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SOME RESULTS ABOUT THE HENSTOCK-KURZWEIL
FOURIER TRANSFORM

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Dedicated to Prof. Vladimir A. Borovikov on the first anniversary of his death

Abstract. We consider the Fourier transform in the space of Henstock-Kurzweil integrable functions. We prove that the classical results related to the Riemann-Lebesgue lemma, existence and continuity are true in appropriate subspaces.

Keywords: Fourier transform, Henstock-Kurzweil integral, bounded variation functions

MSC 2010: 42A38, 26A39, 26A45

1. INTRODUCTION

Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$, its Fourier transform at $s \in \mathbb{R}$ is defined by $\hat{f}(s) = \int_{-\infty}^{\infty} e^{-ixs} f(x) dx$. Here the integral is the Henstock-Kurzweil integral, which is equivalent to the Denjoy and Perron integrals.

The study of the Fourier transform in the space of the Henstock-Kurzweil integrable functions has been recently developed by E. Talvila [3]. He has shown some theorems on existence and continuity for the Fourier transform in certain subspaces. In general, neither existence nor continuity nor the Riemann-Lebesgue lemma are valid in the space of the Henstock-Kurzweil integrable functions.

These facts motivate us to look at a subspace of the Henstock-Kurzweil integrable functions that is not contained in the space of Lebesgue integrable functions and on which these classical properties are valid.

Notation 1.1. Let I be a finite or infinite closed interval. We work on the following subspaces:

- $\mathcal{HK}(I) = \{f; f \text{ is Henstock-Kurzweil integrable on } I\}$.
- $\mathcal{HK}_{\text{loc}}(\mathbb{R}) = \{f; f \in \mathcal{HK}(I) \text{ for each finite closed interval } I\}$.
- $\mathcal{BV}(I) = \{f; f \text{ is of bounded variation on } I\}$.
If $f \in \mathcal{BV}(I)$, $V_I f$ is the total variation of f on I .
- $\mathcal{BV}([\pm\infty]) = \{f; f \in \mathcal{BV}([a, \infty]) \cap \mathcal{BV}([-\infty, b]) \text{ for some } a, b \in \mathbb{R}\}$.
- $\mathcal{BV}_0([\pm\infty]) = \{f \in \mathcal{BV}([\pm\infty]); \lim_{|x| \rightarrow \infty} f(x) = 0\}$.
- $L(I) = \{f; f \text{ is Lebesgue integrable on } I\}$.

Main results 1.2. Our main results are the following:

- $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \subseteq \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$ and $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \not\subseteq L(\mathbb{R})$.
- An existence theorem for \hat{f} on \mathbb{R} when f is in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$.
- Continuity of \hat{f} on $\mathbb{R} \setminus \{0\}$ for functions $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$.
- A Riemann-Lebesgue lemma in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$.

In the following sections we prove these results.

2. THE $\mathcal{HK}(I) \cap \mathcal{BV}(I)$ SUBSPACE

If I is a compact interval, we know that

$$\mathcal{BV}(I) \subset L(I) \subset \mathcal{HK}(I),$$

and consequently $\mathcal{HK}(I) \cap \mathcal{BV}(I) \subset L(I)$.

Now, if I is unbounded, the first two observations which we have are

$$(2.1) \quad \mathcal{BV}(I) \not\subseteq L(I)$$

and

$$(2.2) \quad L(I) \not\subseteq \mathcal{HK}(I) \cap \mathcal{BV}(I).$$

Really, it is easy to demonstrate that the function $f(x) = 1/x$ defined in $[1, \infty]$ is of bounded variation with

$$V_{[1, \infty]} f = 1$$

and

$$\int_1^{\infty} \frac{1}{x} dx = \infty.$$

This implies that (2.1) is true.

To verify (2.2), we consider the function $f: [0, \infty] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x} \sin(1/x) & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0, x \in (1, \infty] \end{cases}$$

which is in $L([0, \infty]) \setminus \mathcal{BV}([0, \infty])$.

