BOUNDS CONCERNING THE ALLIANCE NUMBER

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(Received September 26, 2008)

Abstract. P.Kristiansen, S.M.Hedetniemi, and S.T.Hedetniemi, in Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004), 157–177, and T.W.Haynes, S.T.Hedetniemi, and M.A.Henning, in Global defensive alliances in graphs, Electron. J. Combin. 10 (2003), introduced the defensive alliance number $a(G)$, strong defensive alliance number $\hat{a}(G)$, and global defensive alliance number $\gamma_a(G)$. In this paper, we consider relationships between these parameters and the domination number $\gamma(G)$. For any positive integers $a, b,$ and $c$ satisfying $a \leq c$ and $b \leq c$, there is a graph $G$ with $a = a(G)$, $b = \gamma(G)$, and $c = \gamma_a(G)$. For any positive integers $a, b,$ and $c$, provided $a \leq b \leq c$ and $c$ is not too much larger than $a$ and $b$, there is a graph $G$ with $\gamma(G) = a$, $\gamma_a(G) = b$, and $\gamma_{\hat{a}}(G) = c$. Given two connected graphs $H_1$ and $H_2$, where order$(H_1) \leq$ order$(H_2)$, there exists a graph $G$ with a unique minimum defensive alliance isomorphic to $H_1$ and a unique minimum strong defensive alliance isomorphic to $H_2$.

Keywords: defensive alliance, global defensive alliance, domination number

MSC 2010: 05C69

1. Introduction

Recall that a dominating set of a graph $G$ is a set of vertices $S \subseteq V(G)$ so that for every vertex $v \in V(G)$, either $v \in S$ or $v$ is adjacent to some vertex in $S$. The minimum order of a dominating set for $G$ is the domination number of $G$, denoted $\gamma(G)$.

In [1] and [2], Kristiansen, Hedetniemi, and Hedetniemi and Haynes, Hedetniemi, and Henning introduced defensive alliances, strong defensive alliances, and global defensive alliances. Their primary motivation was the study of war-time alliances between nations. A set $S$ of vertices in a graph $G$ is a defensive alliance if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. Hence, each vertex (nation) in $S$ has at least as many neighboring vertices in its alliance, including itself, as it does neighboring vertices outside its alliance. A defensive alliance $S$ is strong if the inequality is strict.
for every $v \in S$, that is, $|N[v] \cap S| > |N(v) \cap (V - S)|$. An alliance is \textit{global} if $S$ is also a dominating set for the graph $G$.

A minimum defensive alliance is called an \textit{a-set}, and the order of a minimum defensive alliance in $G$ is denoted $a(G)$. Similarly, a minimum strong defensive alliance is an $\hat{a}$-\textit{set}, with order $\hat{a}(G)$, and a minimum global defensive alliance is an $\gamma_a$-\textit{set}, with order $\gamma_a(G)$. The order of a minimum strong global alliance in $G$ is denoted $\gamma_{\hat{a}}(G)$. An \textit{a}-set or an $\hat{a}$-set always induces a connected subgraph, since any component of a defensive alliance is a defensive alliance.

Several relationships follow naturally from these definitions, including the following:

- $a(G) \leq \hat{a}(G)$,
- $\gamma(G) \leq \gamma_a(G) \leq \gamma_{\hat{a}}(G)$,
- $a(G) \leq \gamma_a(G)$,
- $\hat{a}(G) \leq \gamma_{\hat{a}}(G)$.

In this paper, we consider whether there are other, less obvious, relationships between these parameters, and whether any pair of positive integers can be achieved as one of the relationships above by some graph $G$.

In the first section, we show that a general construction for $G$ is possible for each of the inequalities

- $\gamma(G) \leq \gamma_a(G)$,
- $a(G) \leq \gamma_a(G)$,
- $\gamma(G) \leq \gamma_a(G) \leq \gamma_{\hat{a}}(G)$,

although for the last inequality, we will need an additional upper bound on the value for $\gamma_{\hat{a}}(G)$. In the second section, we focus on building graphs around arbitrary given subgraphs so that the subgraphs are induced by \textit{a}-sets, $\hat{a}$-sets, and $\gamma_a$-sets. In particular, we show that, given any two connected graphs $H_1$ and $H_2$ with order($H_1$) $\leq$ order($H_2$), there is a graph $G$ whose unique $\textit{a}$-set induces $H_1$ as a subgraph and whose unique $\hat{a}$-set induces $H_2$ as a subgraph. Furthermore, given any connected graph $H$, there is a graph $G$ whose unique $\gamma_a$-set induces a subgraph isomorphic to $H$.

