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MULTIPLICITY RESULTS FOR SIGN-CHANGING SOLUTIONS OF AN OPERATOR EQUATION IN ORDERED BANACH SPACE

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Abstract. In this paper, we prove some multiplicity results for sign-changing solutions of an operator equation in an ordered Banach space. The methods to show the main results of the paper are to associate a fixed point index with a strict upper or lower solution. The results can be applied to a wide variety of boundary value problems to obtain multiplicity results for sign-changing solutions.

Keywords: sign-changing solution, operator equation in ordered Banach space, fixed point index

MSC 2010: 47H07, 47H10

1. INTRODUCTION

In recent years, many authors studied the existence of sign-changing solutions for various nonlinear problems, see [4]–[10] and the references therein. For example, by using the fixed point index method, the authors of [12] obtained a result of at least one sign-changing solution for the three-point boundary value problem

\[
\begin{aligned}
&y''(t) + f(y) = 0, \quad 0 \leq t \leq 1, \\
y(0) = 0, \quad \alpha y(\eta) = y(1), 
\end{aligned}
\]

where \( f \in C(\mathbb{R}, \mathbb{R}) \), \( \alpha \in [0, 1) \), \( \eta \in (0, 1) \).

Recently, the authors of [11] considered the four-point boundary value problem

\[
\begin{aligned}
&y''(t) + f(t, y(t), y'(t)) = 0, \quad 0 < t < 1, \\
y(0) = \alpha_1 y(\eta_1), \quad y(1) = \alpha_2 y(\eta_2),
\end{aligned}
\]

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where $0 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2 \leq 1$, $0 < \eta_1 < \eta_2 < 1$, $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$. They obtained in [11] the existence of at least four sign-changing solutions for the four-point boundary value problem (1.2).

By using the Leray-Schauder degree and the fixed point index method, the authors of [10] obtained the existence of at least two sign-changing solutions for some $m$-point boundary value problems. In [14] the authors obtained by using bifurcation theory some results concerning infinite sign-changing solutions for some $m$-point boundary value problems.

Let $E$ be a real Banach space which is ordered by a cone $P$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$. Consider the operator equation in a real Banach space $E$

$$x = Ax,$$

where $A = KF, K: E \rightarrow E$ is a completely continuous linear operator, $F: E \rightarrow E$ is a nonlinear continuous and bounded operator.

Let $\bar{x}$ be a non-zero solution of the operator equation (1.3). If $\bar{x} \in (-P)$ or $\bar{x} \in P$ or $\bar{x} \in E \setminus ((-P) \cup P)$, then we say $\bar{x}$ is a negative or positive or sign-changing solution of the equation (1.3), respectively. The purpose of this paper is to prove some multiplicity results for sign-changing solutions of the equation (1.3). The methods to show the results are to associate a fixed point index with a strict upper or lower solution. The results can be applied to a wide variety of boundary value problems to obtain multiplicity results for sign-changing solutions.

2. Main results

Let $\theta$ denote the zero element of $E$. In this section we will always assume that $P$ is a solid normal cone, $e \in P \setminus \{\theta\}$ and $\|e\| < 1$. Let $Q = \{x \in P; x \geq \|x\|e\}$. Then $Q$ is also a cone of $E$.

**Definition 2.1** [12]. An operator $T: D(T) \supset P \rightarrow E$ is called $e$-positive if for every $u \in P \setminus \{\theta\}$, there are numbers $\alpha = \alpha(u)$, $\beta = \beta(u) > 0$ such that

$$\alpha e \leq Tu \leq \beta e.$$

**Definition 2.2** [10]. An operator $T: D(T) \subset E \rightarrow E$ is called $e$-continuous at $x_0 \in D(T)$ if for every $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$-\varepsilon e \leq Tx - Tx_0 \leq \varepsilon e.$$
for every $x \in \mathcal{D}(T)$ with $\|x - x_0\| < \delta$. The operator $T$ is called $e$-continuous on $\mathcal{D}(T)$ if $T$ is $e$-continuous at every $x \in \mathcal{D}(T)$. From [4, Lemma 5.2] we have the following result.

