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ON SOME SINGULAR SYSTEMS OF PARABOLIC FUNCTIONAL EQUATIONS

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Abstract. We will prove existence of weak solutions of a system, containing non-local terms $u, w$.

Keywords: parabolic functional equation, singular system, monotone type operator

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1. Introduction

We will consider initial-boundary value problems for the system

\begin{align}
D_t u - \sum_{i=1}^{n} D_i [a_i(t, x, u(t, x), Du(t, x) + g(w(t, x)) Dw(t, x); u, w)] \\
+ a_0(t, x, u(t, x), Du(t, x) + g(w(t, x)) Dw(t, x); u, w) &= G, \\
D_t w &= F(t, x; u, w) \text{ in } Q_T = (0, T) \times \Omega \subset \mathbb{R}^{n+1}, \quad T \in (0, \infty)
\end{align}

where the functions

\[ a_i : Q_T \times \mathbb{R}^{n+1} \times L^{p_1}(0, T; V_1) \times L^2(Q_T) \to \mathbb{R} \]

(with a closed linear subspace $V_1$ of the Sobolev space $W^{1,p_1}(\Omega)$, $2 \leq p_1 < \infty$) satisfy conditions which are generalizations of the usual conditions for quasilinear parabolic differential equations considered when using the theory of monotone type operators. Further,

\[ F : Q_T \times L^{p_1}(0, T; V_1) \times L^2(Q_T) \to \mathbb{R} \]

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satisfies a Lipschitz condition. In the second part of the paper the case \( g = 0 \) and in the third part the general case will be considered.

Such problems with \( g = 0 \) arise, e.g., when considering diffusion and transport in porous media with variable porosity, see [4], [6]. In [6] a nonlinear system was numerically studied which consisted of a parabolic, an elliptic and an ordinary DE, describing the reaction-mineralogy-porosity changes in porous media. System (1.1), (1.2) is the case when the pressure is assumed to be constant. The case of general \( g \) was motivated by non-Fickian diffusion in viscoelastic polymers and by spread of morphogens (see [7], [8]). In [2], [5] similar degenerate systems of parabolic differential equations were considered without functional dependence and with more special differential equations, by using other methods.

2. Case \( g = 0 \)

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain having the uniform \( C^1 \) regularity property (see [1]) and let \( p_1 \geq 2 \) be a real number. Denote by \( W^{1,p_1}(\Omega) \) the usual Sobolev space of real valued functions with the norm

\[
\| u \| = \left[ \int_\Omega (|Du|^{p_1} + |u|^{p_1}) \right]^{1/p_1}.
\]

Let \( V_1 \subset W^{1,p_1}(\Omega) \) be a closed linear subspace containing \( C_0^\infty(\Omega) \). Denote by \( L^{p_1}(0,T;V_1) \) the Banach space of the set of measurable functions \( u: (0,T) \to V_1 \) such that \( \| u \|_{L^{p_1}(0,T;V_1)} \) is integrable, and define the norm by

\[
\| u \|_{L^{p_1}(0,T;V_1)}^{p_1} = \int_0^T \| u(t) \|_{V_1}^{p_1} \, dt.
\]

For the sake of brevity we denote \( L^{p_1}(0,T;V_1) \) by \( X_T^T \). The dual space of \( X_T^T \) is \( L^{q_1}(0,T;V_1^\ast) \) where \( 1/p_1 + 1/q_1 = 1 \) and \( V_1^\ast \) is the dual space of \( V_1 \) (see, e.g., [10], [11]). Further, let \( X^T = X_T^T \times L^2(Q_T) \).

On functions \( a_i \) we assume:

(A1) The functions \( a_i: Q_T \times \mathbb{R}^{n+1} \times X^T \to \mathbb{R} \) satisfy the Carathéodory conditions for arbitrary fixed \((u, w) \in X_T \) (\( i = 0, 1, \ldots, n \)).

