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A THIRD ORDER BOUNDARY VALUE PROBLEM SUBJECT TO
NONLINEAR BOUNDARY CONDITIONS

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Abstract. Utilizing the theory of fixed point index for compact maps, we establish new results on the existence of positive solutions for a certain third order boundary value problem. The boundary conditions that we study are of nonlocal type, involve Stieltjes integrals and are allowed to be nonlinear.

Keywords: positive solution, nonlinear boundary conditions, third order problem, cone, fixed point index

MSC 2010: 34B18, 34B10, 47H10, 47H30

1. INTRODUCTION

In a very interesting paper [6], Graef and Webb studied the existence of multiple solutions for the nonlinear third order differential equation

$$(1.1) \quad u'''(t) = g(t)f(t, u(t)), \quad t \in (0, 1),$$

subject to the nonlocal boundary conditions (BCs)

$$(1.2) \quad u(0) = 0, \quad u'(p) = 0, \quad u''(1) = \lambda[u''],$$

where $p \in [1/2, 1]$ and $\lambda[\cdot]$ is a linear functional on the space $C[0, 1]$ given by a Stieltjes integral, namely

$$(1.3) \quad \lambda[v] = \int_0^1 v(s) d\Lambda(s),$$

with $d\Lambda$ a signed measure. The formulation (1.3) is quite general and includes, as special cases,

$$\lambda[v] = \sum_{i=1}^m \lambda_i v(\xi_i) \quad \text{and} \quad \lambda[v] = \int_0^1 \lambda(s)v(s) ds,$$

that is, m -point and integral conditions.

Nonlocal boundary conditions, in the case of third order equations, have been studied recently by several authors, see for example the papers by Anderson and Davis [1], Clark and Henderson [4], Palamides and Palamides [24], Palamides and Smyrlis [25], Wang and Ge [26], Yang [31], Yao [33] and references therein.

One motivation given in [6] is that the BCs (1.2) can be seen as a generalization of the BCs that occur in a third order problem studied by Graef and Yang [7] and extended to the higher order case by Graef, Henderson and Yang [8].

The methodology in [6] is to rewrite the BVP (1.1)–(1.2) as a Hammerstein integral equation of the form

$$(1.4) \quad u(t) = \int_0^1 k_\lambda(t, s)g(s)f(s, u(s)) ds.$$

In order to establish existence and nonexistence results for the equation (1.4), Graef and Webb make use of a careful analysis of the Green function k_λ combined with an earlier theory developed by Webb and co-authors [29], [30].

Furthermore, in the paper [6], by making use of the results of [29] that deal with perturbed Hammerstein integral equations of the form

$$(1.5) \quad u(t) = \gamma(t)\tilde{\alpha}[u] + \delta(t)\tilde{\beta}[u] + \int_0^1 k(t, s)g(s)f(s, u(s)) ds,$$

the more general nonlocal BCs

$$u(0) = \tilde{\alpha}[u], \quad u'(p) = 0, \quad u''(1) + \tilde{\beta}[u] = \lambda[u''],$$

where $\tilde{\alpha}[\cdot]$ and $\tilde{\beta}[\cdot]$ are linear functionals on $C[0, 1]$ given by Stieltjes integrals with signed measures, are studied.

In [14] Infante, motivated by earlier work of Guidotti and Merino [9], Infante and Webb [17], [18], Webb [27], [28], and Palamides, Infante and Pietramala [23], studied a thermostat model with nonlinear controllers. The approach used in [14] relied on an extension of the results of [29], valid for equations of the type (1.5), to the context of nonlinear perturbations of the form

$$(1.6) \quad u(t) = \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + \int_0^1 k(t, s)g(s)f(s, u(s)) ds,$$

where H_1, H_2 are continuous functions such that there exist $h_{11}, h_{12}, h_{21}, h_{22} \in [0, \infty)$ with

$$(1.7) \quad h_{11}v \leq H_1(v) \leq h_{12}v \quad \text{and} \quad h_{21}v \leq H_2(v) \leq h_{22}v$$

for every $v \geq 0$. Unlike the results of [29], due to some inequalities involved in the theory, the functionals $\alpha[\cdot]$ and $\beta[\cdot]$ are assumed to be given by *positive* measures.