**Lemma 2.1.** Let $E$ be an ordered Banach space with a solid cone $P$. Let $K : E \to E$ be a compact, $e$-positive, linear operator and let $F : E \to E$ be a map such that, for some $u_0 \in E$, $u_0 = KF(u_0)$. Suppose $F$ is Gâteaux differentiable at $u_0$ with a strictly positive derivative $F'(u_0)$. Denote by $r(T)$ the spectral radius of the operator $T = KF'(u_0)$ and by $h_0$ a positive eigenfunction of $T$ corresponding to $r(T)$. Then there exists a $\tau_0 > 0$ such that, for all $0 < \tau < \tau_0$,

$$ r(T) > 1 \quad \text{implies} \quad \begin{cases} KF(u_0 + \tau h_0) > u_0 + \tau h_0 \\ KF(u_0 - \tau h_0) < u_0 - \tau h_0 \end{cases}$$

and

$$ r(T) < 1 \quad \text{implies} \quad \begin{cases} KF(u_0 + \tau h_0) < u_0 + \tau h_0 \\ KF(u_0 - \tau h_0) > u_0 - \tau h_0 \end{cases}. $$

From [14, Theorem 19.2] we have the following Lemma 2.2.

**Lemma 2.2** (Krein–Rutman). Let $E$ be a Banach space, $P \subset E$ a total cone and let $K \in L(E)$ be compact positive with $r(K) > 0$. Then $r(K)$ is an eigenvalue with a positive eigenvector.

Let us list the following conditions which will be used in this section.

1. **(H$_1$)** $K : E \to E$ is $e$-positive, $e$-continuous and linear completely continuous, $K(P) \subset Q$, $r(K) > 0$.

2. **(H$_2$)** $F : E \to E$ is strictly increasing, bounded and continuous, $F(\theta) = \theta$, $F'(\theta) = \beta_0 I$, where $0 < \beta_0 < (r(K))^{-1}$, $I$ is the identical operator of $E$, $F'(\theta)$ denotes the Fréchet derivative of $F$ at $\theta$.

We have the following main results.

**Theorem 2.1.** Suppose that (H$_1$) and (H$_2$) hold. Moreover, let there exist $u_1, v_1 \in E \setminus ((-P) \cup P)$ and $m_0 > 0$ such that

$$ -m_0 e \leq u_1 < v_1 \leq m_0 e $$

and $u_1 < Au_1, Av_1 < v_1$. Then (1.3) has at least three sign-changing solutions $x_1, x_2$ and $x_3$. Also, (1.3) has at least one positive solution $x_4$ and one negative solution $x_5$.

**Proof.** The proof is completed in four steps.
Step 1. Clearly, \( A : E \to E \) is a strictly increasing operator. From the Krein-Rutman Theorem, the eigenvalue \( r(K) \) of the operator \( K \) has a corresponding positive eigenvector \( h_0 \). Since \( K : E \to E \) is \( e \)-positive, then there are numbers \( \alpha_{h_0}, \beta_{h_0} > 0 \) such that

\[
(2.1) \quad \alpha_{h_0} e \leq h_0 \leq \beta_{h_0} e.
\]

By Lemma 2.1, there exists \( \tau_0 > 0 \) such that for every \( \tau \in (0, \tau_0] \)

\[
(2.2) \quad -\tau h_0 < A(-\tau h_0), \quad A(\tau h_0) < \tau h_0.
\]

We claim that there exist \( \tau_1, \tau_2 \in (0, \tau_0] \) such that

\[
(2.3) \quad -\tau_1 h_0 \not\leq v_1, \quad \tau_2 h_0 \not\geq u_1.
\]

