Ilie Barza; Dorin Ghisa
Blaschke product generated covering surfaces


Persistent URL: [http://dml.cz/dmlcz/140652](http://dml.cz/dmlcz/140652)

**Terms of use:**

© Institute of Mathematics AS CR, 2009

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*
Abstract. It is known that, under very general conditions, Blaschke products generate branched covering surfaces of the Riemann sphere. We are presenting here a method of finding fundamental domains of such coverings and we are studying the corresponding groups of covering transformations.

Keywords: Blaschke product, covering surface, covering transformation, fundamental domain, Cantor set

MSC 2010: 30D50, 14H30

1. Introduction

Every finite or infinite sequence \((a_k), a_k \in D := \{z \in \mathbb{C}; |z| < 1\}\) defines a Blaschke product

\[ w = B(z) = \prod_{k=1}^{n \leq \infty} b_k(z), \]

where

\[ b_k(z) = \frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z}. \]

It is known (see for example [8]) that if \(n = \infty\) then the condition \(\sum_{n=1}^{\infty} (1-|a_n|) < \infty\) is sufficient for the product (1) to converge uniformly on compact subsets of \(\mathbb{C} \setminus (E \cup A)\), where \(E\) is the set of cluster points of zeros \(a_k\) of \(B\), \(E \subset \partial D\), and \(A = \{z \in \mathbb{C}; z = 1/\bar{a}_k, k = 1, 2, 3, \ldots\}\). We have studied in [3] and [4] Blaschke products for which \(E\) is a generalized Cantor set (see [7]). The word generalized used here has the meaning
that we allow more liberty of choice for the open arcs to be removed. In particular we allow some of them to have common ends and therefore the corresponding closed interval between them to be a unique point. In other words, \( E \) is allowed to have isolated points. We will also allow \( E \) to have only isolated points or to be the empty set. Obviously, in this last case \( B \) is a finite Blaschke product. The statements which follow in the subsequent sections do not depend on what type of generalized Cantor set is \( E \).

We have shown (see [3] and [4]) that for a Blaschke product whose cluster set \( E \) of zeros is a generalized Cantor set (for more about this concept, see for example [6]), there is a partition of \( W = \hat{\mathbb{C}} \setminus E \) into regions (fundamental “domains”) \( \Omega_k \) which are mapped continuously and bijectively by \( B \) onto \( \hat{\mathbb{C}} \). The mapping is conformal in the interior of every \( \Omega_k \). The local injectivity of \( B \) is violated just on a set of points (branch points) which is finite in the case of a finite Blaschke product, and which has its cluster points in \( E \) in the case of an infinite Blaschke product. These properties define \((W, B)\) as a (branched) covering surface of \( \hat{\mathbb{C}} \) (see [2]). The regions \( \Omega_k \) accumulate at every point \( e^{i\alpha} \in E \).

For a Blaschke product of order \( n \), there are exactly \( n \) regions \( \Omega_k \). When \( B(z) = \frac{\bar{a}/|a| \cdot (a - z)/(1 - \bar{a}z)}{|a|^2 - \omega_k} \), these are regions bounded by the arcs

\[
\{ z = z_k(\lambda); \quad z_k(\lambda) = \frac{\omega_k \lambda - r}{\omega_k \lambda r - 1} e^{i\theta}, \quad \lambda \geq 0 \}, \quad k = 0, 1, 2, \ldots, n - 1,
\]

where \( a = re^{i\theta} \) and \( \omega_k \) are the roots of order \( n \) of unity.

Since \( z_k(\lambda) \) are Moebius transformations and \( \lambda \) varies in an interval, these regions are bounded by arcs of a circle or by lines, as can be seen in Figures 1 and 2, where \( a = 0.5 \) and \( a = 0.8 \) respectively. We notice how some of the regions get smaller as \( a \) becomes closer to the unit circle.