Next, we will prove that $\mathcal{HK}(I) \cap \mathcal{BV}(I) \not\subseteq L(I)$.

Proposition 2.1. *Let $\varphi: [a, \infty] \rightarrow \mathbb{R}$ be a non-negative function which is decreasing to zero when $x \rightarrow \infty$. If $\varphi \notin \mathcal{HK}([a, \infty])$, then the functions $\varphi(t) \sin t$ and $\varphi(t) \cos t$ are in $\mathcal{HK}([a, \infty]) \setminus L([a, \infty])$.*

Proof. We will demonstrate that $\varphi(t) \sin t \notin L([a, \infty])$. The proof that $\varphi(t) \cos t \notin L([a, \infty])$ can be done in a similar way.

Suppose that n_0 is the first natural number for which $a < (1 + 4n_0)\pi/4$. For $t \in [a, \infty]$ we have

$$|\sin t| \geq \frac{1}{\sqrt{2}} \quad \text{if and only if } t \in \bigcup_{k=n_0}^{\infty} [(1 + 4k)\pi/4, (3 + 4k)\pi/4].$$

Let $n \in \mathbb{N}$ with $n \geq n_0$. Since $(3 + 4n)\pi/4 < (1 + n)\pi$, we have

$$\begin{aligned} (2.3) \quad \int_a^{(1+n)\pi} \varphi(t) |\sin t| dt &\geq \frac{1}{\sqrt{2}} \sum_{k=n_0}^n \int_{(1+4k)\pi/4}^{(3+4k)\pi/4} \varphi(t) dt \\ &\geq \frac{1}{\sqrt{2}} \sum_{k=n_0}^n \int_{(1+4k)\pi/4}^{(3+4k)\pi/4} \varphi((3 + 4k)\pi/4) dt \\ &= \frac{\pi}{2\sqrt{2}} \sum_{k=n_0}^n \varphi((3 + 4k)\pi/4) \\ &\geq \frac{\pi}{2\sqrt{2}} \sum_{k=n_0}^n \varphi((1 + k)\pi). \end{aligned}$$

On the other hand,

$$\begin{aligned} (2.4) \quad \int_a^{(1+n)\pi} \varphi(t) dt &= \int_a^{n_0\pi} \varphi(t) dt + \int_{n_0\pi}^{(1+n)\pi} \varphi(t) dt \\ &= \int_a^{n_0\pi} \varphi(t) dt + \sum_{k=n_0}^n \int_{k\pi}^{(1+k)\pi} \varphi(t) dt \\ &\leq \int_a^{n_0\pi} \varphi(t) dt + \pi \sum_{k=n_0}^n \varphi(k\pi). \end{aligned}$$

Since $\varphi \notin \mathcal{HK}([a, \infty))$, we have $\int_a^\infty \varphi(t) dt = \infty$ and (2.4) implies

$$(2.5) \quad \sum_{k=n_0}^{\infty} \varphi(k\pi) = \infty.$$

Using (2.5) and letting $n \rightarrow \infty$ in (2.3), we conclude that $\varphi(t) \sin t \notin L([a, \infty))$.
For any $x \in [a, \infty)$,

$$\left| \int_a^x \sin t dt \right| \leq 2 \quad \text{and} \quad \left| \int_a^x \cos t dt \right| \leq 2.$$

Hence according to [1, Theorem 16.10] (Chartier-Dirichlet) we have that $\varphi(t) \sin t$ and $\varphi(t) \cos t$ are in $\mathcal{HK}[a, \infty)$. \square

Example 2.2. For any $a > 0$,

$$\frac{\sin t}{t} \in \mathcal{HK}([a, \infty)) \setminus L([a, \infty)).$$

Proposition 2.3. Let $1 > \alpha > 0$. The function $f_\alpha: [\pi^{1/\alpha}, \infty) \rightarrow \mathbb{R}$ defined as

$$f_\alpha(t) = \frac{\sin(t^\alpha)}{t}$$

satisfies

- (a) $f_\alpha \in \mathcal{HK}[\pi^{1/\alpha}, \infty) \setminus L([\pi^{1/\alpha}, \infty))$,
- (b) $f_\alpha \in \mathcal{BV}([\pi^{1/\alpha}, \infty))$.