2. CONSTRUCTIONS FOR INEQUALITIES RELATED TO ALLIANCES

Since every global alliance set is also a dominating set, we know that $\gamma(G) \leq \gamma_a(G)$ for any graph $G$. Every global alliance set is also a defensive alliance set, so $a(G) \leq \gamma_a(G)$. In fact, any three positive integers satisfying these inequalities are achievable as the alliance, domination, and global alliance number of some graph $G$. 

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Theorem 2.1. For any positive integers $a$, $b$, and $c$ with $a \leq c$ and $b \leq c$, there exists a connected graph $G$ such that $a(G) = a$, $\gamma(G) = b$, and $\gamma_a(G) = c$.

Proof. Since the path $P_2$ has the desired properties when $c = 1$, we assume $c \geq 2$.

Case I. $b = 1$.

Construct the graph $G$ by starting with $K_a$ and $K_{2c-a}$. Let $U$ be the vertices of $K_a$, $W$ be a set of $a$ of the vertices of $K_{2c-a}$, and $X$ be the remaining vertices of $K_{2c-a}$. Join every vertex of $U$ to every vertex of $W$.

It is straightforward to see that the vertices of $U$ form a minimum defensive alliance. Since each vertex of $W$ is adjacent to every other vertex, the domination number is 1. A set consisting of all of the vertices of $W$ and $c-a$ of the vertices of $X$ form a minimum global alliance, so $\gamma_a(G) = c$.

Case II. $a = 1$ and $b \geq 2$.

Let $P_b: u_1, u_2, \ldots, u_b$ be a path of order $b$. Then the graph $G$ is obtained from $P_b$ by joining new vertices $v_i$ to $u_i$ for $i \in \{1, \ldots, b-1\}$ and adding $2(c-b)+2$ new vertices $z_1, \ldots, z_{2(c-b)+2}$ to $G$ and joining each $z_i$ to $u_b$. The graph $G$ is shown below.

Observe that $\{u_1, \ldots, u_b\}$ is the minimum dominating set, so $\gamma(G) = b$. Also, observe that $\{u_1, \ldots, u_b, z_1, \ldots, z_{c-b}\}$ is a dominating set and alliance which realizes the minimum cardinality $\gamma_a(G) = c$. Any one of the end-vertices is a defensive alliance, so $a(G) = 1$.

Case III. $a = b = c = 2$.

The graph $C_4$ has the desired property.

Case IV. $a \geq 2$, $b \geq 2$, $c \geq 3$, and $b < c$.

Subcase IV(a). $a = c$.

Start with the complete graph $K_{2a-1}$. Add $b$ new vertices $v_1, v_2, \ldots, v_b$. Join each of the $b$ new vertices to two vertices of $K_{2a-1}$, so that $v_i$ and $v_j$ have no common neighbor for $i \neq j$, and $\deg(v_i) = 2$ for all $i$. Any defensive alliance must contain a vertex of $K_{2a-1}$ and, hence, at least $a$ vertices; any $a$ vertices of $K_{2a-1}$ will be a defensive alliance. Any dominating set must contain either $v_i$ or a neighbor of $v_i$ for
each $i$, so $\gamma(G) = b$. A set of $a$ vertices from $K_{2a-1}$, including a neighbor of each $v_i$, $1 \leq i \leq b$, will be a global dominating set. (Note: $b < a$.)

Subcase IV(b). $a < c$.

