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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS OF
nth-ORDER m-POINT BOUNDARY VALUE PROBLEMS

SIHUA LIANG, JIHUI ZHANG, Jiangsu

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Abstract. The paper deals with the existence of multiple positive solutions for the boundary value problem
\[
\begin{align*}
(\varphi(p(t)u^{(n-1)}(t)))' + a(t)f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) &= 0, \quad 0 < t < 1, \\
\varphi(u^{(i)}(0)) &= 0, \quad i = 0, 1, \ldots, n - 3, \\
u^{(n-2)}(0) &= \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \quad u^{(n-1)}(1) = 0,
\end{align*}
\]
where \(\varphi: \mathbb{R} \rightarrow \mathbb{R}\) is an increasing homeomorphism and a positive homomorphism with \(\varphi(0) = 0\). Using a fixed-point theorem for operators on a cone, we provide sufficient conditions for the existence of multiple positive solutions to the above boundary value problem.

Keywords: boundary-value problems, positive solutions, fixed-point theorem, cone

MSC 2010: 34B18

1. INTRODUCTION

In this paper we introduce a new operator which improves and generates a \(p\)-Laplace operator for some \(p > 1\), and we study the existence of multiple positive solutions for the \(n\)th-order \(m\)-point nonlinear boundary value problem of the form
\[
(\varphi(p(t)u^{(n-1)}(t)))' + a(t)f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1,
\]
with the boundary value condition
\[
\begin{align*}
\varphi(u^{(i)}(0)) &= 0, \quad i = 0, 1, \ldots, n - 3, \\
u^{(n-2)}(0) &= \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \quad u^{(n-1)}(1) = 0,
\end{align*}
\]
where $\varphi: \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with a positive homomorphism with $\varphi(0) = 0$. Here $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$ and $\alpha_i$ satisfy $\alpha_i \in [0, +\infty)$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $p \in C([0, 1], (0, +\infty))$, $f \in C([0, 1] \times [0, +\infty)^{n-1}, [0, +\infty))$.

A projection $\varphi: \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and a positive homomorphism if the following conditions are satisfied:

(1) if $x \leq y$, then $\varphi(x) \leq \varphi(y)$ for all $x, y \in \mathbb{R}$;
(2) $\varphi$ is a continuous bijection and its inverse is also continuous;
(3) $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}_+$.

In the above definition, condition (3) can be replaced by the following stronger condition:

(4) $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}$, where $\mathbb{R} = (-\infty, +\infty)$.

Remark 1.1. If conditions (1), (2) and (4) hold, then $\varphi$ is homogeneous and generates a $p$-Laplace operator, i.e., $\varphi(x) = |x|^{p-2}x$ for some $p > 1$.

Remark 1.2. It is well known that a $p$-Laplacian operator is odd. However, the operator which we defined above is not necessarily odd, see Example 5.1.

The multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il’in and Moiseev [6]. Since then, nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [1], [3], [4], [5], [16] and references therein. Recently, the existence and multiplicity of positive solutions for the $p$-Laplacian operator, i.e., $p(t) \equiv 1$ and $\varphi(x) = |x|^{p-2}x$ for some $p > 1$, have received wide attention, see [2], [13], [14], [15], [17] and references therein. We know that the oddness of a $p$-Laplacian operator is key to the proof. However, in this paper we define a new operator which improves and generalizes a $p$-Laplacian operator for some $p > 1$ and $\varphi$ is not necessarily odd. Moreover for increasing homeomorphism and positive homomorphism operator research has proceeded very slowly, see [10], [11]. Especially the existence of multiple positive solutions for $n$th-order $m$-point boundary value problems still remains unknown.