(A2) There exist bounded (nonlinear) operators \( g_1: X^T \to \mathbb{R}^+ \) and \( k_1: X^T \to L^{q_1}(Q_T) \) such that

\[
|a_i(t, x, \zeta_0, \zeta; u, w)| \leq g_1(u, w)[|\zeta_0|^{p_1-1} + |\zeta|^{p_1-1}] + [k_1(u, w)](t, x), \quad i = 0, 1, \ldots, n
\]

for a.e. \((t, x) \in Q_T\), every \((\zeta_0, \zeta) \in \mathbb{R}^{n+1}\) and \((u, w) \in X^T\).
\[(A_3) \quad \sum_{i=1}^{n} [a_i(t, x, \zeta_0, \zeta; u, w) - a_i(t, x, \zeta_0, \zeta^*; u, w)](\zeta_i - \zeta_i^*) \geq [g_2(u)](t)|\zeta - \zeta^*|^{p_1}, \quad t \in (0, T) \]

where
\[
(g_2(u))(t) \geq \frac{c_2}{1 + \|u\|^{\sigma}_{X_1^1}}
\]

with some constants \(c_2 > 0, 0 \leq \sigma < p_1 - 1\).

(A_4) There exists a (nonlinear) operator \(k_2 : X^T \to L^1(Q_T)\) such that
\[
\sum_{i=0}^{n} a_i(t, x, \zeta_0, \zeta; u, w)\zeta_i \geq [g_2(u)](t)|\zeta_0|^{p_1} + |\zeta|^{p_1} - |k_2(u, w)|(t, x)
\]
for a.e. \((t, x) \in Q_T\), all \((\zeta_0, \zeta) \in \mathbb{R}^{n+1}, (u, w) \in X^T\) and
\[
\|k_2(u, w)\|_{L^1(Q_T)} \leq c_3 [\|u\|^\lambda + \|w\|^\mu + 1]
\]
with some nonnegative constants \(\lambda < p_1 - \sigma, \mu < 2\).

(A_5) There exists \(\delta \in (0, 1]\) such that if \((u_k) \to u\) in \(L^{p_1}(0, T; W^{1-\delta, p_1} (\Omega))\), a.e. in \(Q_T\), \((\zeta_0^k) \to \zeta_0\), \((w_k) \to w\) weakly in \(L^2(Q_T)\) then for \(i = 0, 1, \ldots, n\), a.e. \((t, x) \in Q_T\), and all \(\zeta \in \mathbb{R}^n\) we have
\[a_i(t, x, \zeta_0, \zeta; u_k, w_k) - a_i(t, x, \zeta_0, \zeta; u, w) \to 0, \quad k_1(u_k, w_k) \to k_1(u, w) \text{ in } L^1(Q_T).\]

(See (A_1).) Further, if conditions \((\zeta^k) \to \zeta\), \((w_k) \to w\) a.e. in \(Q_T\) are satisfied, too, then
\[a_i(t, x, \zeta_0^k, \zeta^k; u_k, w_k) \to a_i(t, x, \zeta_0, \zeta; u, w), \quad i = 1, \ldots, n\]
for a.e. \((t, x) \in Q_T\) and
\[a_0(t, x, \zeta_0^k, \zeta^k; u_k, w_k) \to a_0(t, x, \zeta_0, \zeta; u, w)\]
for a.e. \((t, x) \in Q_T\), in the last case assuming also that \((Du_k) \to Du\) a.e. in \(Q_T\).

Assumptions on \(F\): \(Q_T \times \mathbb{R} \times X^T \to \mathbb{R}\):

(F_1) For each fixed \((u, w) \in X^T\), \(F(\cdot, u; u, w) \in L^2(Q_T)\).

(F_2) \(F\) satisfies the following (global) Lipschitz condition: there exists a constant \(K\) such that for each \(t \in (0, T]\), \((u, \bar{w}), (u, \bar{w}^*) \in X^T\) we have
\[
(2.3) \quad \int_{Q_t} e^{-2c\tau} |F(\tau, x, \bar{w}(\tau, x)e^{c\tau}; u, \bar{w}e^{c\tau}) - F(\tau, x, \bar{w}^*(\tau, x)e^{c\tau}; u, \bar{w}^*e^{c\tau})|^2 d\tau dx \\
\leq K \int_{Q_t} |\bar{w}(\tau, x) - \bar{w}^*(\tau, x)|^2 d\tau dx
\]
for each positive number $c$. Further, there is a constant $K_0$ such that
\[ \int_{Q_T} |F(t,x,0;u,0)|^2 \, dt \, dx \leq K_0(\|u\|_{L^p(0,T;W^{1,q}(\Omega))} + 1). \]

(F3) If $(u_k) \to u$ in $L^p(0,T;W^{1,q}(\Omega))$, a.e. in $Q_T$, $(\eta_k) \to \eta$ and $(w_k) \to w$ in $L^2(Q_T)$, a.e. in $Q_T$, then for a.e. $(t,x) \in Q_T$
\[ F(t,x,\eta_k;u_k,w_k) \to F(t,x,\eta;u,w). \]