Here we focus on the boundary value problem (BVP)

$$\begin{aligned} u'''(t) &= g(t)f(t, u(t)), \quad t \in (0, 1), \\ u(0) &= H_1(\alpha[u]), \quad u'(p) = H_2(\beta[u]), \quad u''(1) = \lambda[u''], \quad p \in [1/2, 1], \end{aligned}$$

where the functions H_1, H_2 and the functionals $\alpha[\cdot]$ and $\beta[\cdot]$ are as above.

BVPs with nonlinear BCs have been studied recently by several authors, see for example the papers by Cabada, Minhós and Santos [3], Franco and O'Regan [5], Infante [11], [12], [14], Infante and Pietramala [16], Kong and Wang [19], Minhós [22], Yang [32] and references therein.

Here we utilize some of the results of [6] to show that our BVP fits exactly the framework of [14].

We prove, via the classical fixed point index theory, the existence of multiple positive solutions.

2. SOME PRELIMINARY RESULTS ON THE INTEGRAL EQUATION

We first recall some results from [14]. The assumptions made on the terms that occur in the perturbed Hammerstein integral equation

$$u(t) = \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + \int_0^1 k(t, s)g(s)f(s, u(s)) \, ds := Tu(t),$$

are as follows:

- $f: [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous.
- $k: [0, 1] \times [0, 1] \rightarrow [0, \infty)$ is continuous.
- There exist a subinterval $[a, b] \subseteq [0, 1]$, a function $\Phi \in L^\infty[0, 1]$, and a constant $c_1 \in (0, 1]$ such that

$$\begin{aligned} k(t, s) &\leq \Phi(s) \quad \text{for } t \in [0, 1] \text{ and almost every } s \in [0, 1], \\ k(t, s) &\geq c_1\Phi(s) \quad \text{for } t \in [a, b] \text{ and almost every } s \in [0, 1]. \end{aligned}$$

- $g\Phi \in L^1[0, 1]$, $g \geq 0$ a.e., and $\int_a^b \Phi(s)g(s) \, ds > 0$.

- A, B are functions of bounded variation. Here dA and dB are *positive* measures and we use the notation

$$\mathcal{K}_A(s) := \int_0^1 k(t, s) dA(t) \text{ and } \mathcal{K}_B(s) := \int_0^1 k(t, s) dB(t).$$

- $\gamma \in C[0, 1], \gamma(t) \geq 0, h_{12}\alpha[\gamma] < 1$. There exists $c_2 \in (0, 1]$ such that

$$\gamma(t) \geq c_2 \|\gamma\| \text{ for } t \in [a, b].$$

- $\delta \in C[0, 1], \delta(t) \geq 0, h_{22}\beta[\delta] < 1$. There exists $c_3 \in (0, 1]$ such that

$$\delta(t) \geq c_3 \|\delta\| \text{ for } t \in [a, b].$$

- $D_2 := (1 - h_{12}\alpha[\gamma])(1 - h_{22}\beta[\delta]) - h_{12}h_{22}\alpha[\delta]\beta[\gamma] > 0$.

Under the above hypotheses, the compact operator T leaves invariant the cone

$$(2.1) \quad K = \left\{ u \in C[0, 1], u \geq 0: \min_{t \in [a, b]} u(t) \geq c \|u\| \right\},$$

where $c = \min\{c_1, c_2, c_3\}$. This type of cone was used first by Krasnosel'skiĭ, see e.g. [20], and D. Guo, see e.g. [10], and later by several authors.

We utilize the classical fixed point index theory for compact maps (see for example [2] or [10]) and we work with the following open bounded sets (relative to K):

$$K_\varrho = \{u \in K: \|u\| < \varrho\}, \quad V_\varrho = \left\{ u \in K: \min_{a \leq t \leq b} u(t) < \varrho \right\}.$$

The set V_ϱ is equal to the set called $\Omega_{\varrho/c}$ in [21] (here c is from (2.1)). A key feature of these sets is that they can be nested, that is

$$K_\varrho \subset V_\varrho \subset K_{\varrho/c}.$$

We make use of the quantity

$$D_1 := (1 - h_{11}\alpha[\gamma])(1 - h_{21}\beta[\delta]) - h_{11}h_{21}\alpha[\delta]\beta[\gamma],$$

and observe that the condition $D_2 > 0$ implies $D_1 > 0$.

