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VOLTERRA SUMMATION EQUATIONS AND SECOND ORDER DIFFERENCE EQUATIONS

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Abstract. The asymptotic and oscillatory behavior of solutions of Volterra summation equation and second order linear difference equation are studied.

Keywords: Volterra summation equations, second order difference equations

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1. Introduction

Qualitative properties of solutions of difference equations are of great importance if we have no closed form solutions. Such properties which are widely applied are the oscillation and asymptotic behavior.

The references [1], [2] present a fairly exhaustive list for the interested reader. Some recent results for Volterra summation equations can be found in [5], [6], [7], [9], [10].

In Section 2 we establish conditions for the oscillation of solutions of equations

$$y(n) = p(n) + \sum_{s=0}^{n-1} L(n,s)y(s), \quad n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$$

and

$$\text{(I)} \qquad \Delta x(n) = p(n) - \sum_{s=0}^{n} L(n,s)g(s,x(s)), \quad n \in \mathbb{N}_0 = \{0,1,2,\ldots\}.$$

Such problems have been handled in the papers [5], [7], [8]. What we hope to accomplish here is to present new assumptions [5] about the function $L(n,s)$ ($L(n,s)$

is nonincreasing in n for every s or $L(n,s)$ is nondecreasing in s for every n or $L(n,s) = L_1(n)L_2(s)$) to obtain oscillatory properties of solutions of Volterra summation equations.

In Section 3, we give conditions under which asymptotic properties (oscillation, convergence) of the linear equation of Volterra type imply some asymptotic properties of solutions of second order linear difference equations

$$\Delta^2 x(n) - a(n)x(n+1) = 0, \quad n \in \mathbb{N}_0, \tag{II}$$

and

$$a_2(n)\Delta^2 x(n) + a_1(n)\Delta x(n) + a_0(n)x(n) = b(n), \quad a_2(n) \neq 0, n \in \mathbb{N}_0. \tag{III}$$

2. Oscillation of Volterra summation equations

In this part of the paper we establish sufficient conditions for the oscillation of solutions of the equations (I) and

$$y(n) = p(n) + \sum_{s=0}^{n-1} L(n,s)y(s) \tag{2.1}$$

where

(i) $\{p(n)\}$ is a sequence of real numbers,
(ii) $L\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}^+$ and $L(n,s) = 0$ for $s > n$,
(iii) $g\colon \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ is continuous and $xg(n,x) > 0$ for $x \neq 0$.

By a solution of equation (2.1) we mean a real sequence $\{y(n)\}$ satisfying equation (2.1) for all $n \in \mathbb{N}$.

A nontrivial solution $\{y(n)\}$ is said to be oscillatory (around zero) if for every positive integer n_0 there exists $n \geqslant n_0$ such that $y(n)y(n+1) \leqslant 0$. Otherwise, the solution is said to be nonoscillatory.

We need the following lemmas in our subsequent analysis.

Lemma 2.1. *Suppose that $\{y(n)\}, \{q(n)\}$ are nonnegative sequences defined on $\mathbb{N}_0$ and $L(n,s)$ is nonincreasing in $n \in \mathbb{N}_0$ for every $s \in \mathbb{N}_0$. If*

$$y(n) \leqslant q(n) + \sum_{s=0}^{n-1} L(n,s)y(s), \tag{2.2}$$

then

$$y(n) \leqslant Q(n)\Big\{1+\sum_{s=0}^{n-1} L(s,s)\exp\Big(\sum_{l=s+1}^{n-1} L(l,l)\Big)\Big\} \tag{2.3}$$

for $n \in \mathbb{N}_0$, *where* $Q(n) = \max\limits_{0\leqslant s\leqslant n} q(s)$.

Proof. Using the fact that $L(n,s)$ is nonincreasing in $n \in \mathbb{N}_0$ for every $s \in \mathbb{N}_0$, we arrive at

$$y(n) \leqslant q(n) + \sum_{s=0}^{n-1} L(s,s)y(s), \ n \in \mathbb{N}_0.$$

Let $v(n) = \sum\limits_{s=0}^{n-1} L(s,s)y(s)$ so that $v(0) = 0$ and

$$\begin{aligned} y(n) &\leqslant q(n) + v(n),\\ \Delta v(n) &= L(n,n)y(n). \end{aligned}$$

Hence we may write

$$v(n+1) = (1 + L(n,n))v(n) + (q(n) + r(n))L(n,n), \ r(n) \leqslant 0.$$

The solution of this equation with the initial condition $v(0) = 0$ is given by

$$v(n) = \sum_{s=0}^{n-1}(q(s) + r(s))L(s,s) \prod_{l=s+1}^{n-1} (1 + L(l,l)).$$

The proof of the lemma is completed by observing that $1 + L(n,n) \leqslant \exp(L(n,n))$, $r(n) \leqslant 0$ and $y(n) \leqslant q(n) + v(n)$. □

Remark 1. If $\limsup\limits_{n\to\infty} Q(n) < \infty$ and $\limsup\limits_{n\to\infty} \sum\limits_{s=0}^{n-1} L(s,s) < \infty$, then all $\{y(n)\}$ are bounded for $n \to \infty$.