Proof. (a) This is a consequence of [3, Lemma 23].

(b) Let $x \in (\pi^{1/\alpha}, \infty)$. We know that $f'_\alpha \in \mathcal{HK}([\pi^{1/\alpha}, x])$. Now since

$$f'_\alpha(t) = \frac{\alpha \cos(t^\alpha)}{t^{2-\alpha}} - \frac{\sin(t^\alpha)}{t^2},$$

we have that

$$(2.6) \quad |f'_\alpha(t)| \leq \frac{\alpha}{t^{2-\alpha}} + \frac{1}{t^2}.$$

The function $g(t) = \alpha/t^{2-\alpha} + 1/t^2$ satisfies $g \in \mathcal{HK}([\pi^{1/\alpha}, x])$, hence by (2.6) and [1, Theorem 7.7] we conclude that $f'_\alpha \in L([\pi^{1/\alpha}, x])$ and

$$\begin{aligned} \int_{\pi^{1/\alpha}}^x |f'_\alpha| &\leq \int_{\pi^{1/\alpha}}^x \left(\frac{\alpha}{t^{2-\alpha}} + \frac{1}{t^2} \right) dt \\ &= \left(\frac{1}{\alpha-1} \right) [x^{\alpha-1} - \pi^{(\alpha-1)/\alpha}] - \frac{1}{x} + \frac{1}{\pi^{1/\alpha}}. \end{aligned}$$

Consequently, by [1, Theorem 7.5],

$$V_{[\pi^{1/\alpha}, x]} f_\alpha \leq \left(\frac{1}{\alpha - 1} \right) [x^{\alpha-1} - \pi^{(\alpha-1)/\alpha}] - \frac{1}{x} + \frac{1}{\pi^{1/\alpha}}.$$

Therefore, as $1 - \alpha > 0$, we have that

$$V_{[\pi^{1/\alpha}, \infty]} f_\alpha \leq \frac{1}{(1 - \alpha)\pi^{(1-\alpha)/\alpha}} + \frac{1}{\pi^{1/\alpha}}.$$

Thus, $f_\alpha \in \mathcal{BV}([\pi^{1/\alpha}, \infty])$. □

Similarly, we can prove that for $1 > \alpha > 0$, the function $g_\alpha: [-\infty, -\pi^{1/\alpha}] \rightarrow \mathbb{R}$ defined as

$$g_\alpha(t) = \frac{\sin(-t)^\alpha}{-t}$$

belongs to $\mathcal{HK}([-\infty, -\pi^{1/\alpha}]) \cap \mathcal{BV}([-\infty, -\pi^{1/\alpha}]) \setminus L([-\infty, -\pi^{1/\alpha}])$.

Let $h \in \mathcal{BV}([-\pi^{1/\alpha}, \pi^{1/\alpha}])$. For $1 > \alpha > 0$, the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} h(x) & \text{if } x \in (-\pi^{1/\alpha}, \pi^{1/\alpha}), \\ \frac{\sin |t|^\alpha}{|t|} & \text{if } x \in (-\infty, -\pi^{1/\alpha}] \cup [\pi^{1/\alpha}, \infty) \end{cases}$$

is in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \setminus L(\mathbb{R})$. With this example and Proposition 2.3 we have the following theorem.

Theorem 2.4. *There exists a function f in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R}) \setminus L(\mathbb{R})$.*

Now, since $\mathcal{BV}(\mathbb{R}) \subset \mathcal{BV}([\pm\infty])$, we have immediately the next corollary.

Corollary 2.5. $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty]) \not\subset L(\mathbb{R})$.