Construct the graph $G$ as follows. Start with the complete graphs $K_a$ and $K_{2c-a-1}$. Let $U$ be the vertices of $K_a$, let $W$ be $a - 1$ of the vertices of $K_{2c-a-1}$, and let $X$ be $V(K_{2c-a-1}) - W$. Notice that $X$ is not empty and has even order. Join every vertex of $U$ to every vertex of $W$. Add $b$ new vertices $v_1, v_2, v_3, \ldots, v_b$. Join each $v_i$ to either one vertex of $U$ and one vertex of $W$ or to two vertices of $X$, so that for each $i$, $\deg v_i = 2$, and for each $i$ and $j$, $i \neq j$, vertices $v_i$ and $v_j$ have no common neighbors. In particular, $v_1$ should be joined to two vertices of $X$ and $v_2$ should be joined to one vertex of $U$ and one vertex of $W$.

We leave it for the reader to verify that $U$ is a minimum defensive alliance, though possibly not unique.

Since no two $v_i$ and $v_j$ with $i \neq j$ have a common neighbor, any dominating set must contain at least $b$ vertices, including either $v_i$ or a neighbor of $v_i$ for each $i$. Now, $v_2$ is adjacent to some $w \in W$ which dominates the rest of the graph, so there is a dominating set with $b$ vertices.

It is straightforward to check that the set consisting of $W$, one vertex from $U$, and $c - a$ vertices from $X$, including at least one neighbor of each $v_i$, is a minimum global alliance set of order $c$.

Case V. $a \geq 2$ and $b = c \geq 3$.

Construct $G$ as follows. Start with a complete graph on $c$ vertices $v_1, v_2, \ldots, v_c$, and $\lceil \frac{1}{2}c \rceil$ copies of $K_{2a-2}$. Join $v_1$ to $a - 1$ of the vertices in the first $K_{2a-2}$ and join $v_2$ to the other $a - 1$ vertices. Similarly, for each $i$, $2 \leq i \leq 2 \lceil \frac{1}{2}c \rceil$, join $v_{2i-1}$ to $a - 1$ of the vertices in the $i$th copy of $K_{2a-2}$ and join $v_{2i}$ to the other $a - 1$ vertices. Also, add $2 \lceil \frac{1}{2}c \rceil$ new vertices $u_1, u_2, \ldots, u_2 \lceil \frac{1}{2}c \rceil$. Join $u_i$ to the same $a - 1$ vertices in a copy of $K_{2a-1}$ as $v_i$, and also join $u_i$ to $v_i$. If $c$ is odd, add $2 \lceil \frac{1}{2}c \rceil$ new vertices $w_1, w_2, \ldots, w_{c-1}$. For each $i$, $1 \leq i \leq c - 1$, join $w_i$ to $v_c$ and to $u_i$.

When $c$ is even, the set $N[u_1] - \{v_1\}$, the closed neighborhood of $u_1$ except for $v_1$, is a minimum alliance set with $a$ vertices. When $c$ is odd, the set $N[u_1] - \{v_1, w_1\}$ is a minimum alliance set with $a$ vertices.

The set $\{v_1, v_2, \ldots, v_c\}$ is a minimum dominating set with $c$ vertices, and a minimum global alliance, so we have $\gamma(G) = \gamma_a = c = b$. 

Based simply on the definitions, the domination number, global alliance number, and strong global alliance number must satisfy $\gamma(G) \leq \gamma_a(G) \leq \gamma_\hat{a}(G)$ for any graph $G$. Given any three positive integers $a \leq b \leq c$, is there a graph $G$ so that $\gamma(G) = a$, $\gamma_a(G) = b$, and $\gamma_\hat{a}(G) = c$?

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First, suppose \( b = 1 \). If \( G \) is a graph with \( \gamma_a(G) = 1 \), then there is a single vertex \( u \in V(G) \) so that \( \{u\} \) is a dominating set and a defensive alliance. Since \( \{u\} \) is a dominating set, every other vertex of \( G \) is adjacent to \( u \). Since \( \{u\} \) is a defensive alliance, there must be at most one vertex adjacent to \( u \). Thus, \( G = K_1 \) or \( K_2 \), and \( c = 1 \) or \( c = 2 \).