Using Lemma 2.1, one may easily conclude the following lemmas.

Lemma 2.2. *Suppose that* $\{y(n)\}, \{q(n)\}$ *are nonnegative sequences defined on* $\mathbb{N}_0$ *and* $L(n,s)$ *is nondecreasing in* $s \in \mathbb{N}_0$ *for every* $n \in \mathbb{N}_0$. *If*

$$y(n) \leqslant q(n) + \sum_{s=0}^{n-1} L(n,s)y(s), \tag{2.4}$$

then

$$y(n) \leqslant q(n) + L(n,n)\sum_{s=0}^{n-1} q(s)\exp\Big(\sum_{l=s+1}^{n-1} L(l,l)\Big). \tag{2.5}$$

Corollary 1. *Let $q(n) \leqslant L(n,n)$ for $n \in \mathbb{N}_0$, then*

$$y(n) \leqslant L(n,n)\Big[1+\sum_{s=0}^{n-1} L(s,s)\exp\Big(\sum_{l=s+1}^{n-1} L(l,l)\Big)\Big]. \tag{2.6}$$

Remark 2. If $\limsup\limits_{n\to\infty} L(n,n) < \infty$ and $\limsup\limits_{n\to\infty} \sum\limits_{s=0}^{n-1} L(s,s) < \infty$, then all $\{y(n)\}$ are bounded for $n \to \infty$.

Lemma 2.3. *Suppose that $\{y(n)\}, \{q(n)\}$ are nonegative sequences defined on $\mathbb{N}_0$ and*

$$L(n,s) = L_1(n)L_2(s).$$

If

$$y(n) \leqslant q(n) + \sum_{s=0}^{n-1} L(n,s)y(s), \tag{2.7}$$

then

$$y(n) \leqslant q(n) + L_1(n)\sum_{s=0}^{n-1} q(s)L_2(s)\exp\Big(\sum_{l=s+1}^{n-1} L_1(l)L_2(l)\Big). \tag{2.8}$$

Remark 3. Let $q(n) \leqslant L_1(n)$ for $n \in \mathbb{N}_0$, then

$$y(n) \leqslant L_1(n)\Big\{1+\sum_{s=0}^{n-1} L_1(s)L_2(s)\exp\Big(\sum_{l=s+1}^{n-1} L_1(l)L_2(l)\Big)\Big\}.$$

If $\limsup\limits_{n\to\infty} L_1(n) < \infty$, $\limsup\limits_{n\to\infty} \sum\limits_{s=0}^{n-1} L_1(s)L_2(s) < \infty$, then all $\{y(n)\}$ are bounded for $n \to \infty$.

Theorem 2.4. *Assume that*

1° *$L(n,s)$ is nonincreasing in $n \in \mathbb{N}_0$ for every $s \in \mathbb{N}_0$,*

2° $\limsup\limits_{n\to\infty} Q(n) < \infty$, $Q(n) = \max\limits_{0\leqslant s\leqslant n} |p(s)|$, $\limsup\limits_{n\to\infty} \sum\limits_{s=0}^{n-1} L(s,s) < \infty$.

Then all unbounded solutions of equation (2.1) *are oscillatory.*

Proof. Suppose there is an unbounded nonoscillatory solution $\{y(n)\}$ of (2.1). So there exists an $n_0 \in \mathbb{N}_0$ such that either $y(n) > 0$ or $y(n) < 0$ for all $n \geqslant n_0$. Now from (2.1) we have

$$0 \leqslant |y(n)| \leqslant |p(n)| + \sum_{s=0}^{n-1} L(n,s)|y(s)|,\ n \in \mathbb{N}_0. \tag{2.9}$$

From (2.9) we have for $n \geqslant n_0$

$$
\begin{aligned}
|y(n)| &\leqslant Q(n) + \sum_{s=0}^{n_0-1} L(s,s)|y(s)| + \sum_{s=n_0}^{n-1} L(s,s)|y(s)| \\
&\leqslant M + Q(n) + \sum_{s=n_0}^{n-1} L(s,s)|y(s)|,
\end{aligned}
$$

where $M = \sum\limits_{s=0}^{n_0-1} L(s,s)|y(s)|$.