We observe that $\mathcal{BV}(\mathbb{R}) \subset \mathcal{BV}([\pm\infty])$ properly, because instead of the function h in $\mathcal{BV}([-\pi^{1/\alpha}, \pi^{1/\alpha}])$ we can take a function in $\mathcal{HK}([-\pi^{1/\alpha}, \pi^{1/\alpha}]) \setminus \mathcal{BV}([-\pi^{1/\alpha}, \pi^{1/\alpha}])$.

3. AN EXISTENCE THEOREM FOR $\hat{f}(s)$ IN $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$

A part from Proposition 2.1(b) in [3] by E. Talvila tells us that, if $f \in \mathcal{HK}_{\text{loc}}(\mathbb{R}) \cap \mathcal{BV}_0([\pm\infty])$, then $\hat{f}(s)$ exists for all $s \in \mathbb{R}$. If $s \neq 0$, then the result is true. However, under these conditions, it is not necessarily true for $\hat{f}(0)$. For example, the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in (-1, 1), \\ 1/x & \text{if } x \in (-\infty, -1] \cup [1, \infty) \end{cases}$$

is in $\mathcal{HK}_{\text{loc}}(\mathbb{R}) \cap \mathcal{BV}_0([\pm\infty])$ but $\hat{f}(0)$ does not exist.

In order to have the existence of $\hat{f}(0)$, we need that $f \in \mathcal{HK}(\mathbb{R})$.

We will demonstrate that the Fourier transform exists in $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$ for every $s \in \mathbb{R}$.

Theorem 3.1. *If $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$, then $\hat{f}(s)$ exists for all $s \in \mathbb{R}$.*

Proof. The result is true for $s = 0$ because $f \in \mathcal{HK}(\mathbb{R})$. Now let $s \neq 0$; since $\mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty]) \subset \mathcal{HK}_{\text{loc}}(\mathbb{R}) \cap \mathcal{BV}_0([\pm\infty])$, by [3, Proposition 2.1 (b)] it follows that $\hat{f}(s)$ exists. □

4. CONTINUITY OF \hat{f}

We know that the continuity of the Lebesgue-Fourier transform on \mathbb{R} is a consequence of the dominated convergence theorem and that the Lebesgue integral is absolute. Now to prove the continuity of the Henstock-Kurzweil Fourier transform we can not use the same arguments, because the Henstock-Kurzweil integral is not absolute. Two results about this are given in the following theorems. The first of them is an immediate consequence of [3, Theorem 5].

Theorem 4.1. *Let f be a function with support in a compact interval such that $f \in \mathcal{HK}(\mathbb{R})$. Then \hat{f} is continuous on \mathbb{R} .*

Theorem 4.2. *If $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}([\pm\infty])$, then \hat{f} is continuous on $\mathbb{R} \setminus \{0\}$.*

Proof. Let $t_0 \in \mathbb{R} \setminus \{0\}$ and consider $a < 0$ and $b > 0$ such that $f \in \mathcal{BV}(-\infty, a] \cap \mathcal{BV}[b, \infty)$. If we show that $\widehat{f\chi_{(-\infty, a]}}$, $\widehat{f\chi_{[a, b]}}$ and $\widehat{f\chi_{[b, \infty)}}$ are continuous at t_0 , then \hat{f} is continuous at t_0 , because

$$\hat{f}(t) = \widehat{f\chi_{(-\infty, a]}}(t) + \widehat{f\chi_{[a, b]}}(t) + \widehat{f\chi_{[b, \infty)}}(t) \text{ for all } t \in \mathbb{R}.$$

By Theorem 4.1, $\widehat{f\chi_{[a,b]}}$ is continuous at t_0 . To prove that $\widehat{f\chi_{(-\infty,a]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous at t_0 we will use [3, Proposition 6(a)]. The conditions f is Henstock-Kurzweil integrable on \mathbb{R} and f is of bounded variation on $(-\infty, a] \cup [b, \infty)$ imply that $\lim_{|x| \rightarrow \infty} f(x) = 0$. Now since $t_0 \neq 0$, there exist $K > 0$ and $\delta > 0$ such that if $|t - t_0| < \delta$, then $1/|t| < K$. Thus for all $|t - t_0| < \delta$,

$$\left| \int_u^v e^{-ixt} dx \right| \leq \frac{2}{|t|} < 2K \quad \text{for all } [u, v] \subseteq \mathbb{R}.$$

Therefore, by [3, Proposition 6(a)], $\widehat{f\chi_{(-\infty,a]}}$ and $\widehat{f\chi_{[b,\infty)}}$ are continuous at t_0 . \square

5. THE RIEMANN-LEBESGUE LEMMA

First we give a corollary proved by Talvila in [2].