By contradiction, assume that \(-\tau h_0 \leq v_1\) for all \( \tau \in (0, \tau_0] \). Letting \( \tau \to 0 \), we have \( \theta \leq v_1 \), which contradicts \( v_1 \in E \setminus ((-P) \cup P) \). The second relation can be proved analogously. Hence, (2.3) holds. Let \( u_2 = -\tau_1 h_0 \) and \( v_2 = \tau_2 h_0 \). From (2.1)–(2.3), we have

\[
\begin{align*}
-u_2 &< Au_2, \quad Av_2 < v_2, \quad u_2 \not\geq v_1, \quad v_2 \not\leq u_1
\end{align*}
\]

and

\[
(2.4) \quad -m_1 e \leq u_2 < v_2 \leq m_1 e,
\]

where \( m_1 \) is a positive number.

Step 2. Let \( D_1 = Au_1 + Q \). Then \( D_1 \) is a closed convex subset of \( E \). For any \( x \in D_1 \) we have \( x \geq Au_1 > u_1 \). Since \( F \) is strictly increasing and \( K : P \to Q \), we have

\[
Ax - Au_1 = K(Fx - Fu_1) \geq \|K(Fx - Fu_1)\|e = \|Ax - Au_1\|e,
\]

that is

\[
Ax \geq \|Ax - Au_1\|e + Au_1.
\]

This implies that \( Ax \in D_1 \), and so \( A(D_1) \subset D_1 \).

Let sets \( \Omega_{10}, \Omega_{11} \) and \( \Omega_{12} \) be defined by

\[
\begin{align*}
\Omega_{10} &= \{x \in D_1 ; \ Ax \not\leq u_2\}, \\
\Omega_{11} &= \{x \in D_1 ; \ there \ exists \ \tau > 0 \ such \ that \ Ax \leq Av_1 - \tau e\}, \\
\Omega_{12} &= \{x \in D_1 ; \ Ax \not\leq u_2, Ax \not\leq v_1\}.
\end{align*}
\]
From (2.4) we have

\[(2.5) \quad u_2 - u_1 \leq (m_1 + m_0)e.\]

For any \( x \in D_1 \), if \( \|x\| \geq m_1 + m_0 + \|Au_1\| \), then

\[ x \geq \|x - Au_1\|e + Au_1 \geq (\|x\| - \|Au_1\|)e + u_1 \geq u_2 \]

and so

\[ Ax \geq Au_2 > u_2. \]

This implies \( \Omega_{10} \) is a bounded set. Clearly, \( \Omega_{11} \subset \Omega_{10} \), \( \Omega_{12} \subset \Omega_{10} \) and \( \Omega_{11} \cap \Omega_{12} = \emptyset \). Thus, \( \Omega_{11}, \Omega_{12} \) and \( \Omega_{10} \) are three bounded sets. Since \( K \) is \( e \)-positive, we have \( Au_1 \in \Omega_{11} \subset \Omega_{10} \), and thus \( \Omega_{11} \) and \( \Omega_{10} \) are two nonempty bounded sets. We claim that \( \Omega_{12} \neq \emptyset \). Indeed, if \( \Omega_{12} = \emptyset \), then \( D_1 = S_{11} \cup S_{12} \), where \( S_{11} = \{x \in D_1; \ Ax \leq v_1\} \) and \( S_{12} = \{x \in D_1; \ Ax \geq u_2\} \). Since \( u_2 \not\leq v_1 \), we have \( S_{11} \cap S_{12} = \emptyset \). \( S_{11} \) is a nonempty closed set since \( Au_1 \in S_{11} \). Take \( z_0 \in D_1 \) with \( \|z_0\| \geq m_0 + m_1 + \|Au_1\| \). By (2.5) we have

\[ z_0 \geq \|z_0 - Au_1\|e + Au_1 \geq (\|z_0\| - \|Au_1\|)e + Au_1 \geq (m_0 + m_1)e + u_1 \geq u_2 \]

and so \( Az_0 \geq Au_2 > u_2 \), \( z_0 \in S_{12} \). \( S_{12} \) is a nonempty closed set. Hence, the connected set \( D_1 \) can be represented as a union of two disjoint nonempty closed sets \( S_{11} \) and \( S_{12} \), which is a contradiction. Therefore, \( \Omega_{12} \neq \emptyset \).