The following lemma gives a condition allowing the index to be 0 on the set V_ϱ .

Lemma 1 [14]. Assume that there exists $\varrho > 0$ such that

$$(2.2) \quad f_{\varrho, \varrho/c} \left(\left(\frac{c_2 \|\gamma\|}{D_1} (1 - h_{21} \beta[\delta]) + \frac{c_3 \|\delta\|}{D_1} h_{11} \beta[\gamma] \right) \int_a^b \mathcal{K}_A(s) g(s) ds \right. \\ \left. + \left(\frac{c_2 \|\gamma\|}{D_1} h_{21} \alpha[\delta] + \frac{c_3 \|\delta\|}{D_1} (1 - h_{11} \alpha[\gamma]) \right) \int_a^b \mathcal{K}_B(s) g(s) ds + \frac{1}{M} \right) > 1,$$

where

$$f_{\varrho, \varrho/c} = \inf \left\{ \frac{f(t, u)}{\varrho} : (t, u) \in [a, b] \times [\varrho, \varrho/c] \right\} \quad \text{and} \quad \frac{1}{M} = \inf_{t \in [a, b]} \int_a^b k(t, s) g(s) ds.$$

Then the fixed point index, $i_K(T, V_\varrho)$, is 0.

The next result gives a sufficient condition for the index to be 1 on the set K_ϱ .

Lemma 2 [14]. Assume that there exists $\varrho > 0$ such that

$$(2.3) \quad f^{0, \varrho} \left(\left(\frac{\|\gamma\|}{D_2} (1 - h_{22} \beta[\delta]) + \frac{\|\delta\|}{D_2} h_{12} \beta[\gamma] \right) \int_0^1 \mathcal{K}_A(s) g(s) ds \right. \\ \left. + \left(\frac{\|\gamma\|}{D_2} h_{22} \alpha[\delta] + \frac{\|\delta\|}{D_2} (1 - h_{12} \alpha[\gamma]) \right) \int_0^1 \mathcal{K}_B(s) g(s) ds + \frac{1}{m} \right) < 1,$$

where

$$f^{0, \varrho} = \sup \left\{ \frac{f(t, u)}{\varrho} : (t, u) \in [0, 1] \times [0, \varrho] \right\} \quad \text{and} \quad \frac{1}{m} = \sup_{t \in [0, 1]} \int_0^1 k(t, s) g(s) ds.$$

Then $i_K(T, K_\varrho) = 1$.

3. THE BOUNDARY VALUE PROBLEM

Now we turn our attention to the BVP

$$(3.1) \quad u'''(t) = g(t)f(t, u(t)), \quad t \in (0, 1),$$

$$(3.2) \quad u(0) = H_1(\alpha[u]), \quad u'(p) = H_2(\beta[u]), \quad u''(1) = \lambda[u''], \quad p \in [1/2, 1].$$

In what follows we assume that $\lambda[1] < 1$ and by a solution of the BVP (3.1)–(3.2) we mean a solution of the corresponding perturbed integral equation

$$(3.3) \quad u(t) = H_1(\alpha[u]) + tH_2(\beta[u]) + \int_0^1 k_\lambda(t, s)g(s)f(s, u(s)) ds,$$

where k_λ is the Green function associated to the BCs

$$u(0) = 0, \quad u'(p) = 0, \quad u''(1) = \lambda[u''],$$

that is,

$$k_\lambda(t, s) := \left(tp - \frac{1}{2}t^2 \right) \left(1 + \frac{\Lambda(s)}{1 - \lambda[1]} \right) - t(p - s)\chi_{[0,p]}(s) + \frac{(t - s)^2}{2}\chi_{[0,t]}(s),$$

where

$$\Lambda(s) := \int_0^s d\Lambda(t) \quad \text{and} \quad \chi_I(t) := \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

The function k_λ was investigated in Section 2 of [6] and a key property is given by the following theorem.