Applying Lemma 2.1 and assumption 2° to the last inequality, we obtain that $\{y(n)\}$ is bounded as $n \to \infty$. This contradiction completes the proof of the theorem.

R e m a r k 4. Suppose that the conditions of the theorem are satisfied. Then all nonoscillatory solutions of (2.1) are bounded.

E x a m p l e. Consider

$$
x(n) = \frac{1}{(n+1)^4(n+2)} + \frac{1}{(n+1)^3} \sum_{s=0}^{n-1} (s+1)x(s), \ n \in \mathbb{N}_0.
$$

Clearly, all conditions of Theorem 2.4 are satisfied. Hence all nonoscillatory solutions of the equation are bounded.

In particular,

$$
x(n) = \frac{1}{(n+1)^2(n+2)}
$$

is a bounded nonoscillatory solution of the equation.

Theorem 2.5. *Assume that*

1° $L(n,s)$ *is nonincreasing in* $n \in \mathbb{N}_0$ *for every* $s \in \mathbb{N}_0$,

2° $\limsup\limits_{n\to\infty} \sum\limits_{s=n_0}^{n-1} L(s,s) < \infty$,

3° $\limsup\limits_{n\to\infty} p(n) = \infty, \ \liminf\limits_{n\to\infty} p(n) = -\infty.$

Then all bounded solutions of (2.1) *are oscillatory.*

P r o o f. Let $\{y(n)\}$, $n \in \mathbb{N}$, be bounded solutions of (2.1) such that $|y(n)| \leqslant K$ for $n \in \mathbb{N}$. We claim that $\{y(n)\}$ is oscillatory. If not, it is nonoscillatory. So, there exists an $n_0 > 0$, $n_0 \in \mathbb{N}$, such that for $n \geqslant n_0$, either $y(n) > 0$ or $y(n) < 0$. Let

$y(n) > 0$ for $n \geqslant n_0$. From (2.1) we get for $n \geqslant n_0$

$$
\begin{aligned}
y(n) &= p(n) + \sum_{s=0}^{n_0-1} L(n,s)y(s) + \sum_{s=n_0}^{n-1} L(n,s)y(s) \\
&\leqslant p(n) + K\sum_{s=0}^{n_0-1} L(s,s) + K\sum_{s=n_0}^{n-1} L(s,s).
\end{aligned}
\tag{2.10}
$$

The last two summations on the righthand side of (2.10) are finite.

Since $y(n) > 0$ and 3° holds, we obtain a contradiction. This completes the proof. □

R e m a r k 5. Theorem 2.5 may be formulated as follows:

Suppose that the conditions of Theorem 2.5 are satisfied. Then all nonoscillatory solutions of (2.1) are unbounded.

R e m a r k 6. It is not difficult to write the equation

$$
y(n+1) = A(n)y(n) + \sum_{s=0}^{n} K(n,s)y(s) + p(n) \tag{$*$}
$$

as an equation of the form (2.1) and then to deduce the asymptotic properties of the solutions of ($*$) from the asymptotic properties of (2.1).

Denote

$$
L(n+1,s) = \begin{cases} K(n,n) + A(n) & \text{for } s = n, \\ K(n,s) & \text{for } s < n. \end{cases}
$$

Then

$$
y(n+1) = \sum_{s=0}^{n} L(n+1,s)y(s) + p(n).
$$

Next, the asymptotic behavior of oscillatory and nonoscillatory solutions of equation (I) will be studied.

Theorem 2.6. *Let $g\colon \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ be continuous and $xg(n,x) > 0$ for $x \neq 0$. Suppose that $0 < x_1 \leqslant x_2$ implies that $g(n,x_1) \leqslant g(n,x_2)$ for fixed $n \in \mathbb{N}_0$ and $L(n,s)$ satisfies assumption* (ii).

Let

$$
\sum_{s=n_1}^{n} \sum_{l=0}^{n_1-1} L(s,l) \ \text{and} \ \sum_{s=n_1}^{n} \sum_{l=n_1}^{s} L(s,l)g(l,K) \tag{2.11}
$$

be bounded for $n_1 \in \mathbb{N}$ and $K > 0$.

If

$$\lim_{n\to\infty}\sum_{s=n_1}^{n} p(s)=\infty, \tag{2.12}$$

then all bounded solutions of (I) *are oscillatory.*

Proof. Let $\{x(n)\}, n\in\mathbb{N}_0$ be a bounded solution of (I) such that $|x(n)|\leqslant K$ for $n\in\mathbb{N}_0$. We claim that $\{x(n)\}$ is oscillatory. If not, it is nonoscillatory. So there exists an $n_1>0$ such that for $n\geqslant n_1$ either $x(n)>0$ or $x(n)<0$.