It is easy to see that \( \Omega_{10} \) and \( \Omega_{12} \) are two open subsets of \( D_1 \). For any \( x_0 \in \Omega_{11} \) there exists \( \tau' > 0 \) such that \( Ax_0 \leq Av_1 - \tau'e \). By (H1) and (H2), \( A: E \to E \) is \( e \)-continuous. Hence, there exists \( \delta_0 > 0 \) such that

\[ -\frac{\tau'}{2}e \leq Ax - Ax_0 \leq \frac{\tau'}{2}e \]

for all \( x \in D_1 \) with \( \|x - x_0\| < \delta_0 \). Thus, for all \( x \in D_1 \) with \( \|x - x_0\| < \delta_0 \)

\[ Ax \leq Ax_0 + \frac{\tau'}{2}e \leq Av_1 - \frac{\tau'}{2}e. \]

This implies that \( x \in \Omega_{11} \). Hence, \( \Omega_{11} \) is a nonempty open subset of \( D_1 \).

Step 3. Now we will show that

\[(2.6) \quad x \neq \lambda Ax + (1 - \lambda)Au_1, \quad x \in \partial D_1, \Omega_{11}, \quad \lambda \in [0, 1],\]

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where \( \partial_{D_1} \Omega_{11} \) denotes the boundary of \( \Omega_{11} \) in \( D_1 \). Suppose this is not the case. Then there exist \( x_0 \in \partial_{D_1} \Omega_{11} \) and \( \lambda_0 \in [0,1] \) such that \( x_0 = \lambda_0 Ax_0 + (1 - \lambda_0)Au_1 \). Since \( Au_1 \in \Omega_{11} \), we have \( \lambda_0 \in (0,1) \). It is easy to see that \( Ax_0 \leq Av_1 < v_1 \), and so

\[
x_0 = \lambda_0 Ax_0 + (1 - \lambda_0)Au_1 < \lambda_0 v_1 + (1 - \lambda_0)Au_1 \leq v_1.
\]

Consequently, we have

\[
Av_1 - Ax_0 \geq \|Av_1 - Ax_0\| e,
\]

that is

\[
Ax_0 \leq Av_1 - \|Av_1 - Ax_0\| e.
\]

This implies that \( x_0 \in \Omega_{11} \), which contradicts \( x_0 \in \partial_{D_1} \Omega_{11} \). Thus, (2.6) holds.

Next we will show that \( A \) has at least one fixed point \( x_1 \in \Omega_{11} \). Clearly,

\[
u_1 < Au_1 \leq x_1 = Ax_1 \leq Av_1 < v_1.
\]

From (2.8) and the fact that \( u_1, v_1 \in E \setminus ((-P) \cup P) \) we see that \( x_1 \in E \setminus ((-P) \cup P) \).

Next we will show that \( A \) has at least one other fixed point \( x_2 \in \text{Cl}_{D_1} \Omega_{10} \setminus \Omega_{11} \), where \( \text{Cl}_{D_1} \Omega_{10} \) denotes the closure of \( \Omega_{10} \) in \( D_1 \). Assume on the contrary that \( A \) has no fixed point on \( \text{Cl}_{D_2} \Omega_{10} \setminus \Omega_{11} \). We claim that

\[
x - Ax \neq \lambda e, \ x \in \partial_{D_1} \Omega_{10}, \lambda \geq 0.
\]