In [11], Liu and Zhang studied the existence of positive solutions of the quasi-linear differential equation

$$
\begin{cases}
(\varphi(x'))' + a(t)f(x(t)) = 0, & t \in (0, 1), \\
x(0) - \beta x'(0) = 0, & x(1) + \delta x'(1) = 0,
\end{cases}
$$

subject to linear mixed boundary conditions by a simple application of a fixed-point index theorem in cones, where $\varphi: \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and a positive homomorphism with $\varphi(0) = 0$. 16
Wang and Hou [15] studied the boundary value problem
\[
\begin{aligned}
&\left(\varphi_p(u')(t) + a(t)f(t,u) = 0, \quad 0 < t < 1, \\
&\varphi_p(u'(0)) = \sum_{i=1}^{n-2} a_i \varphi_p(u'(\xi_i)), \quad u(1) = \sum_{i=1}^{n-2} b_i u(\xi_i),
\end{aligned}
\]
where \(\varphi_p(s) = |s|^{p-2}s, \ p > 1\); the authors proved the existence of multiple positive solutions to the above boundary value problem by using a fixed-point theorem for operators on a cone.

In [17], Zhou and Su studied the quasi-linear equation with a \(p\)-Laplacian operator
\[
\begin{aligned}
&\left(\varphi_p(u^{(n-1)})(t) + g(t)f(u(t),u'(t),\ldots,u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \\
u^{(i)}(0) = 0, \quad 0 \leq i \leq n-3, \\
u^{(n-1)}(0) - B_0(u^{(n-1)}(\xi)) = 0, \quad n \geq 3, \\
u^{(n-1)}(0) - B_1(u^{(n-1)}(\eta)) = 0, \quad n \geq 3,
\end{aligned}
\]
where \(\varphi_p(s)\) is a \(p\)-Laplacian operator. They used the fixed-point index theory to find conditions for the existence of one solution, and of multiple solutions.

In a recent paper [8], using lower and upper solutions methods, Kong and Kong established results for solutions and positive solutions of the following two non-homogeneous boundary value conditions problems:
\[
\begin{aligned}
&x''(t) + f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\
x'(0) - m \sum_{i=1}^{m} \alpha_i x'(\xi_i) = \lambda_1, \quad x(1) - m \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2,
\end{aligned}
\]
and
\[
\begin{aligned}
&x''(t) + f(t,x(t),x'(t)) = 0, \quad t \in (0,1), \\
x(0) - m \sum_{i=1}^{m} \alpha_i x(\xi_i) = \lambda_1, \quad x(1) - m \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2,
\end{aligned}
\]
respectively.

In [12], using Mawhin’s coincidence degree theory, the author studied the more generalized BVPs for higher order differential equations with \(p\)-Laplacian subjected to non-homogeneous BCs, in which the nonlinearity \(f\) contains \(t, x, \ldots, x^{(n-1)}\).

In [7], the authors proved the existence of positive and/or negative solutions of a class of four-point boundary-value problems for fourth order ordinary differential equations by using the continuum property (connectedness and compactness) of the solutions funnel (Knesser’s Theorem), combined with the corresponding vector field.

Remark 1.3. On the one hand, we emphasize that the results of the papers [12], [15], [17] are not replaced by \(\varphi\) which we defined above; on the other hand, the assumptions and approach in the paper are different from the paper [7], [8], [12] and the function \(\varphi\) which we defined above is more comprehensive and general than the \(p\)-Laplace operator.
But whether or not we can obtain multiple positive solutions of the $n$th-order $m$-point boundary value problem (1.1) and (1.2) still remains unknown. So the goal of the present paper is to improve and generalizes $p$-Laplacian operator and establish some criteria for the existence of multiple positive solutions by means of the classical fixed-point theorem for compact maps.

We shall assume that the following conditions are satisfied.

$(C_1)$ $f \in C([0,1] \times [0, +\infty)^{n-1}, [0, +\infty))$, and $\alpha_i$ satisfy $0 < \sum_{i=1}^{m-2} \alpha_i < 1$;

$(C_2)$ $p \in C([0,1], (0, +\infty))$ is a nondecreasing function;

$(C_3)$ $a(t)$ is a nonnegative measurable function defined in $(0, 1)$ and $a(t)$ does not identically vanish on any subinterval of $(0, 1)$ and

$$0 < \int_0^1 a(t) \, dt < +\infty.$$

2. Some definitions and fixed point theorems

In this section we provide background definitions from the cone theory in Banach spaces.