Let $x(n)>0$ for $n\geqslant n_1$. From (I) we get for $n\geqslant n_1$

$$\begin{aligned}\Delta x(n) &= p(n)-\sum_{s=0}^{n_1-1}L(n,s)g(s,x(s))-\sum_{s=n_1}^{n}L(n,s)g(s,x(s))\\ &\geqslant p(n)-M\sum_{s=0}^{n_1-1}L(n,s)-\sum_{s=n_1}^{n}L(n,s)g(s,K),\end{aligned}$$

where $M=\sup\limits_{n\in\langle 0,n_1-1\rangle}|g(n,x(n))|$. So

$$\begin{aligned}x(n+1)\geqslant x(n_1)+\sum_{s=n_1}^{n}p(s)-M\sum_{s=n_1}^{n}\sum_{l=0}^{n_1-1}L(s,l)\\ -\sum_{s=n_1}^{n}\sum_{l=n_1}^{s}L(s,l)g(l,K).\end{aligned}$$

In view of conditions (2.11), the last two summations on the righthand side are finite. Since $x(n)>0$ and (2.12) holds, we obtain a contradiction.

Let $x(n)<0$ for $n\geqslant n_1$. Again from (I) we get for $n\geqslant n_1$

$$\Delta x(n)\geqslant p(n)-M\sum_{s=0}^{n_1-1}L(n,s).$$

So

$$x(n+1)\geqslant x(n_1)+\sum_{s=n_1}^{n}p(s)-M\sum_{s=n_1}^{n}\sum_{l=0}^{n_1-1}L(s,l).$$

Hence $x(n)>0$ for large n, a contradiction. This completes the proof. □

Theorem 2.7. *Let $g(n,x)$ be monotonic increasing in x for fixed $n \in \mathbb{N}_0$. Let $L(n,s)$ satisfy condition* (ii)*.*

If for large $n \in \mathbb{N}_0$

$$p(n) - \sum_{s=0}^{n} L(n,s)q(s,\lambda) > 0, \quad \lambda > 0,$$

then all bounded solutions of (I) *are nonoscillatory.*

Proof. Let $\{x(n)\}$ be a bounded solution of (I) on $\mathbb{N}_0$ such that $|x(n)| \leqslant K$, $n \in \mathbb{N}_0$.

From the given condition it follows that there exists an $n_1 \in \mathbb{N}_0$ such that

$$p(n) - \sum_{s=0}^{n} L(n,s)g(s,K) > 0 \quad \text{for } n \geqslant n_1.$$

From (I) for $n \geqslant n_1$ we obtain

$$\begin{aligned} \Delta x(n) &= p(n) - \sum_{s=0}^{n} L(n,s)g(s,x(s)) \\ &\geqslant p(n) - \sum_{s=0}^{n} L(n,s)g(s,K) > 0. \end{aligned}$$

Hence $\{x(n)\}$ is monotonic increasing and consequently $\{x(n)\}$ is nonoscillatory. $\square$

Theorem 2.8. *Assume that $xg(n,x) > 0$ for $x \neq 0$ and let $L(n,s)$ satisfy condition* (ii)*. Further assume that*

$$\sum_{s=0}^{n} p(s) \quad \textit{and} \quad \sum_{s=n_1}^{n} \sum_{l=0}^{n_1} L(s,l)$$

are bounded. Then all unbounded solutions of (I) *are oscillatory.*

Proof. Let $\{x(n)\}$ be an unbounded solutions of (I) on $\mathbb{N}_0$. Let $\{x(n)\}$ be nonoscillatory. So it is ultimately positive or ultimately negative. Let $\{x(n)\}$ be ultimately positive.

So there exists an $n_1 \in \mathbb{N}$ such that $x(n) > 0$ for $n \geqslant n_1$.

For $n \geqslant n_1$ we have by (I)

$$\begin{aligned}\Delta x(n) &= p(n) - \sum_{s=0}^{n_1-1} L(n,s)g(s,x(s)) - \sum_{s=n_1}^{n} L(n,s)g(s,x(s))\\ &\leqslant p(n) + M\sum_{s=0}^{n_1-1} L(n,s)\end{aligned}$$

where $M = \sup\limits_{0\leqslant n\leqslant n_1-1} |g(n,x(n))|$.