In fact, assuming contrary, there exist \( x_0 \in \partial_{D_1} \Omega_{10} \) and \( \lambda_0 \geq 0 \) such that \( x_0 - Ax_0 = \lambda_0 e \). The fact that \( A \) has no fixed point on \( \partial_{D_1} \Omega_{10} \subset \text{Cl}_{D_1} \Omega_{10} \setminus (\Omega_{11} \cup \Omega_{12}) \) implies that \( \lambda_0 > 0 \). Since \( x_0 \in \partial_{D_1} \Omega_{10} (x_0 \notin \Omega_{10}) \), we have \( Ax_0 \geq u_2 \), and so \( x_0 = Ax_0 + \lambda_0 e \geq u_2 + \lambda_0 e \). By (H\(_1\)) and (H\(_2\)) we have

\[
Ax_0 \geq A(u_2 + \lambda_0 e) \geq Au_2 + \|A(u_2 + \lambda_0 e) - Au_2\| e.
\]

Let \( \gamma_0 = \|A(u_2 + \lambda_0 e) - Au_2\| > 0 \). Then there exists \( \delta_0 > 0 \) small enough such that

\[
-\frac{\gamma_0}{2}e \leq Ax - Ax_0 \leq \frac{\gamma_0}{2}e
\]

for any \( x \in D_1 \) with \( \|x - x_0\| < \delta_0 \). Thus, we have for any \( x \in D_1 \) with \( \|x - x_0\| < \delta_0 \)

\[
Ax \geq Ax_0 - \frac{\gamma_0}{2}e \geq Au_2 + \frac{\gamma_0}{2}e > Au_2 > u_2.
\]

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Take $z_0 \in \Omega_{10}$ with $\|z_0 - x_0\| < \delta_0$. Then we have $Az_0 > u_2$, which is a contradiction. Thus, (2.9) holds. For any $x \in D_1$ and $\lambda \geq 0$ we have

$$Ax + \lambda e \geq \|Ax - Au_1\|e + \lambda e + Au_1$$
$$\geq \|Ax - Au_1\|e + Au_1 + \lambda(1 - \|e\|)e$$
$$\geq \|Ax - Au_1\|e + Au_1.$$ 

Thus, $Ax + \lambda e \in D_1$ for all $x \in D_1$. Let $a = \sup_{x \in \text{Cl}_D_1\Omega_{10}} \|Ax\|$ and $b = \sup_{x \in \text{Cl}_D_1\Omega_{10}} \|x\|$. Take $s_0 > 0$ such that $s_0 \|e\| > a + b$. Let an operator $A_1$ be defined by $A_1x = Ax + s_0 e$ for all $x \in \text{Cl}_D_1\Omega_{10}$. Then we have

$$\|A_1x\| \geq s_0 \|e\| - \|Ax\| > b \geq \|x\|, \quad x \in \text{Cl}_D_1\Omega_{10}.$$ 

From the solution property of the fixed point index we have

(2.10) \hspace{1cm} i(A_1, \Omega_{10}, D_1) = 0.

Let $H(t, x) = (1 - t)Ax + tA_1x$ for all $(t, x) \in [0, 1] \times \text{Cl}_D_1\Omega_{10}$. From (2.9) we see that $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial_D_1\Omega_{10}$. Then, by the homotopy invariance property of the fixed point index and (2.10), we have

(2.11) \hspace{1cm} i(A, \Omega_{10}, D_1) = i(A_1, \Omega_{10}, D_1) = 0.

From (2.7) and (2.11) we have

$$i(A, \Omega_{12}, D_1) = i(A, \Omega_{10}, D_1) - i(A, \Omega_{11}, D_1) = -1.$$ 

Therefore, $A$ has at least one fixed point in $\Omega_{12} \subset \text{Cl}_D_1\Omega_{10} \setminus \Omega_{11}$, which is a contradiction. The contradiction obtained proves that $A$ has at least one fixed point $x_2 \in \text{Cl}_D_1\Omega_{10} \setminus \Omega_{11}$. Now we show that $u_2 \not< x_2$. Indeed, if $u_2 < x_2$, then we have

$$Ax_2 - Au_2 \geq \|Ax_2 - Au_2\|e.$$ 

Let $\gamma_1 = \|Ax_2 - Au_2\| > 0$. Since $A$ is $e$-continuous, there exists $\delta_1 > 0$ such that for any $x \in E$ with $\|x - x_2\| < \delta_1$