Definition 2.1. Let $(E, \| \cdot \|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

(a) if $y \in P$ and $\lambda \geq 0$, then $\lambda y \in P$;

(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by $P$ on $E$ by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. A map $\alpha$ is said to be a nonnegative, continuous, concave functional on a cone $P$ of a real Banach space $E$, if

$$\alpha: P \rightarrow [0, \infty)$$

is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.3. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

To obtain positive solutions of (1.1) and (1.2) the following fixed-point theorem in cones is fundamental.
Theorem 2.1 [9]. Let $K$ be a cone in a Banach space $X$. Let $D$ be an open bounded set with $D_k = D \cap K \neq \emptyset$ and $\overline{D_k} \neq K$. Let $T : \overline{D_k} \to K$ be a compact map such that $x \neq Tx$ for $x \in \partial D_k$. Then the following results hold:

1. If $\|Tx\| \leq \|x\|$ for $x \in \partial D_k$, then $i_k(T, D_k) = 1$.
2. Suppose there is $e \in K$, $e \neq 0$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(T, D_k) = 0$.
3. Let $D^1$ be open in $X$ such that $\overline{D^1} \subset D_k$. If $i_k(T, D_k) = 1$ and $i_k(T, D^1_k) = 0$, then $T$ has a fixed point in $D_k \setminus \overline{D^1_k}$. The same result holds if $i_k(T, D_k) = 0$ and $i_k(T, D^1_k) = 1$.

3. The preliminary lemmas

The basic space used in this paper is $E = \{u \in C^{2n-2}[0,1]: u^{(i)}(0) = 0, i = 0, 1, \ldots, n - 3\}$. Thus $E$ is a Banach space when endowed with the norm $\|u\| = \sup_{t \in [0,1]} |u^{(n-2)}(t)|$. By a solution of (1.1), (1.2), we mean a function $u \in C^n[0,1]$ which satisfies (1.1), (1.2).

We can easily get the following lemmas which are useful in the proof of our main results.

Lemma 3.1. Assume that (C$_1$)–(C$_3$) hold. Then boundary value problem (1.1) and (1.2) has a solution $u \in C^n[0,1]$ if and only if $u(t)$ solves the equation

$$u(t) = \int_0^t \int_0^{\zeta_1} \cdots \int_0^{\zeta_{n-3}} w(\zeta_{n-2}) \, d\zeta_{n-2} \, d\zeta_{n-3} \cdots d\zeta_1,$$

and

$$\inf_{t \in [0,1]} u^{(n-2)}(t) \geq \gamma \|u\|$$

where

$$w(t) = \int_0^t \frac{1}{p(s)} \varphi^{-1} \left( \int_s^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \, d\tau \right) \, ds$$

$$+ \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} p(s)^{-1} \varphi^{-1} \left( \int_s^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \, d\tau \right) \, ds,$$

$$\gamma = \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i},$$

$$1 - \sum_{i=1}^{m-2} \alpha_i + \sum_{i=1}^{m-2} \alpha_i \xi_i.$$
Proof. Necessity. First suppose that \( u \in C^0[0,1] \) is a solution of problem (1.1) and (1.2). From the equation of the boundary condition we have

\[-\varphi(p(1)u^{(n-1)}(1)) + \varphi(p(t)u^{(n-1)}(t)) = \int_t^1 a(s)f(s,u(s),u'(s),\ldots,u^{(n-2)}(s))\,ds.\]

From the boundary value condition \( u^{(n-1)}(1) = 0 \) we have

\[u^{(n-1)}(t) = \frac{1}{p(t)}\varphi^{-1}\left(\int_t^1 a(s)f(s,u(s),u'(s),\ldots,u^{(n-2)}(s))\,ds\right).\]  

Integrating (3.3) from 0 to \( t \), we have

\[u^{(n-2)}(t) - u^{(n-2)}(0) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,d\tau\right)\,ds.\]  

Substituting \( u^{(n-2)}(0) = \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i) \) into (3.4), we obtain

\[u^{(n-2)}(t) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,d\tau\right)\,ds + \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i).\]  

It is easy to see by simple calculation that

\[u^{(n-2)}(t) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,d\tau\right)\,ds + \sum_{i=1}^{m-2} \alpha_i \int_0^t \varphi^{-1}\left(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,d\tau\right)\,ds \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i}.\]  

Then by integrating Eq. (3.5) \( n-2 \) times on \( [0,1] \) we find that for any \( t \in [0,1] \), \( u(t) \) can be expressed by equation (3.1).