We will consider the remaining cases in the following proof. First, however, we introduce a useful construction. For any integers \( i, j, \) and \( k \) with \( i \geq 1, 0 \leq j \leq i - 1, \) and \( j \geq 2k - 1 \), we construct a graph \( H(i, j, k) \) with order \( i \), minimum degree \( j \), and containing a clique on \( k \) vertices, each of which has degree \( j \) in the graph as a whole. Notice that \( i \geq 2k \). Start with \( K_k \cup K_{i-k} \). Then add \( k(j - k + 1) \) edges between the two complete graphs, distributed as evenly as possible. Thus, each vertex in \( K_k \) will have degree \( (k - 1) + (j - k + 1) = j \) and each vertex in \( K_{i-k} \) will have degree at least \( i - k - 1 + \lceil k(j - k + 1)/(i - k) \rceil \). Since \( i \geq 2k \) and \( i > j \), clearly \( (i - j - 1)(i - 2k) \geq 0 \). With a little arithmetic, this inequality is equivalent to \( i - k - 1 + k(j - k + 1)/(i - k) \geq j \). Since the right hand side is an integer, we can take the floor function of the left hand side and the inequality will still hold.

**Theorem 2.2.** Let \( a, b, \) and \( c \) be three positive integers with \( a \leq b \leq c, 2 \leq b, \) and \( c \leq \frac{1}{2}(ab + 2b - a[b/a]) \). Then there exists a graph \( G \) such that \( \gamma(G) = a, \gamma_a(G) = b, \) and \( \gamma_a(G) = c \).

**Proof.** We construct \( G \) as follows. We start with \( K_b \) and partition the vertices of \( K_b \) into \( a \) sets \( S_1, S_2, \ldots, S_a \) as nearly equal in size as possible, so \(|S_i| = [b/a] \) or \([b/a] + 1 \) for each \( i \).

Let \( q = |(c-b)/a| \). Define \( a \) additional graphs \( W_1, W_2, \ldots, W_a \) as follows. If \( q = 0 \), that is, \( c - b < a \), then \( W_i \) is the graph with no edges on \( b \) vertices for \( 1 \leq i \leq c - b \) and \( W_j \) is the graph with no edges on \( b - 1 \) vertices for \( c - b < j \leq a \). Otherwise, using the construction described prior to this theorem, define \( W_i = H(b, [b/a] + 2q - 1, q + 1) \), a graph of order \( b \) with minimum degree \( [b/a] + 2q - 1 \) and clique size at least \( q + 1 \), for \( 1 \leq i \leq c - b - qa \) and \( W_j = H(b, [b/a] + 2q - 3, q) \) for \( c - b - qa < i \leq a \).

This is possible provided \([b/a] + 2q - 1 \leq b - 1 \) or, if \( a \) divides \( c - b \), \([b/a] + 2q - 3 \leq b - 1 \). By substituting \([(c-b)/a]\) for \( q \) and solving for \( c \), we see that the first inequality is satisfied if \( c \leq \frac{1}{2}(ab + 2b - a[b/a]) \). Notice that, due to the floor function in the definition of \( q \), \( c \leq \frac{1}{2}(ab + 2b - a[b/a]) \) implies \([b/a] + 2q - 1 \leq b - 1 \) but not vice versa.

Now, join every vertex of \( W_i \) to every vertex of \( S_i \) for \( 1 \leq i \leq a \).

We will show that the set formed by selecting a single entry from each set \( S_i \) with \( 1 \leq i \leq a \) is a minimum dominating set, that the vertices of \( K_b \) form a minimum
global alliance set, and that the vertices of $K_b$ along with $[(c - b)/a] + 1$ vertices from each $W_i$, $1 \leq i \leq c - b - qa$, and $[(c - b)/a]$ vertices from each $W_j$, $c - b - aq < j \leq a$, forms a minimum global strong alliance set.

Claim 1. $\gamma(G) = a$.

Notice that the $a$ sets $W_1, W_2, \ldots, W_a$ are disjoint, with the property that for any two vertices $w \in W_i$ and $w' \in W_j$, $i \neq j$, $w$ and $w'$ are not adjacent and have no common neighbor. Thus, any dominating set must contain at least $a$ vertices.

Now, choose one vertex from each set $S_i$, $1 \leq i \leq a$. This is a dominating set.