Moreover, we have found that the invariants of \( B \), i.e. the mappings \( U_k: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) with the property that \( B \circ U_k = B \), are Moebius transformations of the form

\[
U_k(z) = \frac{a(1 - \omega_k) - (|a|^2 - \omega_k)z}{1 - |a|^2 \omega_k - \bar{a}(1 - \omega_k)z}.
\]

They form a cyclic group of order \( n \) with respect to composition, where \( U_k \circ U_{k'} = U_{k+k' \mod n} \). In particular, \( U_k \circ U_{n-k} = U_0 \), where \( U_0(z) = z \). In this case \( U_k \) are cyclically permuting the regions \( \Omega_k \) in the sense that every \( \Omega_{k'} \) is mapped conformally onto an \( \Omega_{k+k' \mod n} \). Indeed, it can be easily checked that

\[
U_k(z_k(\lambda)) = z_{k+k' \mod n}(\lambda) \quad \text{for every} \quad k, k' \in \{0, 1, \ldots, n - 1\}.
\]

This means that for every \( k' \in \{0, 1, \ldots, n - 1\} \) the arcs \( \{ z; \quad z = z_k(\lambda), \lambda \geq 0 \} \) and \( \{ z; \quad z = z_{k'+1 \mod n}(\lambda), \lambda \geq 0 \} \) are mapped bijectively by \( U_k \) onto the arcs.
\{z; \ z = z_{k+k'} (\text{mod } n)(\lambda), \lambda \geq 0\} \text{ and } \{z; \ z = z_{k+k'+1} (\text{mod } n)(\lambda), \lambda \geq 0\} \text{ respectively.}

Then, by the conformal correspondence theorem (see [8], page 154), the domain \(\Omega_{k'}\) bounded by the first two arcs is mapped by \(U_k\) conformally onto the domain \(\Omega_{k+k'} (\text{mod } n)\) bounded by the other two arcs.

We can say even more, namely that all \(U_k\) have the fixed points \(\alpha\) and \(1/\bar{\alpha}\). Moreover, for every \(z \in \hat{\mathbb{C}}\) and every \(k = 0, 1, 2, \ldots, n-1\), we have \(U_k(1/\bar{z}) = 1/\bar{U}_k(z)\); in particular, all \(U_k\) map \(\partial D\) onto \(\partial D\). As a consequence, every \(z \in \hat{\mathbb{C}}\) has exactly \(n\) pre-images by \(B\), if we consider that \(\alpha\) and \(1/\bar{\alpha}\) are values taken with multiplicity \(n\) (indeed, they belong to every one of the \(n\) fundamental domains \(\Omega_k\)).

We were trying to draw similar conclusions for Blaschke products of a more general form.

The technique of simultaneous continuation, which has been described in [3], allowed us to prove the existence of fundamental domains \(\Omega_k\) for any Blaschke product whose cluster set of zeros \(E\) is a generalized Cantor set.

Let us repeat here the main features of the technique. If \(B\) is a Blaschke product of degree \(n\), then there are exactly \(n\) distinct solutions \(e^{i\alpha_k}\) of the equation \(B(z) = 1\). They determine a partition of \(\partial D\) into \(n\) half-open arcs \(\Gamma_k\). Let \(b_1, b_2, \ldots, b_q\) be the solutions of the equation \(B'(z) = 0\) situated in \(D\) and let \(w_j = B(b_j), j = 1, 2, \ldots, q\). We might have \(w_j = w_{j'}\) even if \(b_j \neq b_{j'}\). In particular, \(w_j = 0\) for all multiple zeros \(a_j\) of \(B\), for which we have obviously \(B'(a_j) = 0\).

We connect \(w = 1\) and all the points \(w_j\) by a polygonal line \(\eta\) with no self-intersection, then we perform continuations \(\gamma_k\) over \(\eta\) from every point \(e^{i\alpha_k}\). The arcs \(\gamma_k\) and \(\Gamma_k\) determine a partition of \(D\) into sets \(A_k\) whose interiors are mapped conformally by \(B\) onto slit unit discs. The fundamental domains \(\Omega_k\) are \(A_k \cup \mathring{A}_k\), where \(\mathring{A}_k\) is symmetric to \(A_k\) with respect to the unit circle.
The case of an infinite Blaschke product is reduced to the finite case by using an exhaustion sequence of \( \mathbb{C} \setminus E \).

Figure 3 above exhibits computer generated fundamental domains for the Blaschke product defined by the triple zero \( 0.4 + 0.3i \) and the double zero \( 0.5 - 0.6i \). In Figure 4 we added the triple zero \( 0.9 - 0.2i \) in order to show that, due to its closeness to the unit circle, the addition of this zero affects in a visible way only the domain containing it. This and the previous examples might contribute to a better understanding of the geometry of infinite Blaschke product mappings.

2. INVARANTS OF \( B \) ON \( \partial D \)

We introduce the notation needed in order to account for all the removed open arcs. At every stage \( m \) we remove \( 2^{m-1} \) open arcs.