Sufficiency. If \( u(t) \) is a solution of (3.1) direct differentiation of (3.1) we have

\[u^{(n-2)}(t) = \int_0^t \frac{1}{p(s)}\varphi^{-1}\left(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,d\tau\right)\,ds + \sum_{i=1}^{m-2} \alpha_i \int_0^t \varphi^{-1}\left(\int_s^1 a(\tau)f(\tau,u(\tau),u'(\tau),\ldots,u^{(n-2)}(\tau))\,d\tau\right)\,ds \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i}.\]  

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Obviously, for $t \in [0, 1]$ we have
\[
(\varphi(p(t)u^{(n-2)}(t)))' + a(t)f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0,
\]
\[
u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n - 3,
\]
\[
u^{(n-2)}(0) = \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \quad \nu^{(n-1)}(1) = 0.
\]
Finally, we show that (3.2) holds. It is clear that $\nu^{(n-1)}(t) \geq 0$, which implies that
\[
\|\nu\| = \nu^{(n-2)}(1), \quad \min_{t \in [0, 1]} \nu^{(n-2)}(t) = \nu^{(n-2)}(0).
\]
On the other hand, for given $s_1, s_2 \in [0, 1]$ with $s_1 \leq s_2$, one can prove that $\nu^{(n-1)}(s_2) \leq \nu^{(n-1)}(s_1)$. Hence, $\nu^{(n-1)}(t)$ is nonincreasing on $[0, 1]$.

So, for every $i \in \{1, 2, \ldots, m - 2\}$ we have
\[
\frac{\nu^{(n-2)}(1) - \nu^{(n-2)}(0)}{1} \leq \frac{\nu^{(n-2)}(\xi_i) - \nu^{(n-2)}(0)}{\xi_i},
\]
\[\text{i.e., } \nu^{(n-2)}(\xi_i) - \nu^{(n-2)}(0) \geq \xi_i \nu^{(n-2)}(1) - \xi_i \nu^{(n-2)}(0).
\]
Therefore,
\[
\sum_{i=1}^{m-2} \alpha_i \nu^{(n-2)}(\xi_i) - \sum_{i=1}^{m-2} \alpha_i \nu^{(n-2)}(0) \geq \sum_{i=1}^{m-2} \alpha_i \xi_i \nu^{(n-2)}(1) - \sum_{i=1}^{m-2} \alpha_i \xi_i \nu^{(n-2)}(0).
\]
This together with the boundary value $\nu^{(n-2)}(0) = \sum_{i=1}^{m-2} \alpha_i \nu^{(n-2)}(\xi_i)$ implies that
\[
\nu^{(n-2)}(0) \geq \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i + \sum_{i=1}^{m-2} \alpha_i \xi_i} \nu^{(n-2)}(1).
\]
The proof is complete. \(\square\)

**Lemma 3.2.** Suppose that conditions (C1)–(C3) hold. Then the solution $\nu(t)$ of problem (1.1), (1.2) satisfies
\[
\nu(t) \leq \nu'(t) \leq \ldots \leq \nu^{(n-3)}(t) \leq \nu^{(n-2)}(t), \quad 0 \leq t \leq 1.
\]

**Proof.** If $\nu(t)$ is the solution of problem (1.1), (1.2), then $\nu^{(n-2)}(t)$ is a concave function and $\nu^{(i)}(t) \geq 0$, $i = 0, 1, 2, \ldots, n - 2$, $t \in [0, 1]$. Thus we have
\[
\nu^{(i)}(t) = \int_0^t \nu^{(i+1)}(s) \, ds \leq t \nu^{(i+1)}(t) \leq \nu^{(i+1)}(t), \quad i = 0, 1, \ldots, n - 3,
\]
i.e., $\nu(t) \leq \nu'(t) \leq \ldots \leq \nu^{(n-2)}(t)$, $t \in [0, 1]$. \(\square\)

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To establish the existence of multiple positive solutions of problem (1.1) and (1.2), we construct a cone \( K \) in \( E \) by

\[ K = \{ u \in E : u^{(n-2)}(t) \geq 0, \min_{0 \leq t \leq 1} u^{(n-2)}(t) \geq \gamma \| u \| \}, \]

where \( \gamma \) is defined by Lemma 3.1. Obviously, \( K \) is a cone of \( E \).