So, for $n \geqslant n_1$ we obtain

$$0 < x(n+1) \leqslant x(n_1) + \sum_{s=n_1}^{n} p(s) + M\sum_{s=n_1}^{n}\sum_{s=0}^{n_1-1} L(n,s).$$

Hence $\{x(n)\}$ is bounded, a contradiction. Analogously for $\{x(n)\}$ ultimately negative. Thus the theorem is proved. □

Theorem 2.9. *Let $0 < x_1 \leqslant x_2$ imply that $g(n,x_1) \leqslant g(n,x_2)$ for each fixed $n \in \mathbb{N}_0$, let $g(n,-x) = -g(n,x)$, and let $L(n,s)$ satisfy condition* (ii). *Further assume that*

$$\sum_{s=0}^{n} p(n) \text{ and } \sum_{s=n_1}^{n}\sum_{l=0}^{n_1-1} L(s,l)$$

are bounded.

If $\lim\limits_{n\to\infty} \sum\limits_{s=n_1}^{n}\sum\limits_{l=n_1}^{s} L(s,l)g(l,\lambda) = \infty$ for $\lambda > 0$, then there are no nontrivial bounded solutions.

P r o o f. Let $\{x(n)\}$ be a nonoscillatory solution of (I) on $\mathbb{N}_0$ that is bounded away from zero as $n \to \infty$. So there exist an $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that for $n \geqslant n_0$ we have $|x(n)| \geqslant \varepsilon$. Let $\{x(n)\}$ be ultimately positive; then there exists an $n_1 > n_0$ such that $x(n) > 0$ for $n \geqslant n_1$. Hence $x(n) \geqslant \varepsilon$ for $n \geqslant n_1$. Now for $n \geqslant n_1$ we have

$$\begin{aligned}\Delta x(n) &= p(n) - \sum_{s=0}^{n_1-1} L(n,s)g(s,x(s)) - \sum_{s=n_1}^{n} L(n,s)g(s,x(s))\\ &\leqslant p(n) + M\sum_{s=0}^{n_1-1} L(n,s) - \sum_{s=n_1}^{n} L(n,s)g(s,\varepsilon)\end{aligned}$$

where $M = \sup\limits_{0\leqslant s\leqslant n_1-1} |g(s,x(s))|$.

Hence

$$x(n+1) \leqslant x(n_1) + \sum_{s=n_1}^{n} p(s) + M \sum_{s=n_1}^{n} \sum_{l=0}^{n_1-1} L(s,l) - \sum_{s=n_1}^{n} \sum_{l=n_1}^{s} L(s,l) g(l,\varepsilon).$$

It is easy to see that $0 \leqslant \limsup_{n\to\infty} x(n+1) < 0$, a contradiction. The proof of the case $x(n) < 0$ for $n \geqslant n_1 > n_0$ is similar. The theorem is proved. $\square$

Theorem 2.10. *Let $g(n,x)$ be monotonic increasing in x for fixed n. Let $L(n,s)$ satisfy condition* (ii). *Let*

$$\sum_{s=n_0}^{n} \sum_{l=0}^{n_0-1} L(s,l), \qquad \sum_{s=n_0}^{n} \sum_{l=n_0}^{s} L(s,l) g(l,\lambda)$$

be bounded for $n_0 \in \mathbb{N}$ and $\lambda > 0$.

If $\lim_{n\to\infty} \sum_{s=0}^{n} p(s) = \infty$, then no oscillatory solution of (I) *such that the set $\{n \in \mathbb{N}\colon x(n) = 0\}$ is unbounded, goes to zero as $n \to \infty$.*

P r o o f. Let $\{x(n)\}$ be an oscillatory solution of (I) on $\mathbb{N}_0$ such that the set $\{n \in \mathbb{N}\colon x(n) = 0\}$ is unbounded. Let $\lim_{n\to\infty} x(n) = 0$. So for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|x(n)| < \varepsilon$ for $n \geqslant n_0$.

Let $m_n \in \mathbb{N}$ be a sequence of zeros of $\{x(n)\}$ such that $m_n \to \infty$ as $n \to \infty$. Choose n large enough so that $m_n > n_0$. From (I) we get for $n \geqslant n_0$

$$\begin{aligned}\Delta x(n) &= p(n) - \sum_{s=0}^{n} L(n,s) g(s,x(s)) \\ &= p(n) - \sum_{s=0}^{n_0-1} L(n,s) g(s,x(s)) - \sum_{s=n_0}^{n} L(n,s) g(s,x(s)),\end{aligned}$$

so

$$x(n+1) \geqslant x(n_0) + \sum_{s=n_0}^{n} p(s) - M \sum_{s=n_0}^{n} \sum_{l=0}^{n_0-1} L(s,l) - \sum_{s=n_0}^{n} \sum_{l=n_0}^{s} L(s,l) g(s,\varepsilon)$$