Let us put all these arcs in a sequence

\[
I_n = \{ z = e^{i\theta}; \ \theta_n < \theta < \theta'_n \}, \quad n = 1, 2, \ldots
\]

and set

\[
E = \partial D \setminus \bigcup_{n=1}^\infty I_n.
\]

We have also

\[
\partial D \setminus E = \bigcup_{n=1}^\infty I_n.
\]
For every such removed open arc there are infinitely many disjoint half-open sub-arcs (see [3]):

$$\Gamma_{nj} = \{ z = e^{i\theta}; \alpha_{nj} \leq \theta < \alpha_{n,j+1} \}, \ j \in \mathbb{Z}, \ \lim_{j \to +\infty} \alpha_{nj} = \theta'_n, \ \lim_{j \to -\infty} \alpha_{nj} = \theta_n.$$  

The sub-arcs $\Gamma_{nj}$ are mapped by $B$ continuously and bijectively onto the unit circle in the $w$-plane with $B(e^{i\alpha_{nj}}) = 1$.

If $B$ is a finite Blaschke product of order $n$, then $E = \emptyset$ and $\partial D$ is the disjoint union of exactly $n$ arcs $\Gamma_k$ which are mapped by $B$ continuously and bijectively onto $\partial D$ (see [5]). Let us denote by $\Psi_{nj}$ the inverse mappings of $B|\Gamma_{nj}$ and associate with every bijection $\chi: \mathbb{N}^* \times \mathbb{Z} \to \mathbb{N}^* \times \mathbb{Z}$ a mapping $U\chi: \partial D \setminus E \to \partial D \setminus E$ defined in the following way. If $(n', j') = \chi(n, j)$, then for every $\Gamma_{nj}$ we set

$$U\chi|\Gamma_{nj} = \Psi^{-1}_{n',j'} \circ \Psi_{nj}|\Gamma_{nj}, \tag{2.1}$$

It can be easily checked that for every $\Gamma_{nj}$,

$$B \circ U\chi|\Gamma_{nj}(e^{i\theta}) = B(\Psi_{n',j'}^{-1}(\Psi_{nj}(e^{i\theta}))) = \Psi_{nj}^{-1}(e^{i\theta}) = B(e^{i\theta}). \tag{2.2}$$

Therefore

$$B \circ U\chi = B \text{ on } \partial D \setminus E. \tag{2.3}$$

We notice that $U\chi$ are continuous functions in every $\Gamma_{nj}$, but they may fail to be continuous at the points $e^{i\alpha_{nj}}$ if the images by $U\chi$ of $\Gamma_{nj-1}$ and $\Gamma_{nj}$ are not adjacent arcs. On the other hand, all $U\chi_k$ with $\chi_k$ of the form $\chi_k(n, j) = (n, j + k), \ k \in \mathbb{Z},$ are continuous in every $I_n$. Our purpose is to extend analytically $U\chi$, therefore we will deal in the next section only with functions of the type $U\chi_k$. Nonetheless, the following theorem may be of some interest.

**Theorem 1.** The set of mappings $\{U\chi\}$ is a group with respect to composition. If $B$ is a Blaschke product of degree $n$, then $U\chi$ realizes a permutation of the $n$ half-open arcs and $\{U\chi_k\}$ is a cyclic subgroup of order $n$. Moreover, in the infinite case, $\{U\chi_k; \chi_k(n, j) = (n', j + k)\}$ are infinite cyclic subgroups for every given bijection $n \to n'$ of $\mathbb{N}^*$. Here $k$ varies in $\mathbb{Z}$.

**Proof.** Indeed, with $(n', j') = \chi(n, j)$ and $(n'', j'') = \chi'(n', j')$, there is $\chi''$ such that $(n'', j'') = \chi' \circ \chi(n, j) = \chi(n, j)$; therefore

$$U\chi''|\Gamma_{nj} = \Psi_{n'',j''}^{-1} \circ \Psi_{nj}^{-1}|\Gamma_{nj} = \Psi_{n'',j''}^{-1} \circ \Psi_{n',j'}^{-1} \circ \Psi_{nj}^{-1}|\Gamma_{nj} = U\chi'|\Gamma_{n',j'} \circ U\chi|\Gamma_{nj}.$$  

177
In other words,

\begin{equation}
U_{\chi'} \circ U_{\chi} = U_{\chi'} \circ U_{\chi}.
\end{equation}

In particular, if \( \chi_0 \) is the identity mapping, then

\begin{equation}
U_{\chi} \circ U_{\chi_0} = U_{\chi_0} \circ U_{\chi} = U_{\chi}
\end{equation}

for every \( \chi \), i.e. \( U_{\chi_0} \) is the identity element of the group.