Claim 2. $\gamma_a(G) = b$.

As noted in Claim 1, any dominating set of $G$ must contain either a vertex of $W_i$ or a vertex of $S_i$ for each $i$, $1 \leq i \leq a$. Suppose a vertex $u \in S_i$ is in a global alliance set. Since $|N[u]| = 2b$ or $2b - 1$, we must have at least $b$ vertices in the set, counting $u$. Suppose the vertices $w_1, w_2, \ldots, w_r$ from a specific set $W_i$ are in a global alliance set, but no vertex of $S_i$ is in the set. Then each $w_i$ has at least $|S_i|$ enemies and at most $r$ friends, including itself, so $r \geq |S_i|$. Thus, if there are no vertices from any $S_i$ in the set, then there must be at least $\sum_{i=1}^{a} |S_i| = b$ vertices from $\bigcup_{i=1}^{a} W_i$ in the set. Either way, $\gamma_a(G) \geq b$.

Notice that $\bigcup_{i=1}^{a} S_i$ is a global alliance set of order $b$.

Claim 3. $\gamma_a(G) = c$.

Again, any dominating set must contain at least one vertex of $W_i \cup S_i$ for each $i$. If $W_i$ is an empty graph on $b$ or $b - 1$ vertices, then any strong alliance set which contains a vertex of $W_i$ must also contain a vertex of $S_i$. We may assume, then, that we need at least one vertex $u_i$ from each $S_i$ in this case. We will also need at least $\left\lceil \frac{1}{3} \deg(u_i) \right\rceil = b - 1$ or $b$ additional vertices from $N[u_i] = W_i \cup K_b$. If $|W_i| = b - 1$ and if $\bigcup_{j=1}^{a} S_j = K_b$ is contained in the strong alliance, then no vertex of $W_i$ is needed; each vertex in $S_i$ has $b$ allies and $b - 1$ enemies. However, if $|W_i| = b$, then any strong alliance which contains $S_i$ must contain at least one vertex of $W_i$ as well.

For $W_i = H(b, [b/a] + 2q - 1, q + 1)$, any strong dominating set which contains a vertex $u \in S_i$ must contain $\left\lceil \frac{1}{3} \deg(u) \right\rceil = b$ neighbors of $u$, including at least one vertex $w \in W_i$. And any strong dominating set which contains $w \in W_i$ must contain at least half of the neighbors of $w$, at least $[b/a] + q$ vertices, including at least $q$ vertices in $W_i$, not counting $w$, or $q + 1$ total vertices in $W_i$. Similarly, for $W_i = H(b, [b/a] + 2q - 3, q)$, any strong dominating set must contain at least $q$ vertices of $W_i$.

Thus, at a minimum, we will need all $b$ vertices of $\bigcup_{i=1}^{a} S_i$, one vertex from each $W_i$ which is an empty graph on $b$ vertices, $q + 1$ vertices from each $H(b, [b/a] + 2q - 1,$
q + 1), and q vertices from each $H(b, \lfloor b/a \rfloor + 2q - 3, q)$. If we add these, we have at least c vertices. Thus, $\gamma_a(G) \geq c$. Such a set will be a strong global alliance set provided the vertices from each $W_i$ form a clique in that $W_i$ and have the minimum degree in $W_i$. By our construction of $W_i$, such a set can be found. 

It is not known whether the condition $c \leq \frac{1}{2}(ab + 2b - a\lfloor b/a \rfloor)$ is necessary. However, $\gamma_a(G)$ can be bounded above by a formula in terms of $\gamma_a(G)$. We mention one such upper bound.

**Observation 2.3.** For any graph $G$, $\gamma_a(G) \leq \gamma_a(G)(1 + \gamma_a(G))$.

To see this bound, suppose that $\gamma_a(G) = b$, and let $S$ be a subgraph of order $b$ which is a global alliance set. Then each vertex of $S$ has at most $b$ neighbors outside of $S$. Since $S$ is a dominating set, $G$ has at most $b(1 + b)$ vertices. Clearly, $V(G)$ is a strong global alliance set.