where $M = \sup\limits_{0 \leqslant s \leqslant n_0-1} |g(s, x(s))|$. Hence

$$x(m_n) \geqslant x(n_0) + \sum_{s=n_0}^{m_n-1} p(s) - M \sum_{s=n_0}^{m_n-1} \sum_{l=0}^{n_0-1} l(s,l) \\ - \sum_{s=n_0}^{m_n-1} \sum_{l=0}^{s} L(s,l) g(s,\varepsilon),$$

that is

$$\sum_{s=n_0}^{m_n-1} p(s) \leqslant M \sum_{s=n_0}^{m_n-1} \sum_{l=0}^{n_0-1} L(s,l) + \sum_{s=n_0}^{m_n-1} \sum_{l=0}^{s} L(s,l) g(s,\varepsilon) - x(n_0).$$

Consequently $\lim\limits_{n\to\infty} \sum\limits_{s=n_0}^{m_n-1} p(s) < \infty$, a contradiction. This completes the proof of the theorem. □

3. Main results

Proposition [7]. *Suppose that there exist $A(n)$, $a_0(n)$, $a_1(n)$, $a_2(n)$, $B(n)$, $a_2(n) \neq 0$, $A(n) \neq 1$ for $n \geqslant n_0 \geqslant 0$. Moreover, suppose that there exists a solution $\{y(n)\}$ of the equation*

$$y(n) = f(n) + \sum_{s=n_0}^{n-1} K(n,s) y(s) \tag{3.1}$$

where

$$\begin{aligned} f(n) &= c + \frac{c_1}{g(n)} + \frac{1}{g(n)} \sum_{s=n_0}^{n-1} B(s) \Delta g(s), \\ g(n) &= \prod_{s=n_0}^{n-1} \Big(1 + \frac{1}{A(s)-1}\Big), \\ K(n,s) &= \frac{\Delta g(s)}{g(n)} \varphi(s) - \psi(s), \qquad (3.2) \\ \varphi(n) &= A(n)\Big(\psi(n) - \frac{a_1(n-1)}{a_2(n-1)}\Big) + 1 + \Delta A(n-1), \\ \psi(n) &= \Delta^2 A(n-1) + A(n+1) \frac{a_0(n)}{a_2(n)} - \Delta\Big(A(n) \frac{a_1(n-1)}{a_2(n-1)}\Big), \end{aligned}$$

$c, c_1 = \text{const.}$

Then $\{y(n)\}$ *satisfies the difference equation*

$$a_2(n)\Delta^2 y(n) + a_1(n)\Delta y(n) + a_0(n)y(n) = b(n), \quad b(n) = \frac{a_2(n)\Delta B(n)}{A(n+1)}. \tag{3.3}$$

Theorem 3.1. *Suppose that*

1° $K(n,s)$ *satisfies condition* (ii),

2° $K(n,s)$ *is nonincreasing in* $n \in \mathbb{N}$ *for every* $s \in \mathbb{N}$,

3° $\limsup\limits_{n\to\infty} Q(n) < \infty$, $Q(n) = \max\limits_{n_0 \leqslant s \leqslant n} |f(s)|$,

4° $\limsup\limits_{n\to\infty} \sum\limits_{s=n_0}^{n-1} K(s,s) < \infty$.

Then the difference equation (3.3) *has unbounded oscillatory solutions.*

P r o o f. See Theorem 2.4 and Proposition. □

Theorem 3.2. *Suppose that*

1° $K(n,s)$ *satisfies conditions* 1°, 2° *of Theorem* 3.1,

2° $\limsup\limits_{n\to\infty} \sum\limits_{s=n_0}^{n-1} K(s,s) < \infty$,

3° $\limsup\limits_{n\to\infty} f(n) = \infty$, $\liminf\limits_{n\to\infty} f(n) = -\infty$.

Then the difference equation (3.3) *has bounded oscillatory solutions.*

P r o o f. See Theorem 2.5 and Proposition.

Now we consider the equation

$$\Delta^2 x(n) - a(n)x(n+1) = 0 \tag{3.4}$$

$n \in \mathbb{N}$, $\{a(n)\}$ is a sequence defined for $n \in \mathbb{N}$, $a(n) \neq 0$ for all $n \in \mathbb{N}$. We shall prove a theorem about asymptotic properties of solutions of equation (3.4). In the proof of this theorem we shall use theorems from this part of the paper and the following theorem. □

Theorem 3.3 [7]. *Suppose that*

1° *there exist functions* A, a_2, a_1, a_0, b *for* $n \geqslant n_0$,

2° $A(n) > 1$, $a_2(n) \neq 0$ *for* $n \geqslant n_0$,

3° $\sum\limits_{n=n_0}^{\infty} 1/A(n) = \infty$,

4° $\lim\limits_{n\to\infty} \varphi(n) = 0$,

5° $\sum\limits_{n=n_0}^{\infty} A(n+1)b(n)/a_2(n) = s \quad (|s| < \infty)$,

6° $\sum\limits_{n=n_0}^{\infty} |\psi(n)| < \infty$

where φ, ψ are defined in part III (3.2). *Then there exists a solution* $\{y(n)\}$ *of difference equation* (3.3) *such that* $\lim_{n\to\infty} y(n) = 1$.