$$-\frac{\gamma_1}{2} e \leq Ax - Ax_2 \leq \frac{\gamma_1}{2} e$$

and so

$$Ax \geq Ax_2 - \frac{\gamma_1}{2} e \geq Au_2 + \frac{\gamma_1}{2} e > Au_2 > u_2.$$
This implies that \( B(x_2, \delta_1) \cap \Omega_{10} = \emptyset \), which contradicts \( x_2 \in \text{Cl}_{D_1} \Omega_{10} \). Thus, \( u_2 \not\in x_2 \), and so \( x_2 \not\in \emptyset \). If \( x_2 \leq \emptyset \), since \( u_1 < Au_1 \leq Ax_2 = x_2 \), we have \( u_1 < \theta \), which contradicts \( u_1 \in E \setminus ((-P) \cup P) \). Therefore, \( x_2 \in E \setminus ((-P) \cup P) \), and \( x_2 \) is a sign-changing solution of (1.3).

**Step 4.** Let \( D_2 = Av_1 - Q \), \( \Omega_{20} = \{ x \in D_2; Ax \not\leq v_2 \} \) and \( \Omega_{21} = \{ x \in D_2; \text{there exists } \tau > 0 \text{ such that } Ax \geq Au_1 + \tau e \} \). Essentially the same argument as in Step 3 shows that \( A \) has at least one fixed point \( x_3 \in \text{Cl}_{D_2} \Omega_{20} \setminus \Omega_{21} \) and \( x_3 \) is a sign-changing solution of (1.3).

Finally, we shall show the existence of positive solutions and negative solutions. Let \( D_3 = Q(= A\theta + Q) \), \( \Omega_{30} = \{ x \in D_3; Ax \not\geq u_1 \} \) and \( \Omega_{31} = \{ x \in D_3; \text{there exists } \tau > 0 \text{ such that } Ax \leq Au_2 - \tau e \} \). Then \( A \) has at least one fixed point \( x_4 \) such that \( x_4 \in \text{Cl}_{D_3} \Omega_{30} \setminus \Omega_{31} \) and \( x_4 \) is a positive solution of (1.3).

Let \( D_4 = -Q(= A\theta - Q) \), \( \Omega_{40} = \{ x \in D_4; Ax \not\geq v_1 \} \) and \( \Omega_{41} = \{ x \in D_4; \text{there exists } \tau > 0 \text{ such that } Ax \geq Au_2 + \tau e \} \). Then \( A \) has at least one fixed point \( x_5 \) such that \( x_5 \in \text{Cl}_{D_4} \Omega_{40} \setminus \Omega_{41} \) and \( x_5 \) is a negative solution of (1.3). This completes the proof. □

**Remark 2.1.** The position of \( u_1, u_2, v_1 \) can be illustrated roughly by the following figure.

[Diagram of \( \theta, P, D_1 = Au_1 + Q, u_2, x_2, Au_1 + P, v_1, x_1 \)]

**Remark 2.2.** The position of \( u_1, u_2, v_1, v_2 \) and \( x_1, x_2, x_3, x_4, x_5 \) in Theorem 2.1 can be illustrated roughly by the following figure.

[Diagram of \( \theta, x_5, u_2, v_2, x_4, x_3, x_1, v_1, x_2 \)]
Remark 2.3. The two pairs of strict lower and upper solutions \( u_1, v_1 \) and \( u_2, v_2 \) in Theorem 2.1 satisfy \( u_1 \not< v_2 \) and \( u_2 \not< v_1 \). We say these two pairs of strict lower and upper solutions are parallel to each other. We should point out that this condition was first put forward in [14]. The above \( u_1, v_2 \) and \( u_2, v_1 \) are also two pairs of non-well-ordered upper and lower solutions. For other discussions concerning the non-well-ordered upper and lower solutions, the reader is referred to [11, 5.4B].