If \( \chi_k(n, j) = (n, j + k) \), then \( \chi_{k'} \circ \chi_k(n, j) = (n, j + k + k') = \chi_{k+k'}(n, j) \), hence

\[ U_{\chi_k} \circ U_{\chi_{k'}} = U_{\chi_{k+k'}}, \quad k \in \mathbb{Z}. \]

3. Analytic extensions of the functions \( U_{\chi_k} \)

Theorem 6 of [6] can be easily extended to the case when \( E \) is a generalized Cantor set. More exactly, we have

**Theorem 2.** Let \( K \) be a compact subset of \( \partial D \setminus E \) (in the topology of \( \partial D \)). Then there is a neighborhood \( V \) of \( K \) (in \( \mathbb{C} \)) such that every function \( U_{\chi_k} \) can be extended analytically to \( V \). The extended functions still verify the identity \( B \circ U_{\chi_k} = B \).

**Proof.** For every \( z \in W \), the derivative of \( B \) is given by

\begin{equation}
B'(z) = -B(z) \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{(a_n - z)(1 - \bar{a}_n z)}.
\end{equation}

If \( \zeta = e^{i\theta} \in \partial D \setminus E \), then \( (a_n - \zeta)(1 - \bar{a}_n \zeta) = -\zeta(a_n - \zeta)(\bar{a}_n - \bar{\zeta}) = -\zeta |a_n - \zeta|^2 \) and \( |B(\zeta)| = 1 \). Thus

\begin{equation}
|B'(\zeta)| = \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|a_n - \zeta|^2} > 0.
\end{equation}

Consequently, the local inverse theorem (see [1], p. 131) can be applied at the point \( \zeta \) and we conclude that there is a neighborhood \( V_\zeta \) of \( \zeta \), \( V_\zeta \subset W \) such that \( B \) maps \( V_\zeta \) conformally onto a domain \( W_\zeta \) from the \( w \)-plane. Therefore, there is an analytic local inverse \( \varphi_\zeta : W_\zeta \rightarrow V_\zeta \) of \( B \). On the other hand, there is a couple \( (n, j) \) such that \( \zeta \in \Gamma_{n, j} \). On \( \Gamma_{n, j} \) the inverse of \( B \) is \( \Psi_{n, j} \), in other words \( \varphi_\zeta \) is an analytic extension of \( \Psi_{n, j}|_{V_\zeta \cap \Gamma_{n, j}} \). There is a finite covering \( \{ V_{\zeta_1}, V_{\zeta_2}, \ldots, V_{\zeta_p} \} \) of \( K \) and \( \Psi_{n, j}|_{V_\zeta \cap \Gamma_{n, j}} \) can be extended analytically in each of them, therefore it can be extended analytically.
in a neighborhood of $K$. The same is true for $U_{\chi_k}$ defined by the formula (2.1). Let us denote by $\varphi_n$ the analytic extension of $U_{\chi_k}|_{\Gamma_n}$ in a neighborhood in $\mathbb{C}$ of $K \cap I_n$ and let $\varphi_K$ be defined on $K$ by $\varphi_K(z) = \varphi_n(z)$ for every $n$, where $z \in K \cap I_n$. Then $\varphi_K$ is an analytic extension of $U_{\chi_k}$ in a neighborhood $V$ of $K$ and for every $\zeta \in \Gamma_{n,j} \subset K \cap I_n$ we have

\begin{equation}
B(\varphi_K(\zeta)) = B(U_{\chi_k}(\zeta)) = B(\zeta).
\end{equation}

By virtue of the functional relations theorem (see [1], p. 288), the identity (3.3) is true in $V$. □

4. The covering transformations of $(W, B)$

For simplicity of notation, we drop in this section the subscript $k$ in $\chi_k$. The analytic functions $\varphi_\chi$ extending $U_\chi|_{\Gamma_{n,j}}$ are defined in a neighborhood of every arc $\Gamma_{n,j}$. Let us denote by $\gamma_{n,j}$ the arcs obtained as in [3] by simultaneous continuation and let $A_{n,j}$ be the sets bounded by $\Gamma_{n,j} + \gamma_{n,j} - \gamma_{n,j+1}$ to which the arcs $\Gamma_{n,j}$ and $\gamma_{n,j}$, considered as point sets, are added. If $A_{n,j}$ is symmetric to $A_{n,j}$ with respect to the unit circle, we denote $\Omega_{n,j} = A_{n,j} \cup \hat{A}_{n,j}$. Let $D_{n,j}$ be the interior of $\Omega_{n,j}$.