**Remark 3.1.** We note that \( u^i(t) \leq \| u \|, t \in [0,1] \) for all \( u \in K \), \( i = 1, \ldots, n-3 \) by virtue of the definition of \( E \).

Define \( T : K \rightarrow E \) by

\[ (T u)(t) = \int_0^t \int_0^\zeta_1 \cdots \int_0^{\zeta_{n-3}} w(\zeta_{n-2}) \, d\zeta_{n-2} \, d\zeta_{n-3} \cdots d\zeta_1, \]

where

\[
\begin{align*}
  w(t) &= \int_0^t \frac{1}{p(s)} \varphi^{-1} \left( \int_s^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \, d\tau \right) \, ds \\
  &\quad + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left[ \sum_{i=1}^{m-2} \alpha_i \int_0^{\zeta_i} p(s)^{-1} \varphi^{-1} \left( \int_s^1 a(\tau) f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) \, d\tau \right) \, ds \right],
\end{align*}
\]

From (3.6) and Lemma 3.1, it is easy to obtain the following result.

**Lemma 3.3.** Let conditions \( (C_1)-(C_3) \) hold. Then \( T : K \rightarrow K \) is completely continuous.

We define

\[ K_\varrho = \{ u \in K : \| u \| \leq \varrho \}, \]

\[ \Omega_\varrho = \{ u \in K : \min_{t \in [0,1]} u^{(n-2)}(t) < \gamma \varrho \} = \{ u \in E : \gamma \| u \| \leq \min_{t \in [0,1]} u^{(n-2)}(t) < \gamma \varrho \}. \]

**Lemma 3.4** [9]. \( \Omega_\varrho \) has the following properties:

(a) \( \Omega_\varrho \) is open relative to \( K \),
(b) \( K_{\gamma \varrho} \subset \Omega_\varrho \subset K_\varrho \),
(c) \( u \in \partial \Omega_\varrho \) if and only if \( \min_{t \in [0,1]} u^{(n-2)}(t) = \gamma \varrho \),
(d) \( u \in \partial \Omega_\varrho \), then \( \gamma \varrho \leq u(t) \leq \varrho \) for \( t \in [0,1] \).
Now, we introduce the following notation. Let \( i = 1, 2, \ldots, n - 3, \) and

\[
\begin{align*}
&f^0_\gamma = \min \left\{ \min_{t \in [0,1]} \frac{f(t, u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(t)} : u^{(i)}(t) \in [0, \varrho], u^{(n-2)}(t) \in [\gamma \varrho, \varrho] \right\} , \\
f^0_0 = \max \left\{ \max_{t \in [0,1]} \frac{f(t, u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(t)} : u^{(i)}(t) \in [0, \varrho], u^{(n-2)}(t) \in [0, \varrho] \right\} , \\
f^\alpha = \lim_{u^{(n-2)}(t) \to \alpha} \sup_{t \in [0,1]} \frac{f(t, u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(u^{(n-2)})} \\
(f^\alpha = \lim_{u^{(n-2)}(t) \to \alpha} \min \min_{t \in [0,1]} \frac{f(t, u(t), u'(t), \ldots, u^{(n-2)}(t))}{\varphi(u^{(n-2)})},
\end{align*}
\]

\[
\frac{1}{I} = \frac{1}{p(0)} \varphi^{-1} \left( \int_0^1 a(\tau) \, d\tau \right) \left( 1 + \sum_{i=1}^{m-2} \alpha_i \xi_{m-2} / \left( 1 - \sum_{i=1}^{m-2} \alpha_i \right) \right),
\]

\[
\frac{1}{L} = \frac{1}{p(1)} \int_0^1 \varphi^{-1} \left( \int_s^1 a(\tau) \, d\tau \right) \, ds.
\]

4. Main results

The main results of this paper are the following.