Corollary 5.1. *If $|\int_a^x g_n| \leq M$ for all $n \geq 1$ and all $x \in [a, b]$, if each f_n is of bounded variation, if $\lim_{x \rightarrow b^-} f_n(x) = 0$ uniformly in n , if $f_n \rightarrow 0$ on $[a, b]$ and if $V(f_n) \rightarrow 0$, then $\int_a^b g_n f_n \rightarrow 0$.*

Theorem 5.2. *If $f \in \mathcal{HK}(\mathbb{R}) \cap \mathcal{BV}(\mathbb{R})$, then $\lim_{|t| \rightarrow \infty} \hat{f}(t) = 0$.*

Proof. First we will prove that for every sequence $\{t_n\}_{n \in \mathbb{N}} \subseteq [0, \infty)$ such that $n \leq t_n$ for all $n \in \mathbb{N}$ it is true that $\lim_{n \rightarrow \infty} \hat{f}(t_n) = 0$.

Let $\{t_n\}_{n \in \mathbb{N}} \subseteq [0, \infty)$ be a sequence such that $n \leq t_n$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, define $f_n(x) = n^{-1}f(x)$, $g_n(x) = ne^{-ixt_n}$ on $[0, \infty)$ and $f_n(\infty) = 0$, $g_n(\infty) = 0$.

For all $n \in \mathbb{N}$ and all $s \in [0, \infty)$,

$$\left| \int_0^s g_n(x) dx \right| = \left| n \int_0^s e^{-ixt_n} dx \right| \leq \frac{2n}{t_n} \leq 2.$$

Since $f \in \mathcal{BV}([0, \infty]) \cap \mathcal{HK}([0, \infty])$, we have that each f_n is in $\mathcal{BV}([0, \infty]) \cap \mathcal{HK}([0, \infty])$ and

$$\lim_{n \rightarrow \infty} V_{[0, \infty]} f_n = \lim_{n \rightarrow \infty} \frac{1}{n} V_{[0, \infty]} f = 0.$$

We observe too that $\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} n^{-1}f(x) = 0$ for all $x \in [0, \infty]$.

Thus according to Corollary 5.1,

$$\lim_{n \rightarrow \infty} \int_0^\infty f(x) e^{-ixt_n} dx = \lim_{n \rightarrow \infty} \int_0^\infty f_n(x) g_n(x) dx = 0.$$

Using Corollary 5.1 for intervals of the type $(a, b]$ we can prove too that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^0 f(x) e^{-ixt_n} dx = 0.$$

Thus $\lim_{n \rightarrow \infty} \hat{f}(t_n) = 0$.

We now prove that $\lim_{t \rightarrow \infty} \hat{f}(t) = 0$. Suppose that it is not true, then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exists $t_n > n$ such that $|\hat{f}(t_n)| \geq \varepsilon$. The sequence $\{t_n\}_{n \in \mathbb{N}}$ satisfies $\{t_n\}_{n \in \mathbb{N}} \subseteq [0, \infty)$ and $n \leq t_n$ for all $n \in \mathbb{N}$, hence by the first part of this proof we have $\lim_{n \rightarrow \infty} \hat{f}(t_n) = 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $|\hat{f}(t_n)| < \varepsilon$ for all $n \geq n_0$. If we take $n_1 > n_0$ then $\varepsilon \leq |\hat{f}(t_{n_1})| < \varepsilon$, which is a contradiction.

The proof of $\lim_{t \rightarrow -\infty} \hat{f}(t) = 0$ is analogous. \square

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