### 3. Specified alliance and strong alliance sets

In this section, we specify not only the order of the $a$-set, $\hat{a}$-set, and/or $\gamma_a$-set of the graph but also the subgraphs induced by these sets. If a defensive alliance or strong defensive alliance induces a subgraph that is not connected, then any component of that subgraph would be an alliance of smaller order. Thus, any $a$-set or $\hat{a}$-set induces a connected subgraph. Provided that two graphs $H_1$ and $H_2$ are connected, though, the next theorem shows that there is a graph $G$ whose unique $a$-set induces a subgraph isomorphic to $H_1$ and whose unique $\hat{a}$-set induces a subgraph isomorphic to $H_2$.

**Theorem 3.1.** Given $1 \leq a \leq b$ and any two connected graphs $H_1$ and $H_2$ with orders $a$ and $b$ respectively, there exists a connected graph $G$ with the following properties.

(a) $H_1$ is isomorphic to the subgraph induced by the only defensive alliance of $G$ that has minimum cardinality $a(G)$.

(b) $H_2$ is isomorphic to the subgraph induced by the only strong defensive alliance of $G$ that has minimum cardinality $\hat{a}(G)$.

**Proof.** Suppose that $1 \leq a \leq b$ and that $H_1$ and $H_2$ are connected graphs such that $a = |V(H_1)|$ and $b = |V(H_2)|$. Since both $H_1$ and $H_2$ are connected, $\deg(v) \geq 1$ for each vertex $v$ in $H_1$ or $H_2$. Modify $H_1$ and $H_2$ to get the graph $G$ as follows: (1) For every vertex $u \in V(H_1)$ and $i \in \{1, \ldots, \deg_{H_1}(u) + 1\}$, adjoin an end-vertex $y_u^{(i)}$ to $u$. (2) For every vertex $v \in V(H_2)$ and $i \in \{1, \ldots, \deg_{H_2}(v)\}$, adjoin an end-vertex
Add $K_{4b}$ to the new graph and adjoin each vertex labelled $y^{(i)}_u$ and $z^{(i)}_v$ to each vertex in $K_{4b}$. The resulting graph is $G$.

Observe that $V(H_1) \subseteq V(G)$ is a defensive alliance (with cardinality $a(G)$) and that any defensive alliance which contains a vertex of $H_1$ must contain every vertex of $H_1$. Further, one sees that any alliance with vertices in $V(H_2)$ must contain a vertex labelled $z^{(i)}_v$. Also, observe that no alliance of $G$ can contain any vertex labelled $y^{(i)}_u$ or $z^{(i)}_v$ unless it contains at least $1 + 2b$ vertices. Lastly, notice that any alliance of $G$ that is a subset of $V(K_{4b}) \subseteq V(G)$ must also be an alliance of $K_{4b}$ alone. Any such alliance must have cardinality at least $4b/2 = 2b$. With all these observations, one sees that $V(H_1)$ must be the only defensive alliance of $G$ with least cardinality. Similarly, $V(H_2)$ is the only strong defensive alliance of $G$ that has minimum cardinality $\hat{a}(G)$.

**Corollary 3.2.** For any $1 \leq a \leq b$, there exists a connected graph $G$ with $a = a(G) \leq b = \hat{a}(G)$.

Next, we see that any connected graph is the subgraph induced by the unique minimum strong alliance set of some graph. As with a minimum alliance, a minimum strong alliance will always induce a connected subgraph.

**Theorem 3.3.** Given a connected graph $H$, there exists a connected graph $G$ for which $H$ is the subgraph induced by the unique global (respectively, strong global) defensive alliance of $G$ with minimum cardinality $\gamma_a(G)$ (respectively, $\gamma_{\hat{a}}(G)$).

**Proof.** Adjoin every vertex of $K_{\deg_H(v)+1}$ to each vertex $v \in H$. For proof of the strong global result, adjoin every vertex of $K_{\deg_H(v)}$ to each vertex $v \in H$. □
The next result is a variation on Theorem 3.1. In the construction in Theorem 3.1, the two graphs $H_1$ and $H_2$ induced by the $a$-set and the $\hat{a}$-set, respectively, are disjoint. These two sets could also overlap. We would like to know if we can specify $H_1$, $H_2$, and the intersection of the two sets. The next result addresses this question in the case when $H_1$ is a subgraph of $H_2$.