**Theorem 4.1.** Assume that one of the following conditions holds:

\((C_4)\) There exist \( \varrho_1, \varrho_2, \varrho_3 \in (0, \infty) \) with \( \varrho_1 < \gamma \varrho_2 \) and \( \varrho_2 < \varrho_3 \) such that

\[
\begin{align*}
&f^0_\varrho_1 < \varphi(l), \quad f^0_\varrho_2 > \varphi(L), \quad f^0_\varrho_3 < \varphi(l).
\end{align*}
\]

\((C_5)\) There exist \( \varrho_1, \varrho_2, \varrho_3 \in (0, \infty) \) with \( \varrho_1 < \varrho_2 < \gamma \varrho_3 \) such that

\[
\begin{align*}
&f^0_\varrho_1 > \varphi(L), \quad f^0_\varrho_2 < \varphi(l), \quad f^0_\varrho_3 > \varphi(L).
\end{align*}
\]

Then (1.1) and (1.2) has two positive solutions \( u_1, u_2 \) with \( u_1 \in \Omega_{\varrho_2} \setminus \overline{K}_{\varrho_1}, \)

\( u_2 \in K_{\varrho_3} \setminus \overline{\Omega}_{\varrho_2}. \)

**Proof.** We only consider the condition \((C_4)\). If \((C_5)\) holds, then the proof is similar to the case when \((C_4)\) holds. Let \( T \) be a completely continuous operator that was defined by (3.6).
First, we show that $i_k(T, K_{\varphi_1}) = 1$. In fact, $f_0^g < \varphi(l)$ by (3.6) and Remark 3.1 and we have for $u \in K_{\varphi_1}$,

$$
\|Tu\| = |(Tu)^{(n-2)}(1)| = \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau)f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) d\tau\right) ds
$$

$$
\leq \int_0^1 p(0)^{-1} \varphi^{-1}\left(\int_0^1 a(\tau)f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) d\tau\right) ds
$$

$$
\frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} p(s)^{-1} \varphi^{-1}(\int_s^1 a(\tau)f(\tau, u(\tau), u'(\tau), \ldots, u^{(n-2)}(\tau)) d\tau) ds}{1 - \sum_{i=1}^{m-2} \alpha_i}
$$

$$
< \varphi^{-1}(\varphi(l)\varphi(\varphi_1)) \frac{1}{p(0)} \varphi^{-1}\left(\int_0^1 a(\tau) d\tau\right)\left(1 + \sum_{i=1}^{m-2} \alpha_i \frac{\xi_{m-2}}{(1 - \sum_{i=1}^{m-2} \alpha_i)}\right)
$$

$$
= \frac{l\varphi_1}{p(0)} \varphi^{-1}\left(\int_0^1 a(\tau) d\tau\right)\left(1 + \sum_{i=1}^{m-2} \alpha_i \frac{\xi_{m-2}}{(1 - \sum_{i=1}^{m-2} \alpha_i)}\right) = \varphi_1 = \|u\|.
$$

This implies that $\|Tu\| < \|u\|$ for $u(t) \in \partial K_{\varphi_1}$. By (1) of Theorem 2.1, we have $i_k(T, K_{\varphi_1}) = 1$.

Secondly, we show that $i_k(T, \Omega_{\varphi_2}) = 0$.

Let $e^{(n-2)}(t) \equiv 1$ for $t \in [0, 1]$. Then $e \in \partial K_1$, and we claim that

$$
u \neq Tu + \lambda e, \ u \in \partial \Omega_{\varphi_2}, \ \lambda > 0.
$$

In fact, if not, there exist $u_0 \in \partial \Omega_{\varphi_2}$ and $\lambda_0 > 0$ such that $u_0 = Tu_0 + \lambda_0 e$. By Lemma 3.3, Remark 3.1 and (3.6) we have

$$
\gamma_{\varphi_2} = \min_{t \in [0,1]} u_0^{(n-2)}(t) = (Tu_0)^{(n-2)}(t) + \lambda_0 e^{(n-2)}(t) \geq \gamma \|Tu_0\| + \lambda_0
$$

