occurs once as negative and \( p + 1 \) occurs once as positive. Since both \( p - 1 \) and \( p + 1 \) occur only once in the summation, we get \( \{ f(x_1), f(x_{2p}) \} = \{ f(a_{p-1}), f(a_{p+1}) \} \).

**Proof.** Follows from the proof of Theorem 3.1.

**Lemma 3.3.** For any radio \( k \)-coloring of \( P_n \), the following are the only possibilities for the sum \( \sum_{i=2}^{n} d(x_i, x_{i-1}) \) to be \( 2p^2 - 3 \) when \( n = 2p \).

(i) In the summation each of \( \{ 1, 2, 3, \ldots, p - 1 \} \) occurs twice with a negative sign, each of \( \{ p + 1, p + 2, p + 4, \ldots, 2p \} \) occurs twice with a positive sign, \( p \) occurs once as negative and \( p + 3 \) occurs once as positive. Since both \( p \) and \( p + 3 \) occur only once in the summation, we get \( \{ f(x_1), f(x_{2p}) \} = \{ f(a_p), f(a_{p+3}) \} \).

(ii) In the summation each of \( \{ 1, 2, 3, \ldots, p - 3, p - 1, p \} \) occurs twice with a negative sign, each of \( \{ p + 2, p + 3, \ldots, 2p \} \) occurs twice with a positive sign, \( p - 2 \) occurs once as negative and \( p + 1 \) occurs once as positive. Since both \( p - 2 \) and \( p + 1 \) occur only once in the summation, we get \( \{ f(x_1), f(x_{2p}) \} = \{ f(a_{p-2}), f(a_{p+1}) \} \).

(iii) In the summation each of \( \{ 1, 2, 3, \ldots, p - 2, p \} \) occurs twice with a negative sign, each of \( \{ p + 1, p + 3, \ldots, 2p \} \) occurs twice with a positive sign, \( p - 1 \) occurs once as negative and \( p + 2 \) occurs once as positive. Since both \( p - 1 \) and \( p + 2 \) occur only once in the summation, we get \( \{ f(x_1), f(x_{2p}) \} = \{ f(a_{p-1}), f(a_{p+2}) \} \).
(iv) In the summation each of \( \{1 + 2 + \ldots + p - 2\} \) occurs twice with a negative sign, each of \( \{p + 2 + p + 3 + \ldots + 2p\} \) occurs twice with a positive sign, \( p \) and \( p - 1 \) occur once with negative sign and \( p + 1 \) occurs as \(+ (p + 1)\) and \(-(p + 1)\). Since both \( p \) and \( p - 1 \) occur only once in the summation, we get
\[
\{f(x_1), f(x_{2p})\} = \{f(a_p), f(a_{p - 1})\}.
\]

**Proof.** Follows from the proof of Theorem 3.1.

**Lemma 3.4.** Let \( k \) be an even integer and \( n = k + 2 = 2p, n \geq 8 \). Let \( f \) be a minimal radio \( k \)-coloring of \( P_n \) and let \( x_1, x_2, x_3, \ldots, x_n \) be as in Definition 1.2. Then \( \sum_{i=2}^{n} d(x_i, x_{i+1}) = 2p^2 - 1 \) and \( \sum_{i=2}^{n} \varepsilon_i = 2 \).

**Proof.** For any path \( P_m \) and a radio \( k \)-coloring \( g \) we have
\[
g(y_m) = (m - 1)(1 + k) - \sum_{i=2}^{m} d(y_i, y_{i-1}) + \sum_{i=2}^{m} \varepsilon_i + 1
\]
where \( y_1, y_2, y_3, \ldots, y_m \) is an ordering of vertices of \( P_m \) such that \( g(y_i) \leq g(y_{i+1}) \), \( 1 \leq i \leq m \). Equation (1) is true because \( g(y_m) = \sum_{i=2}^{m} (g(y_i) - g(y_{i-1})) + g(y_1) = \sum_{i=2}^{m} (k + 1 - d(y_i, y_{i-1}) + \varepsilon_i) + 1 = (m - 1)(1 + k) - \sum_{i=2}^{m} d(y_i, y_{i-1}) + \sum_{i=2}^{m} \varepsilon_i + 1 \).

Since \( f \) is a minimal radio \( k \)-coloring of \( P_{k+2} \), \( f(x_n) = ac(P_n) = 2p^2 - 4p + 5 \) by Theorem 1.3. By Theorem 3.1, the maximum possible value of \( \sum_{i=2}^{n} d(x_i, x_{i-1}) \) is \( 2p^2 - 1 \). Then from equation (1) we have the following possibilities:

(I) \( \sum_{i=2}^{k+2} d(x_i, x_{i+1}) = 2p^2 - 1 \) and \( \sum_{i=2}^{k+2} \varepsilon_i = 2 \).