3. Applications

Consider the two-point boundary value problem

\[
\begin{aligned}
&u''(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\
u(0) = u(1) = 0,
\end{aligned}
\]

where \( f: [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous and strictly increasing in the second argument, \( f(\cdot, 0) \equiv 0 \).

**Theorem 3.1.** Suppose that there exist \( u_1, v_1 \in C^2[0, 1] \) which are sign-changing on \([0, 1] \), \( m_0 > 0 \) such that \( u_1 \not\equiv v_1 \) on \([0, 1] \), and

\[
\begin{aligned}
&u''_1(t) + f(t, u_1(t)) > 0, \quad 0 < t < 1, \\
u_1(0) \leq 0, u_1(1) \leq 0, \\
v''_1(t) + f(t, v_1(t)) < 0, \quad 0 < t < 1, \\
v_1(0) \geq 0, v_1(1) \geq 0,
\end{aligned}
\]

\[ -m_0 t(1-t) \leq u_1(t) \leq v_1(t) \leq m_0 t(1-t), \quad t \in [0, 1] \]

and

\[ 0 < \beta_0 = \lim_{u \to 0} \frac{f(t, u)}{u} < \pi^2 \quad \text{uniformly on} \ [0, 1]. \]

Then (3.1) has at least three sign-changing solutions. Moreover, (3.1) has at least one non-zero non-negative solution and one non-zero non-positive solution.

**Proof.** Let \( E \) be the Banach space \( C[0, 1] \) with the maximum norm. Let \( P = \{ x \in E; \ x(t) \geq 0, \ t \in [0, 1] \} \). Then \( E \) is a real Banach space and \( P \) is a solid cone of \( E \). Let \( e(t) = t(1-t) \) for \( t \in [0, 1] \) and \( Q = \{ x \in P; \ x(t) \geq \|x\| e(t), \ t \in [0, 1] \} \). \( Q \) is also a cone of \( E \). Let operators \( K, F \) and \( A \) be defined by

\[
\begin{aligned}
(Kx)(t) &= \int_0^1 G(t, s)x(s) \, ds, \quad t \in [0, 1], \ x \in E, \\
(Fx)(t) &= f(t, x(t)), \quad t \in [0, 1], \ x \in E
\end{aligned}
\]

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and $A = KF$, where
\[
G(t, s) = \begin{cases} 
t(1 - s), & t \leq s, \\
s(1 - t), & t > s.
\end{cases}
\]

It is easy to see that
\[
(3.4) \quad e(t)G(\tau, s) \leq G(t, s) \leq e(t), \quad t, s, \tau \in [0, 1].
\]

For each $x \in P \setminus \{\theta\}$, we have from (3.4)
\[
\|x\|e(t) \geq (Kx)(t) \geq (Kx)(\tau)e(t), \quad t, \tau \in [0, 1], \ x \in P,
\]
and thus
\[
\|x\|e(t) \geq (Kx)(t) \geq \|Kx\|e(t), \quad t \in [0, 1], \ x \in P.
\]

This implies that $K$ is $e$-positive. Thus, we have for each $x, y \in E$
\[
-\|x - y\|e(t) \leq (K(x - y))(t) \leq \|x - y\|e(t), \quad t \in [0, 1].
\]

This implies that $K$ is $e$-continuous. The sequence of eigenvalues of $K$ is $\{(n^2\pi^2)^{-1}\}$.

Since $\lim_{x \to 0} f(t, x)/x = \beta_0$, $F$ is Fréchet differentiable at $\theta$ and $r(A'(\theta)) = \beta_0\pi^{-2} < 1$. From (3.2) and (3.3), it is easy to prove that $u_1 < Au_1$ and $Av_1 < v_1$. Consequently, all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, the conclusion of Theorem 3.1 holds.

Remark 3.1. Obviously, Theorem 2.1 can be applied to other types of nonlinear boundary value problems to obtain multiplicity results for sign-changing solutions.

References


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