$$
= \gamma \int_0^1 \frac{1}{p(s)} \varphi^{-1}\left(\int_s^1 a(\tau)f(\tau, u_0(\tau), u_0'(\tau), \ldots, u_0^{(n-2)}(\tau)) d\tau\right) ds + \lambda_0
$$

$$
> \frac{\gamma}{p(1)} \varphi^{-1}(\varphi(L)\varphi(\varphi_2)) \int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau) d\tau\right) ds + \lambda_0
$$

$$
= \frac{\gamma}{p(1)} L\varphi_2 \int_0^1 \varphi^{-1}\left(\int_s^1 a(\tau) d\tau\right) ds + \lambda_0
$$

$$
= \gamma \varphi_2 + \lambda_0.
$$

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This implies that $\gamma \bar{q} > \gamma q + \lambda_0$, which is a contradiction. Hence by (2) of Theorem 2.1, we have $i_k(T, \Omega_{\bar{q}}) = 0$.

Finally, similarly to the proof of $i_k(T, K_{\bar{q}}) = 1$, we can prove that $i_k(T, K_{\bar{q}}) = 1$. Since $q < \gamma \bar{q}$, we have $K_{\bar{q}} \subset K_{\gamma \bar{q}} \subset \Omega_{\bar{q}}$. Therefore, Theorem 2.1 implies that problem (1.1), (1.2) has at least two positive solutions $u_1, u_2$ with $u_1 \in \Omega_{\bar{q}} \setminus K_{\bar{q}}$, $u_2 \in K_{\bar{q}} \setminus \Omega_{\bar{q}}$.

If $(C_5)$ holds, the proof is similar to the above. □

Remark 4.1. From the proof of Theorem 4.1 we know that $i_k(T, K_{\bar{q}}) = 1 \neq 0$, and we can obtain that problem (1.1) and (1.2) has a third nonnegative solution $u_3$ with $u_3 \in K_{\bar{q}}$.

As a special case of Theorem 4.1 we obtain the following result.

**Corollary 4.1.** If there exist $\bar{q}, \bar{q}' \in (0, \infty)$ with $\bar{q}' < \gamma \bar{q}$ such that one of the following conditions holds:

- $(C_6)$ $0 < f_{\bar{q}}' < \varphi(l)$, $f_{\bar{q}} > \varphi(L)$, $0 < f'^{\infty} < \varphi(l)$.
- $(C_7)$ $\varphi(L) < f_{\bar{q}'}' \leq \infty$, $f_{\bar{q}}' < \varphi(l)$, $\varphi(L) < f^{\infty} \leq \infty$.

Then (1.1) and (1.2) has two positive solutions in $K$.

**Proof.** We show that $(C_6)$ implies $(C_4)$. Let $a \in (f^{\infty}, \varphi(l))$. Then there exists $r > a$ such that $\max_{t \in [0,1]} f(t, u, u', \ldots, u^{(n-2)}) \leq a \varphi(u^{(n-2)})$ for $u^{(n-2)} \in [r, \infty)$ since $0 \leq f^{\infty} < \varphi(l)$. Let

$$
\beta = \max \left\{ \max_{t \in [0,1]} f(t, u, u', \ldots, u^{(n-2)}); \ 0 \leq u, u', \ldots, u^{(n-2)} \leq r \right\}
$$

and

$$
\bar{q}_3 > \max \left\{ \varphi^{-1} \left( \frac{\beta}{\varphi(l) - a} \right), \ q \right\}.
$$

Then we have

$$
\sup_{t \in [0,1]} f(t, u, u', \ldots, u^{(n-2)}) \leq a \varphi(u^{(n-2)}) + \beta 
$$

$$
\leq a \varphi(\bar{q}_3) + \beta < \varphi(l) \varphi(\bar{q}_3) \text{ for } u^{(n-2)} \in [0, \bar{q}_3].
$$

This implies that $f_{\bar{q}_3}^{\bar{q}_3} < \varphi(l)$ and $(C_4)$ holds. Similarly, $(C_7)$ implies $(C_5)$. □

By an argument similar to that of Theorem 4.1 we obtain the following results.
Theorem 4.2. Assume that one of the following conditions holds:

\((C_8)\) There exist \(\varrho_1, \varrho_2 \in (0, \infty)\) with \(\varrho_1 < \varrho_2\) such that \(f_0^{\varrho_1} \leq \varphi(l)\) and \(f_0^{\varrho_2} \geq \varphi(L)\).