Theorem 3.1 [6]. *Suppose that $\Lambda(s) \geq 0$ for $s \leq p$ and $\Lambda(s)/(1 - \lambda[1]) \geq -(s - p)/(1 - p)$ for $s > p$, and let*

$$\Phi(s) := \begin{cases} \frac{p^2}{2} + \frac{p^2}{2} \frac{\Lambda(s)}{1 - \lambda[1]}, & s \geq p, \\ \frac{s^2}{2} + \frac{p^2}{2} \frac{\Lambda(s)}{1 - \lambda[1]}, & s < p. \end{cases}$$

Then, for $t \in [0, 1]$ and $s \in [0, 1]$, we have

$$c(t)\Phi(s) \leq k_\lambda(t, s) \leq \Phi(s),$$

where $c(t) := (2tp - t^2)/p^2$.

In order to satisfy the conditions of Section 2, we need

$$h_{12}\alpha[1] < 1, \quad h_{22}\beta[t] < 1, \quad (1 - h_{12}\alpha[1])(1 - h_{22}\beta[t]), -h_{12}h_{22}\alpha[t]\beta[1] > 0,$$

and, by fixing $[a, b] \subset (0, 1)$, we obtain

$$(3.4) \quad c := \min\{a, a(2p - a)/p^2, b(2p - b)/p^2\}.$$

By means of the fixed point index results of Section 2, we can state a result on the existence of one or of two positive solutions. Note that, provided the nonlinearity f possesses a suitable oscillatory behavior, it is possible to state, with arguments similar to those in [21], a theorem on the existence of three or more positive solutions.

Theorem 3.2. Let $[a, b] \subset (0, 1)$ and let c be as in (3.4). Then equation (3.3) has a positive solution in K if one of the following conditions holds.

(S₁) There exist $\varrho_1, \varrho_2 \in (0, \infty)$ with $\varrho_1 < \varrho_2$ such that (2.3) is satisfied for ϱ_1 and (2.2) is satisfied for ϱ_2 .

(S₂) There exist $\varrho_1, \varrho_2 \in (0, \infty)$ with $\varrho_1 < c\varrho_2$ such that (2.2) is satisfied for ϱ_1 and (2.3) is satisfied for ϱ_2 .

Equation (3.3) has at least two positive solutions in K if one of the following conditions holds.

(D₁) There exist $\varrho_1, \varrho_2, \varrho_3 \in (0, \infty)$ with $\varrho_1 < \varrho_2 < c\varrho_3$ such that (2.3) is satisfied for ϱ_1 , (2.2) is satisfied for ϱ_2 and (2.3) is satisfied for ϱ_3 .

(D₂) There exist $\varrho_1, \varrho_2, \varrho_3 \in (0, \infty)$ with $\varrho_1 < c\varrho_2$ and $\varrho_2 < \varrho_3$ such that (2.2) is satisfied for ϱ_1 , (2.3) is satisfied for ϱ_2 and (2.2) is satisfied for ϱ_3 .

The next example illustrates the applicability of our result.

Example 1. Consider the BVP

$$\begin{aligned} u'''(t) &= f(u(t)), \quad t \in (0, 1), \\ u(0) &= H_1(u(1/4)), \quad u'(2/3) = H_2(u(1/2)), \quad u'(3/4) = u'(1), \end{aligned}$$

where the functions H_1, H_2 are chosen in a way similar to that used in [15], that is

$$H_1(w) = \begin{cases} \frac{2}{3}w, & 0 \leq w \leq 1, \\ \frac{1}{3}w + \frac{1}{3}, & w \geq 1, \end{cases} \quad H_2(w) = \begin{cases} \frac{9}{10}w, & 0 \leq w \leq 1, \\ \frac{9}{20}w + \frac{9}{20}, & w \geq 1. \end{cases}$$

In this case we have

$$h_{11} = 1/3, \quad h_{21} = 9/20, \quad h_{12} = 2/3, \quad h_{22} = 9/10.$$

We fix $[a, b] = [1/8, 7/8]$ and, by direct calculation, we obtain

$$D_1 = 23/48, \quad D_2 = 1/30, \quad m = 324/31, \quad M(1/8, 7/8) = 36864/1325.$$

This value for m corrects the typo ($m = 567/55$) present in [6].

Therefore all terms appearing in (2.2) and (2.3) can be computed and the growth assumptions for the nonlinearity f are

$$f^{0,e} < 0.24820 \quad \text{and} \quad f_{\varrho, \varrho/c} > 5.7245.$$

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