**Theorem 3.** The functions $\varphi_\chi$ can be extended analytically to $W$. The extended functions verify the identity $B \circ \varphi_\chi = B$, therefore they are cover transformations of $(W, B)$. Moreover, they are the only cover transformations of $(W, B)$ and they represent conformally every $D_{n,j}$ on a $D_{n,j'}$.

**Proof.** We extend first $\varphi_\chi$ to every $A_{n,j}$ in the following way. Given arbitrary $\varepsilon > 0$, we denote $O_\varepsilon = \bigcup D(b_k, \varepsilon)$, where $D(b_k, \varepsilon)$ are open discs of radius $\varepsilon$ centered at every branch point of $(W, B)$.

Let us denote by $H_D$ the set of branch points of $(W, B)$ situated in $D$. It is known (see [3]) that all the points of $H_D$ belong to the arcs $\gamma_{n,j}$ and $H_D \cap K$ is a finite set for every compact set $K \subset \hat{\mathbb{C}} \setminus E$. Every set $A_{n,j} \setminus O_\varepsilon$ is compact, and therefore it can be covered by a finite number $q = q(n,j)$ of open discs $V_1, V_2, \ldots, V_q$ in which $B$ is injective. For $z \in V_r$, let $\varphi_{n,j,r}(z) = B^{-1}|_{A_{n',j'}}(B(z))$, where $(n', j') = \chi(n, j)$. It is obvious that if $z \in V_r \cap V_{r'}$, then $\varphi_{n,j,r}(z) = \varphi_{n,j,r'}(z)$; therefore there is a unique analytic function $\varphi_{n,j}$ defined in $V = \bigcup_{r=1}^q V_r$, which coincides with $\varphi_{n,j,r}$ in every $V_r$. As $z \in V$ implies $z \in V_r$ for some $r$, we have

\begin{equation}
B \circ \varphi_{n,j}(z) = B \circ \varphi_{n,j,r}(z) = B \circ B^{-1}|_{A_{n',j'}}(B(z)) = B(z).
\end{equation}
Since $\epsilon$ is arbitrary, $\varphi_{n,j}$ are in fact defined on $A_{n,j} \setminus H_D$ and $B \circ \varphi_{n,j}(z) = B(z)$ for every $(n,j) \in \mathbb{N}^* \times \mathbb{Z}$. The functions $\varphi_{n,j}$ are analytic continuations in $A_{n,j}$ of the functions $U_\chi$ from Section 2, since they are defined by a formula similar to (2.1). Consequently, if $U_\chi$ are continuous in $I_n$ and $V, V'$ are neighborhoods of $A_{n,j}$ and $A_{n,j+1}$ respectively, then $\varphi_{n,j}(z) = \varphi_{n,j+1}(z)$ for $z \in V \cap V'$. Consequently, there is a unique function $\varphi_\chi$ defined on $\bigcup_{j=-\infty}^{\infty} A_{n,j}$ such that

\begin{equation}
B \circ \varphi_\chi(z) = B(z).
\end{equation}

This happens for sure if $\chi(n,j) = (n,j+k)$, $k \in \mathbb{Z}$. We can extend $\varphi_\chi$ by symmetry to $\hat{C} \setminus E$ and it will continue to verify the identity (4.2).

Let us suppose now that $U$ is an arbitrary covering transformation of $(W,B)$ over $\mathbb{C}$, i.e. an analytic function $U : W \rightarrow W$ such that

\begin{equation}
B(U(z)) = B(z) \text{ for every } z \in W.
\end{equation}

In particular, the identity (4.3) is true for $|z| = 1$, $z \notin E$. It is known that $|B(z)| = 1$ if and only if $|z| = 1$ and $z \notin E$. Therefore $|B(U(z))| = 1$ if and only if $|U(z)| = 1$ and $U(z) \notin E$. Consequently $|U(z)| = 1$ if and only if $|z| = 1$.