First, a comment about notation. For a graph $H_2$ with subgraph $H_1$, we will use $H_2 - H_1$ as shorthand for the subgraph induced by the vertices $V(H_2) - V(H_1)$. If $u$ is a vertex in $H_1$, we will write $\deg_{H_2 - H_1} u$ for the number of edges joining $u$ to vertices in $H_2 - H_1$. Notice that this is a slight abuse of notation, since $u$ is not in $H_2 - H_1$.

**Theorem 3.4.** Suppose $H_2$ is a connected graph with a proper connected subgraph $H_1$ so that each of the following conditions hold:

1. $H_1$ is a defensive alliance (not necessarily minimum) in $H_2$
2. every vertex of $H_1$ is adjacent to a vertex in $H_2 - H_1$
3. the subgraph of $H_2$ induced by $V(H_2) - V(H_1)$ is connected

Then there exists a graph $G$ so that the unique minimum strong defensive alliance of $G$ is isomorphic to $H_2$ and the unique minimum defensive alliance of $G$ is $H_1$.

**Proof.** Assume all of the conditions hold. We will construct $G$ as follows. For each vertex $v$ that is in $H_2$ and not in $H_1$, attach $\deg_{H_2} v$ new end-vertices. For each vertex $u$ in $H_1$, attach $\deg_{H_1} u - \deg_{H_2 - H_1} u + 1$ new end-vertices. (Notice that $\deg_{H_1} u + 1 \geq \deg_{H_2 - H_1} u$ since $H_1$ is a defensive alliance in $H_2$.) Add a new complete subgraph $K_{2n+1}$, where $n$ is the order of $H_2$. Join each of the new end-vertices to each of the vertices in the complete graph.

**Claim 1.** $H_1$ is a defensive alliance in $G$.

Each vertex $u$ in $H_1$ is defended by itself and $\deg_{H_1} u$ neighbors. It has $\deg_{H_1} u - \deg_{H_2 - H_1} u + 1 + \deg_{H_2 - H_1} u$ enemies. Thus, it is defended.

**Claim 2.** Any other defensive alliance in $G$ has more than $V(H_1)$ vertices.

Suppose a defensive alliance contains a vertex $w$ in $G$ that is not a vertex of $H_2$. Then the alliance must also contain at least $\lceil \frac{1}{2} \deg_G w \rceil$ of the neighbors of $w$. Since every vertex $w$ not in $H_2$ has degree at least $2n$, the alliance must have at least $n + 1 > |V(H_2)| > |V(H_1)|$ vertices. Thus, we may assume without loss of generality that every defensive alliance is a subgraph of $H_2$.

Suppose a vertex $v \in H_2 - H_1$ is in a defensive alliance. Since it has at least $\deg_{H_2} v$ enemies not in $H_2$, it must have at least $\deg_{H_2} v - 1$ allies. Thus, all but one of its neighbors in $H_2$ must also be in the alliance. If the remaining neighbor is not in the alliance, then $v$ has $\deg_{H_2} v + 1$ enemies; so we can conclude that every neighbor of $v$ is in the alliance. Now, since $H_2 - H_1$ is connected, it follows that

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every vertex in $H_2 - H_1$ is in the alliance; and since every vertex of $H_1$ is adjacent to a vertex of $H_2 - H_1$, every vertex of $H_1$ is in the alliance.

Since $H_1$ is a proper subset of $H_2$, this alliance is larger than $H_1$.

Finally, suppose a proper subset of $H_1$ is a defensive alliance in $G$. Since $H_1$ is connected, there must be some $w \in V(H_1)$ which is in the alliance but adjacent to a vertex $u \in V(H_1)$ which is not in the alliance. Then $w$ has at least $\deg_{H_2-H_1}(w) + \deg_{H_1}(w) - \deg_{H_2-H_1}(w) + 1$ enemies, including $u$, and at most $\deg_{H_1}(w) - 1 + 1$ allies, counting itself. This is a contradiction.