In equation (3.4) we represent the function $x(n)$ in the form

$$x_k(n) = \frac{a^{-\frac{1}{4}}(n)}{\prod_{s=0}^{n-2}\bigl(1+\varepsilon_k a^{\frac{1}{2}}(n-s)\bigr)}\, y(n)$$

where $a(n) > 0$ for $n \in \mathbb{N}$, $\varepsilon_k = \mathrm{e}^{k\pi\mathrm{i}}$, $k = 1, 2$, $a(n) \neq 1$, and we obtain the difference equation

$$a_2(n)\Delta^2 y(n) + a_1(n)\Delta y(n) + a_0(n)y(n) = 0 \tag{3.5}$$

where

$$\begin{aligned}
&a_2(n) = a^{-\frac{1}{4}}(n+1),\\
&a_1(n) = 2a^{-\frac{1}{4}}(n+2) - (2+a(n))a^{-\frac{1}{4}}(n+1)(1+\varepsilon_k a^{\frac{1}{2}}(n+2)),\\
&a_0(n) = (1+\varepsilon_k a^{\frac{1}{2}}(n+2))[a^{-\frac{1}{4}}(n)(1+\varepsilon_k a^{\frac{1}{2}}(n+1)) - (2+a(n))a^{-\frac{1}{4}}(n+1)].
\end{aligned}$$

Theorem 3.4. *Suppose that*

1° A, a_2, a_1, a_0 *are defined for* $n \geqslant n_0 \geqslant 0$, $A(n) > 1$, $A(n+1) = a_2(n)$, $b(n) = 0$,

2° $\sum_{n=n_0}^{\infty} 1/a_2(n-1) = \infty$,

3° $\sum_{n=n_0}^{\infty} |\psi(n)| < \infty$, $\psi(n) = \Delta^2 a_2(n-2) + a_0(n) - \Delta a_1(n-1)$,

4° $\lim_{n\to\infty} \varphi(n) = 0$, $\varphi(n) = a_2(n-1)\psi(n) - a_1(n+1) + 1 + \Delta a_2(n-2)$.

Then the difference equation (3.4) *has for* $n \geqslant n_0 \geqslant 0$ *solutions* $\{x_1(n)\}$ *and* $\{x_2(n)\}$ *such that*

$$x_k(n) \sim \frac{a^{-\frac{1}{4}}(n)}{\prod_{s=0}^{n-2}\bigl(1+\varepsilon_k a^{\frac{1}{2}}(n-s)\bigr)}, \quad k = 1, 2.$$

P r o o f. By Theorem 3.3 we obtain under our hypotheses that the difference equation (3.5) has for $n \geqslant n_0 \geqslant 0$ a solution $\{y(n)\}$ such that

$$\lim y(n) = 1 \quad \text{as } n \to \infty.$$

Then the functions

$$x_k(n) = \frac{a^{-\frac{1}{4}}(n)}{\prod_{s=0}^{n-2}\bigl(1+\varepsilon_k a^{\frac{1}{2}}(n-s)\bigr)} y(n)$$

for $k = 1, 2$ satisfy difference equation (3.4) and we have

$$x_k(n) \sim \frac{a^{-\frac{1}{4}}(n)}{\prod\limits_{s=0}^{n-2}\big(1 + \varepsilon_k a^{\frac{1}{2}}(n-s)\big)} \quad \text{as } n \to \infty.$$

The proof is complete. $\square$

One of the most effective techniques to study (3.4) is to make the change of variables

$$x(n) = 2\Big(-\frac{1}{2}\Big)^{n+1} \prod_{s=n_0}^{n-2} (2 + a(s))y(n), \quad n_0 \geqslant 0.$$

Then (3.4) is transformed to

$$a_2(n)\Delta^2 y(n) + a_1(n)\Delta y(n) + a_0(n)y(n) = 0,$$

where

$$\begin{aligned} a_2(n) &= (2 + a(n))(2 + a(n-1)), \\ a_1(n) &= 4(2 + a(n))(2 + a(n-1)), \\ a_0(n) &= 4 + 3(2 + a(n))(2 + a(n-1)). \end{aligned}$$

Theorem 3.5. *Suppose that*

1° *the assumptions of Theorem* 3.4 *are satisfied,*

2° $a(n) \geqslant -1$ *for* $n \geqslant n_0$.