(II) \( \sum_{i=2}^{k+2} d(x_i, x_{i+1}) = 2p^2 - 2 \) and \( \sum_{i=2}^{k+2} \varepsilon_i = 1 \).

(III) \( \sum_{i=2}^{k+2} d(x_i, x_{i+1}) = 2p^2 - 3 \) and \( \sum_{i=2}^{k+2} \varepsilon_i = 0 \).

Next we prove that (II) and (III) cannot occur. Let \( P_n : a_1 a_2 \ldots a_p a_{p+1} \ldots a_{2p-1} a_{2p} \). Consider the coloring \( f \) of the first \( 2p - 1 \) vertices of \( P_n \). Let \( a_{2p} = x_j \) and \( \varepsilon = (f(x_{j+1}) - f(x_j)) - (1 + k - d(x_{j-1}, x_{j+1})) \). Liu and Zhu [6] have given a minimal radio \( k \)-coloring of \( P_{2(p-1)+1} = P_{2p-1} = P_{k+2-1} = P_{k+1} \) whose summation of distances is equal to \( 2(p-1)^2 + 2(p-1) - 2 \) and summation of \( \varepsilon_i \)’s is equal to 0 with the span \( rc_k(P_{k+1}) = 2(p-1)^2 + 3 = 2p^2 - 4p + 5 = rc_k(f) \). So, from Theorem 3.1, we have the following two possibilities for the coloring \( f \) of \( P_n \setminus \{a_{2p}\} \).

(i) \( \left( \sum_{i=2}^{k+2} d(x_i, x_{i-1}) \right) - (d(x_j, x_{j-1}) + d(x_j, x_{j+1})) - d(x_{j-1}, x_{j+1}) = 2(p - 1)^2 + 2(p - 1) - 1 \) and \( \left( \sum_{i=2}^{k+2} \varepsilon_i \right) - (\varepsilon_j + \varepsilon_{j+1} - \varepsilon) = 1 \);
(ii) \[ \left( \sum_{i=2}^{k+2} d(x_i, x_{i-1}) \right) - \left( d(x_j, x_{j-1}) + d(x_j, x_{j+1}) \right) - d(x_{j-1}, x_{j+1}) = 2(p-1)^2 + 2(p-1) - 2 \] and \[ \left( \sum_{i=2}^{k+2} \varepsilon_i \right) - (\varepsilon_j + \varepsilon_{j+1} - \varepsilon) = 0. \]

Suppose \( f \) satisfies case II. Here \( \sum_{i=2}^{k+2} d(x_i, x_{i-1}) = 2p^2 - 2 \). Then the coloring \( f \) of the vertices \( P_n \setminus \{a_{2p}\} \) is either of type (i) or of type (ii). Suppose it is of type (i). Then \( d(x_j, x_{j+1}) + d(x_j, x_{j-1}) - d(x_{j-1}, x_{j+1}) = 2p - 1 \). Let \( x_j = a_r, x_{j+1} = a_s, 1 \leq r, s < 2p \). Since \( x_j = a_{2p} \), we have \( d(x_j, x_{j+1}) + d(x_j, x_{j-1}) - d(x_{j-1}, x_{j+1}) = 2(2p-r) \) or \( 2(2p-s) = 2d(x_j, x_{j+1}) \) or \( 2d(x_j, x_{j-1}) = \frac{1}{2}(2p-1) \) which is not possible because \( \frac{1}{2}(2p-1) \) is not an integer. Suppose the coloring \( f \) of the vertices \( P_n \setminus \{a_{2p}\} \) is of type (ii). Then \( d(x_j, x_{j+1}) + d(x_j, x_{j-1}) - d(x_{j-1}, x_{j+1}) = 2p \). Similarly to the above, we get \( d(x_j, x_{j-1}) \) or \( d(x_j, x_{j+1}) = p \). So \( x_{j-1} \) or \( x_{j+1} = a_p \). Since \( \sum_{i=2}^{k+2} \varepsilon_i = 1 \), we have \( \varepsilon_j + \varepsilon_{j+1} - \varepsilon = 1 \).

Since \( \varepsilon_j \) and \( \varepsilon_{j+1} \) appear in the summation \( \sum_{i=2}^{k+2} \varepsilon_i \), we get \( \varepsilon_j + \varepsilon_{j+1} = 1 \). Similarly, if we consider the coloring \( f \) of the vertices \( P_n \setminus \{a_1 = x_l \text{ (say)}\} \), then \( x_{l-1} \) or \( x_{l+1} = a_{p+1} \) and \( \varepsilon_l + \varepsilon_{l+1} = 1 \). As \( \sum_{i=2}^{k+2} \varepsilon_i = 1 \), we have \( l = j + 1 \) and \( \varepsilon_l = \varepsilon_{j+1} = 1 \) or \( j = l + 1 \) and \( \varepsilon_j = \varepsilon_{l+1} = 1 \). That is, \( |f(a_1) - f(a_{2p})| = 1 + k - d(a_1, a_{2p}) + 1 = 1 \) \( \Rightarrow |f(a_p) - f(a_{p+1})| = 1 \), which is not possible because \( k \geq 6 \).