On the other hand, with the notation of Section 2, $B(U(z)) = 1$ if and only if $z = e^{i\alpha_{n,j}}$. Therefore, if $\alpha_{n,j}$ is given, then $U(e^{i\alpha_{n,j}}) = e^{i\alpha_{n',j'}}$ for some $(n',j') \in \mathbb{N}^* \times \mathbb{Z}$. We cannot have $(n',j') = (n,j)$, since covering transformations have no fixed points. Therefore $U$ realizes a permutation of the points $e^{i\alpha_{n,j}}$, hence of the arcs $\Gamma_{n,j}$. Since $U$ is continuous on $\partial D \setminus E$, the permutation must be a cyclic one. Then

\begin{equation}
B|_{\Gamma_{n',j'}}(U|_{\Gamma_{n,j}}(z)) = B|_{\Gamma_{n,j}}(z)
\end{equation}

and therefore

\begin{equation}
U|_{\Gamma_{n,j}}(z) = B^{-1}|_{\Gamma_{n',j'}}(B|_{\Gamma_{n,j}}(z)).
\end{equation}

In other words,

\begin{equation}
U|_{\Gamma_{n,j}}(z) = \varphi_{n,j}|_{\Gamma_{n,j}}(z).
\end{equation}

By virtue of the functional relations theorem (see [1], p. 288), we have that $U(z) = \varphi_\chi(z)$ throughout $W$ for $\chi(n,j) = (n',j')$.

This theorem tells us that the set $G = \{\varphi_\chi\}$ of conformal self-mappings of $W$ represents the whole group of cover transformations of $(W,B)$ over $\mathbb{C}$.

Obviously, this is true for every type of Blaschke product, regardless of whether it is finite or infinite. □

180
In the first case $G$ is a finite cyclic group. In the infinite case, with $E$ discrete, the group $G$ is finitely generated and has a finite number of infinite cyclic subgroups. The fundamental domains of $G$ are the domains $\Omega_{n,j}$ previously described. Obviously, $\Omega_{n,j}$ are not uniquely determined, since they all depend on the initial choice of the arcs $\Gamma_{n,j}$, as well as on the arcs $\gamma_{n,j}$.

If we start with an equation of the form $B(z) = \lambda, |\lambda| = 1$ instead of $B(z) = 1$, we might arrive at different domains $\Omega'_{n,j}$ and different mappings $\psi_{n,j}$ between them, generating invariants for $B$. However, assuming that for some corresponding half-open arcs $\Gamma'_{n,j}$, in this new situation we have $\Gamma'_{n,j} \cap \Gamma_{n,j} \neq \emptyset$, and having in view the way $\varphi_{n,j}$ and $\psi_{n,j}$ have been constructed, they must coincide on $\Gamma'_{n,j} \cap \Gamma_{n,j}$, and then, by virtue of the functional relations, they coincide throughout $W$.

Consequently, the cover transformations of $(W, B)$ are independent (as they should be!) of the construction we used. Moreover, we can reformulate the final result in [3] as follows:

**Theorem 4.** Suppose that $B$ is an infinite Blaschke product whose cluster point set $E$ of zeros is a generalized Cantor set. Then $B$ generates a branched covering surface $(W, B)$ of the Riemann sphere, where $W = \mathbb{C} \setminus E$. The fundamental domains of the group of covering transformations of $(W, B)$ over $\hat{\mathbb{C}}$ accumulate at every point of $E$, i.e. for every $\zeta \in E$ and every neighborhood $V_{\zeta}$ of $\zeta$ there are infinitely many sets $\Omega_{n,j} \subset V_{\zeta}$ which are mapped by $B$ continuously and bijectively onto $\hat{\mathbb{C}}$. The mappings are conformal in the interior of every $\Omega_{n,j}$. Moreover, if $K \subset \mathbb{C} \setminus E$ is a compact set, then there is a finite covering of $K$ with sets $\Omega_{n,j}$, hence every $w = B(z), z \in K$ has a finite number of pre-images by $B$.

The novelty here resides in the fact that it is transparent how these cover transformations intrinsically depend on $B$. The question arises whether this property is specific to Blaschke products and if not, what other classes of non-univalent functions may display something similar. It would be also interesting to characterize proper subgroups of the group $G$ when $E$ is a Cantor set.

**Acknowledgements.** The authors wish to express their gratitude to all the computer science colleagues who helped generate and handle computer graphics for this paper; particularly to Sasha Costin, Florin Muscutaru and Anders Hedin.

They would like also to thank the referees who made valuable suggestions.
References


Authors’ addresses: Ilie Barza, Department of Mathematics, Karlstad University, SE-65188, Karlstad, Sweden, e-mail: ilie.barza@kau.se; Dorin Ghisa, Department of Mathematics, Glendon College, Toronto, Canada, e-mail: dghisa@yorku.ca.