Then the difference equation (3.4) *has for* $n \geqslant n_0$ *a solution* $\{x_1(n)\}$ *such that*

$$x_1(n) \sim 2\Big(-\frac{1}{2}\Big)^{n+1} \prod_{s=n_0}^{n-2} (2 + a(s)) \quad \text{for} \quad n \to \infty.$$

If in addition we have

(i) $\sum\limits_{n=n_0}^{\infty} (1 + a(n)) < \infty$,

(ii) $\lim\limits_{n\to\infty} (-\frac{1}{2})^{-2n} \prod\limits_{s=n_0}^{n-2} (2 + a(s))^{-2} = \infty$ *then there exists a solution* $\{x_2(n)\}$ *of equation* (3.4) *such that*

$$x_2(n) \sim \frac{1}{2\left(-\frac{1}{2}\right)^{n+1} \prod\limits_{s=n_0}^{n-2} (2 + a(s))} \quad \text{for} \quad n \to \infty.$$

P r o o f. The proof of the first part of Theorem 3.5 is analogous to the proof of Theorem 3.4, where $x_1(n) = 2\left(-\frac{1}{2}\right)^{n+1} \prod_{s=n_0}^{n-2} (2 + a(s))\, y(n)$. We shall prove part two.

The function

$$x_2(n) = \frac{3}{2} x_1(n) \sum_{s=n}^{\infty} \frac{1}{x_1(s) x_1(s+1)}$$

is for $n \geqslant n_0$ the solution of the difference equation (3.4) for which

$$x_2(n)\Delta x_1(n) - x_1(n)\Delta x_2(n) \neq 0.$$

Then

$$\frac{x_2(n)}{2^{-1}\left(-\frac{1}{2}\right)^{-n-1} \prod\limits_{s=n_0}^{n-2} (2+a(s))^{-1}} = \frac{\frac{3}{2} x_1(n) \sum\limits_{s=n}^{\infty} 1/x_1(s)x_1(s+1)}{2^{-1}\left(-\frac{1}{2}\right)^{-n-1} \prod\limits_{s=n_0}^{n-2} (2+a(s))^{-1}}$$

$$\sim \frac{\frac{3}{2} 2\left(-\frac{1}{2}\right)^{n+1} \prod\limits_{s=n_0}^{n-2} (2+a(s)) \sum\limits_{s=n}^{\infty} 1/x_1(s)x_1(s+1)}{2^{-1}\left(-\frac{1}{2}\right)^{-n-1} \prod\limits_{s=n_0}^{n-2} (2+a(s))^{-1}}$$

$$\sim \frac{\frac{3}{2} \sum\limits_{s=n}^{\infty} 1/x_1(s)x_1(s+1)}{\left(-\frac{1}{2}\right)^{-2n} \prod\limits_{s=n_0}^{n-2} (2+a(s))^{-2}} \quad \text{for } n \to \infty.$$

□

Theorem A [6]. *Let $\{s_n\}, \{a_n\}, \{b_n\}$ be given sequences. The hypothesis $\lim\limits_{n\to\infty} s_n = s$ implies $\lim\limits_{n\to\infty} (a_n/b_n) = s$ if*

1a) $|b_n| \to \infty$ *and* $\sum\limits_{s=0}^{n-1} |\Delta b_s| \leqslant K|b_n|$ *or*

1b) $a_n \to 0$, $b_n \to 0$, $b_n \neq 0$ *for infinitely many indices* n *and* $\sum\limits_{s=n}^{\infty} |\Delta b_s| \leqslant K|b_n|$ *(where the constant K does not depend on n),*

2) $\Delta a_n = s_n \Delta b_n$.

Now the assumption of Theorem 3.5 and Theorem A imply that

$$\lim_{n\to\infty} \frac{\frac{3}{2} \sum\limits_{s=n}^{\infty} 1/x_1(s)x_1(s+1)}{\left(-\frac{1}{2}\right)^{-2n} \prod\limits_{s=n_0}^{n-2} (2+a(s))^{-2}} = 3 \cdot \lim_{n\to\infty} \frac{(2+a(n-1))}{[-a(n-1)\,(4+a(n-1))]} = 1$$

and

$$x_2(n) \sim \frac{1}{2\left(-\frac{1}{2}\right)^{n+1} \prod\limits_{s=n_0}^{n-2} (2+a(s))} \quad \text{for} \quad n \to \infty,$$

hence the proof is complete.

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