\((C_9)\) There exist \(\varrho_1, \varrho_2 \in (0, \infty)\) with \(\varrho_1 < \varrho_2\) such that \(f_0^{\varrho_1} \leq \varphi(l)\) and \(f_0^{\varrho_2} \geq \varphi(L)\).

Then (1.1), (1.2) has a positive solution in \(K\).

As a special case of Theorem 4.2 we obtain the following result.

Corollary 4.2. Assume that one of the following conditions holds:

\((H_9)\) \(0 \leq f_0 \leq \varphi(l)\) and \(\varphi(L) < f_\infty \leq \infty\).

\((H_{10})\) \(0 \leq f_\infty < \varphi(l)\) and \(\varphi(L) < f_0 \leq \infty\).

Then (1.1) and (1.2) has a positive solution in \(K\).

Remark 4.2. This result strictly includes the sublinear and superlinear cases.

Theorem 4.1 can be generalized to obtain many solutions.

Theorem 4.3. Suppose that \((C_1)-(C_3)\) hold. Then we have the following assertions.

(1) There exist \(\{\varrho_i\}_{i=1}^{2k_0} \subset (0, \infty)\) with \(\varrho_1 < \gamma \varrho_2 < \varrho_2 < \varrho_3 < \gamma \varrho_4 < \ldots < \varrho_{2k_0}\) such that

\[ f_0^{\varrho_{2k-1}} < \varphi(l), \quad f_0^{\varrho_{2k}} > \varphi(L) \quad (k = 1, 2, \ldots, k_0). \]

Then problem (1.1), (1.2) has at least \(2k_0\) solutions in \(K\).

(2) There exist \(\{\varrho_i\}_{i=1}^{2k_0} \subset (0, \infty)\) with \(\varrho_1 < \varrho_2\) and \(\varrho_2 < \gamma \varrho_3 < \varrho_3 < \gamma \varrho_4 < \ldots < \varrho_{2k_0+2}\) such that

\[ f_0^{\varrho_{2k-1}} > \varphi(l), \quad f_0^{\varrho_{2k}} < \varphi(l) \quad (k = 1, 2, \ldots, k_0). \]

Then problem (1.1), (1.2) has at least \(2k_0 - 1\) solutions in \(K\).

Remark 4.3. If \(\varphi(u) = u\), the problem is a second boundary value problem. If \(\varphi(u) = |u|^{p-2}u, \ p > 1\), the problem is a boundary value problem with a \(p\)-Laplacian. Then our results of Theorem 4.1 and 4.2 are also new.
Example 5.1. As an example we mention the boundary value problem

\[
\begin{aligned}
& (\varphi(p(t)u^{(n-1)}(t)))' + a(t)f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \\
& u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n-3, \\
& u^{(n-2)}(0) = \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \quad u^{(n-1)}(1) = 0,
\end{aligned}
\]

(5.1)

where

\[
\varphi(u) = \begin{cases} 
    \frac{u^6}{1 + u^4}, & u \leq 0, \\
    u^6, & u > 0.
\end{cases}
\]

Here \(\alpha_i, a(t) \in ((0, 1], [0, \infty))\) and \(f\) satisfies the conditions of Theorem 4.1. It is clear that \(\varphi: \mathbb{R} \rightarrow \mathbb{R}\) is an increasing homeomorphism and a positive homomorphism with \(\varphi(0) = 0\).

Remark 5.1. Because the \(p\)-Laplacian operator is odd but the operator which we define by (5.2) is not odd, so the \(p\)-Laplacian operators does not apply to our Example 5.1. Hence we generalize boundary value problems with \(p\)-Laplacian operators and the results [2], [13], [15], [17] do not apply to Example 5.1.

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