**Remark.** A sufficient condition for (2.3) to hold is the following inequality:
\[ \int_{\Omega} |F(\tau,x,w(\tau,x);u,w) - F(\tau,x,w^*(\tau,x);u,w^*)|^2 \, dx \leq K_1 \int_{Q_T} |w(s,x) - w^*(s,x)|^2 \, ds \, dx + K_2 \int_{\Omega} |w(\gamma(\tau),x) - w^*(\gamma(\tau),x)|^2 \, dx, \quad \tau \in (0,T) \]
with some constants $K_1, K_2$ and a function $\gamma \in C^1$ satisfying $\gamma' > 0$, $0 \leq \gamma(\tau) \leq \tau$.

**Definition.** We define an operator $A = (A_1, A_2) : X^T \to (X^T)^*$ by
\[
[A(u,w),(v,z)] = [A_1(u,w),v] + [A_2(u,w),z],
\]
\[
[A_1(u,w),v] = \int_{Q_T} \sum_{i=1}^n a_i(t,x,u(t,x),Du(t,x);u,w)D_i v \, dt \, dx + \int_{Q_T} a_0(t,x,u(t,x),Du(t,x);u,w)v \, dt \, dx,
\]
\[
[A_2(u,w),z] = \int_{Q_T} F(t,x,w(t,x);u,w)z \, dt \, dx,
\]
$(u,w),(v,z) \in X^T$, where the brackets $[\cdot,\cdot]$ mean the dualities in spaces $(X^T)^*, X^T$, $(X^T)^*, X^T, [L^2(Q_T)]^*, [L^2(Q_T)]$, respectively.

**Theorem 2.1.** Assume $(A_1)$–$(A_5)$ and $(F_1)$–$(F_3)$. Then for any $G \in (X^T)^*$, $H \in L^2(Q_T)$ there exists $(u,w) \in X^T$ such that $D_t u \in (X^T)^*$, $D_t w \in L^2(Q_T)$,
\[
D_t u + A_1(u,w) = G, \quad u(0) = 0, \tag{2.4}
\]
\[
D_t w + A_2(u,w) = H, \quad w(0) = 0. \tag{2.5}
\]

**Sketch of the proof.** Define a new unknown function $\tilde{w}$ (instead of $w$) by
\[ \tilde{w}(t,x) = w(t,x)e^{-ct}, \] i.e. $w(t,x) = \tilde{w}(t,x)e^{ct}$
with constant $c > 0$. Further, define a function $\tilde{F}$ and operators $\tilde{A}_1, \tilde{A}_2$ by

$$
\tilde{F}(t, x, \eta; u, \tilde{w}) = e^{-ct}F(t, x, \eta e^{ct}; u, \tilde{w}e^{ct}) + c\eta,
$$

$$
[\tilde{A}_1(u, \tilde{w}), v] = [A_1(u, w), v] = \int_{Q_T} \sum_{i=1}^n a_i(t, x, u(t, x), Du(t, x); u, w)D_i v \, dt \, dx
$$

$$
+ \int_{Q_T} a_0(t, x, u(t, x), Du(t, x); u, w)v \, dt \, dx,
$$

$$
[\tilde{A}_2(u, \tilde{w}), z] = \int_{Q_T} \tilde{F}(t, x, \tilde{w}(t, x); u, \tilde{w})z \, dt \, dx
$$

$$
= \int_{Q_T} e^{-ct}F(t, x, \tilde{w}(t, x)e^{ct}; u, \tilde{w}e^{ct})z \, dt \, dx + c\int_{Q_T} \tilde{w}z \, dt \, dx.
$$

Clearly, $(u, w)$ is a solution of (2.4), (2.5) if and only if $(u, \tilde{w})$ satisfies

(2.6) \quad D_t u + \tilde{A}_1(u, \tilde{w}) = G, \quad u(0) = 0,

(2.7) \quad D_t \tilde{w} + \tilde{A}_2(u, \tilde{w}) = e^{-ct}H = \tilde{H}, \quad \tilde{w}(0) = 0.

By $(A_1)$–$(A_5)$, $(F_1)$, $(F_2)$ the operator $\tilde{A} : X^T \rightarrow (X^T)^*$ is bounded and demi-continuous (see [10], [11]).