Suppose \( f \) satisfies case III. Here \( \sum_{i=2}^{k+2} d(x_i, x_{i-1}) = 2p^2 - 3 \). By Lemma 3.3, \( \{f(a_{p-2}), f(a_{p+1})\} = \{0, 2p^2 - 4p + 5\} \) or \( \{f(a_p), f(a_{p+3})\} = \{0, 2p^2 - 4p + 5\} \) or \( \{f(a_{p-1}), f(a_{p+2})\} \) or \( \{f(a_p), f(a_{p-1})\} = \{0, 2p^2 - 4p + 5\} \). The coloring of \( P_n \setminus \{a_1\} \) is either of type (i) or of type (ii). Suppose it is of type (i). That is, the distance summation is \( 2(p-1)^2 + 2(p-1) - 1 \). So by Theorem 3.1, either of the sets \( \{f(a_p), f(a_{p+1})\}, \{f(a_{p+1}), f(a_{p+2})\} \) is equal to \( \{0, 2p^2 - 4p + 5\} \), which is not possible because to repeat the color 0 (or \( 2p^2 - 4p + 5 \)) its distance from the previous 0 (or \( 2p^2 - 4p + 5 \)) color must be at least \( 2p - 1 \). Suppose the coloring \( P_n \setminus \{a_1\} \) is of type (ii). Then \( d(x_l, x_{l-1}) + d(x_l, x_{l+1}) - d(x_{l-1}, x_{l+1}) = 2p - 1 \Rightarrow 2d(x_l, x_{l-1}) \) or \( d(x_l, x_{l+1}) = 2p - 1 \Rightarrow d(x_l, x_{l-1}) \) or \( d(x_l, x_{l+1}) = \frac{1}{2}(2p - 1) \), which is not possible.

**Theorem 3.5.** For any integer \( n \geq 8 \),

\[
ac'(P_n) \geq \begin{cases} 
2p^2 - 6p + 8 & \text{if } n = 2p, \\
2p^2 - 4p + 6 & \text{if } n = 2p + 1.
\end{cases}
\]

**Proof.** Case I. \( k \) odd and \( n = k + 3 = 2p \).

Let \( f \) be any radio \( k \)-coloring of the path \( P_n: a_1 a_2 a_3 \ldots a_{n-1} a_{n-2p} \) with span.
2p² - 6p + 7. Let \( x_1, x_2, x_3, \ldots, x_{n-1}, x_n \) be the ordering of the vertices such that \( f(x_i) \leq f(x_{i+1}) \), \( 1 \leq i \leq n - 1 \). Consider the coloring \( f \) of the first \( 2(p-1) \) vertices of \( P_n \). Liu and Zhu [6] have given a minimal radio \( k \)-coloring of \( P_{k+1=2(p-1)} \) with

\[
\sum_{i=2}^{2(p-1)} d(x_i, x_{i-1}) = 2(p-1)^2 - 1 \quad \text{and} \quad \sum_{i=2}^{2(p-1)} \varepsilon_i = 0 \text{ (where the span is } 2(p-1)^2 - 2(p-1) + 2 = 2p^2 - 6p + 6). \]

If the span of the coloring \( f \) of \( P_n \setminus \{a_{2p-1}, a_{2p}\} \) is \( 2p^2 - 6p + 7 \), then from Theorem 3.1 we have the following two possibilities for the coloring \( f \) of \( P_n \setminus \{a_{2p-1}, a_{2p}\} \):

(i) \[
\sum_{i=2}^{2(p-1)} d(x_i, x_{i-1}) = 2(p-1)^2 - 1 \quad \text{and} \quad \sum_{i=2}^{2(p-1)} \varepsilon_i = 1,
\]

(ii) \[
\sum_{i=2}^{2(p-1)} d(x_i, x_{i-1}) = 2(p-1)^2 - 2 \quad \text{and} \quad \sum_{i=2}^{2(p-1)} \varepsilon_i = 0.
\]

Then by Theorem 3.1 and Lemma 3.2,

\[
\{f(a_{p-1}), f(a_p)\} \text{ or } \{f(a_{p-1}), f(a_{p+1})\} \text{ or } \{f(a_{p-2}), f(a_{p-1})\} = \{0, 2p^2 - 6p + 7\}.
\]