**Claim 3.** $H_2$ is a strong defensive alliance in $G$.

Consider a vertex $v$ in $H_2 - H_1$. Since $v$ has $\deg_{H_2} v$ allies in $H_2$ and $\deg_{H_2} v$ enemies outside of $H_2$, $v$ is strongly defended. A vertex $u$ in $H_1$ has $\deg_{H_2} u$ allies in $H_2$ and $\deg_{H_1} u - \deg_{H_2-H_1} u + 1 \leq \deg_{H_2} u - 1 + 1$ enemies outside of $H_2$, so $u$ is also strongly defended.

**Claim 4.** Any other strong defensive alliance in $G$ has more than $|V(H_2)|$ vertices.

As before, if a vertex $w \notin H_2$ is in a defensive alliance, so are at least half of its neighbors. Since every vertex not in $H_2$ has degree at least $2n$, this alliance has at least $n + 1$ vertices.

We may assume without loss of generality that any smaller strong defensive alliance is a subgraph of $H_2$. Any strong alliance is also an alliance, so, as argued in Claim 2, no proper subgraph of $H_1$ can be a strong alliance. If we consider $H_1$, then each vertex has one more enemy than ally; thus, $H_1$ is not a strong defensive alliance.

Suppose a vertex $v \in H_2 - H_1$ is in a strong defensive alliance. Since $v$ has $\deg_{H_2} v$ enemies outside of $H_2$, every neighbor of $v$ must also be in the alliance. Just as in Claim 2, it follows that every vertex in $H_2$ must be in the alliance. □

Each of the conditions in the theorem is necessary to the premise of Theorem 3.4.

(i) If $H_1$ is not a defensive alliance in $H_2$, then it cannot be a defensive alliance in $G$, since we can only add more enemies. If $H_1$ is not connected and $H_1$ is a defensive alliance of $G$, then any component of $H_1$ is also a defensive alliance. Similarly, any component of a strong defensive alliance is also a strong defensive alliance.

(ii) We must have every vertex of $H_1$ adjacent to a vertex of $H_2 - H_1$. Consider the graph $H_2$ defined by $V(H_2) = \{a, b, c, d, e, f, g, h, i, j\}$ and

$$E(H_2) = \{ab, ad, ae, ai, bc, bf, cd, ci, dg, ef, eg, fh, fi, gh, gi, gj, ij\},$$

with subgraph $H_1$ induced by $\{g, i, j\}$. Notice that $H_1$ is a connected subgraph of $H_2$ and a defensive alliance of $H_2$, and the graph induced by $H_2 - H_1$ is connected. However, there is no graph $G$ that has $H_2$ as its minimum strong defensive alliance and $H_1$ as its minimum defensive alliance. Suppose there were such a graph $G$. Since
$H_1$ is a defensive alliance in $G$, there cannot be any additional vertices adjacent to $g$ or $i$ since they can barely defend themselves against the rest of $H_2$. Because $H_2$ is a strong defensive alliance, so is the graph induced by $H_2 - \{j\}$. The only vertices of $H_2$ defended by $j$ are $i$ and $g$, but they have no enemies outside of $H_2$.

(iii) Finally, we must have the subgraph induced by $H_2 - H_1$ connected. Consider the graph $H_2$ defined by $V(H_2) = \{a, b, c, d, e, f, g, h, i, j, k\}$ and

$$E(H_2) = \{ab, ac, ae, af, bd, be, cd, cf, df, ef, eg, eh, fg, gj, gi, hi, hj, ik, jk\},$$

with subgraph $H_1$ induced by vertices $e, f, g$. Then $H_1$ is connected and a defensive alliance in $H_2$, and every vertex of $H_1$ is adjacent to a vertex of $H_2 - H_1$. However, there is no graph $G$ with minimum strong alliance $H_2$ and minimum alliance $H_1$. Suppose to the contrary that there is such a $G$. Since $H_1$ is a defensive alliance in $G$, there cannot be any additional vertices adjacent to $e, f$, or $g$. However, as before, if $H_2$ is a strong defensive alliance in $G$, then so is the graph induced by $\{a, b, c, d, e, f\}$. 

![Graph Diagram](image-url)
References


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