Similarly, if we consider the coloring \( f \) of \( P_n \setminus \{a_1, a_2\} \) and if the span of the coloring of \( P_n \setminus \{a_1, a_2\} \) is \( 2p^2 - 6p + 7 \), we get

\[
\{f(a_{p+1}), f(a_{p+2})\} \text{ or } \{f(a_{p+2}), f(a_{p+3})\} \text{ or } \{f(a_{p}), f(a_{p+2})\} = \{0, 2p^2 - 6p + 7\}.
\]

Equations (2) and (3) lead to a contradiction because the color 0 or \( 2p^2 - 6p + 7 \) can be repeated only if its distance is at least \( 2p - 2 \). Suppose the span of the coloring of \( P_n \setminus \{a_1, a_2\} \) is \( 2p^2 - 6p + 6 \) and the span of the coloring of \( P_n \setminus \{a_{2p-1}, a_{2p}\} \) is \( 2p^2 - 6p + 7 \). If the span of the coloring of \( P_n \setminus \{a_1, a_2\} \) is \( 2p^2 - 6p + 6 \), then \( \{f(a_{p+1}), f(a_{p+2})\} = \{0, 2p^2 - 6p + 6\} \). If the span of the coloring of \( P_n \setminus \{a_{2p-1}, a_{2p}\} \) is \( 2p^2 - 6p + 7 \), then \( \{f(a_{p-1}), f(a_p)\} \text{ or } \{f(a_{p-1}), f(a_{p+1})\} \text{ or } \{f(a_{p-2}), f(a_{p-1})\} = \{0, 2p^2 - 6p + 7\} \), which is not possible as the colors \( 2p^2 - 6p + 6 \) and \( 2p^2 - 6p + 7 \) are at least \( 2p - 3 \) apart and the color 0 can be repeated if its distance from the previous 0 color is at least \( 2p - 2 \). Similarly we get a contradiction when the span of the coloring of \( P_n \setminus \{a_1, a_2\} \) is \( 2p^2 - 6p + 7 \) and the span of the coloring of \( P_n \setminus \{a_{2p-1}, a_{2p}\} \) is \( 2p^2 - 6p + 6 \).

Case II. \( k \) even and \( n = k + 3 = 2p + 1 \).

Let \( f \) be any radio \( k \)-coloring of the path \( P_n \): \( a_1 a_2 a_3 \ldots a_{n-1} a_{n=2p+1} \) with span \( 2p^2 - 4p + 5 \). Let \( x_1, x_2, x_3, \ldots, x_{n-1}, x_n \) be the ordering of the vertices such that \( f(x_i) \leq f(x_{i+1}) \), \( 1 \leq i \leq n - 1 \). Consider the coloring of the first \( 2p \) vertices of \( P_n \). Then by Lemma 3.4, this coloring has the sum of distances equal to \( 2p^2 - 1 \) and the sum of \( \varepsilon_i \)'s is equal to 2. So by Theorem 3.1 we have \( \{f(a_p), f(a_{p+1})\} = \{0, 2p^2 - 6p + 7\} \).
{0, 2p^2 - 4p + 5}. Similarly, if we consider the coloring of \( P_n \setminus \{a_1\} \) then we get \( \{f(a_{p+1}), f(a_{p+2})\} = \{0, 2p^2 - 4p + 5\} \). This is a contradiction.

From Theorem 2.1 and Theorem 3.5 we summarize the main result of this paper.

**Theorem 3.6.** For any integer \( n \geq 8 \),

\[
ac'(P_n) = \begin{cases} 
2p^2 - 6p + 8 & \text{if } n = 2p, \\
2p^2 - 4p + 6 & \text{if } n = 2p + 1.
\end{cases}
\]

**Observation 3.7.** For any positive integers \( m \) and \( n \) with \( m < n \) one gets that \( rc_k(P_m) \leq rc_k(P_n) \). Therefore by Theorem 3.6 we have for \( n = k + i, \ i \geq 4 \), that \( rc_{n-i}(P_{(n-i)+3}) \) is \( 2p^2 - 6p + 8 \) for \( (n-i)+3 = 2p \) and \( 2p^2 - 4p + 6 \) for \( (n-i)+3 = 2p+1 \). Since \( (n-i)+3 < n \), we get the following lower bound for \( rc_k(P_n) \) with \( n = k + i, \ i \geq 4 \):

\[
rc_k(P_n) \geq \begin{cases} 
2p^2 - 6p + 8 & \text{if } n = 2p - 3 + i, \\
2p^2 - 4p + 6 & \text{if } n = 2p - 2 + i.
\end{cases}
\]

**References**


Authors’ addresses: Srinivasa Rao Kola, Pratima Panigrahi, Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, India, e-mail: srinivas@maths.iitkgp.ernet.in, pratima@maths.iitkgp.ernet.in.