By $(F_2)$, $\tilde{A}_2$ is monotone for sufficiently large $c > 0$), thus, by using $(A_1)$–$(A_5)$, one can show that $\tilde{A}$ is pseudomonotone with respect to the domain of $L = D_t$:

$$
D(L) = \{(u, \tilde{w}) \in X^T : (D_t u, D_t \tilde{w}) \in (X^T)^*, \quad u(0) = 0, \quad \tilde{w}(0) = 0\},
$$

i.e.

(2.8) \quad (u_k, \tilde{w}_k) \rightharpoonup (u, \tilde{w}) \text{ weakly in } X^T,

(2.9) \quad (Lu_k, L\tilde{w}_k) \rightharpoonup (Lu, L\tilde{w}) \text{ weakly in } (X^T)^*,

and

(2.10) \quad \limsup_{k \to \infty} [\tilde{A}(u_k, \tilde{w}_k), (u_k, \tilde{w}_k) - (u, \tilde{w})] \leq 0

imply

(2.11) \quad \tilde{A}(u_k, \tilde{w}_k) \rightharpoonup \tilde{A}(u, \tilde{w}) \text{ weakly in } (X^T)^*.
Because, by (2.8)

\[(2.12) \quad (u_k) \to u \quad \text{in} \quad L^p(0, T; W^{1,\lambda}(\Omega)) \quad \text{and a.e. in} \quad Q_T\]

for a subsequence (again denoted by \((u_k)\), for simplicity), see, e.g., [10]. We may choose the number \(c > 0\) such that \(c > K\) (see (F2)). We have

\[(2.13) \quad [\tilde{A}_2(u_k, \tilde{w}_k), \tilde{w}_k - \tilde{w}] = [\tilde{A}_2(u_k, \tilde{w}_k) - \tilde{A}_2(u_k, \tilde{w}), \tilde{w}_k - \tilde{w}] + [\tilde{A}_2(u, \tilde{w}) - \tilde{A}_2(u, \tilde{w}), \tilde{w}_k - \tilde{w}] + [\tilde{A}_2(u, \tilde{w}), \tilde{w}_k - \tilde{w}]\]

where by \(c > K\), (F2) the first term on the right hand side is nonnegative, the second term tends to 0 by (2.12), (F2), (F3), Vitali’s theorem, and Cauchy-Schwarz inequality, while, finally, the third term converges to 0 by (2.8). Thus (2.9), (2.13) imply (for a subsequence)

\[(2.14) \quad \limsup_{k \to \infty} [\tilde{A}_1(u_k, \tilde{w}_k), u_k - u] \leq 0.\]

By using (A2), (A3), (A5), Vitali’s theorem and Hölder’s inequality, one obtains from (2.8), (2.14)

\[(2.15) \quad \lim_{k \to \infty} [\tilde{A}_1(u_k, \tilde{w}_k), u_k - u] = 0,\]

hence by (A3) one obtains for a subsequence

\[(2.16) \quad \lim_{k \to \infty} \int_{Q_T} |D^2u_k - Du|^p \, dt \, dx = 0, \quad \text{thus} \quad (D^2u_k) \to Du \quad \text{a.e. in} \quad Q_T.\]

Further, by (2.15), (2.9), (2.13)

\[(2.17) \quad \lim_{k \to \infty} [\tilde{A}_2(u_k, \tilde{w}_k), \tilde{w}_k - \tilde{w}] = 0,\]

thus by assumption (F2) and due to \(c > K\) (for a subsequence)

\[(2.18) \quad \lim_{k \to \infty} \int_{Q_T} |\tilde{w}_k - \tilde{w}|^2 \, dt \, dx \quad \text{and so} \quad (\tilde{w}_k) \to \tilde{w} \quad \text{a.e. in} \quad Q_T.\]

Consequently, from (A5), (2.12), (2.16) one obtains (by using Vitali’s theorem)

\[(2.19) \quad \tilde{A}_1(u_k, \tilde{w}_k) \to \tilde{A}_1(u, \tilde{w}) \quad \text{weakly in} \quad L^q(0, T; V_1^*).\]
Similarly, (2.18), (F₂), (F₃) imply

\[(2.20) \quad \tilde{A}_2(u_k, \tilde{w}_k) \to \tilde{A}_2(u, \tilde{w}) \quad \text{weakly in } L^2(Q_T).\]

Thus (2.19), (2.20) imply (2.11) for a subsequence. Finally, (2.15), (2.17) imply (2.10) (for a subsequence). One can prove in the standard way that the last facts imply (2.10), (2.11) for the original sequence.

Finally, by \((A₄), \text{ (F₂)}\) (for sufficiently large \(c > 0\)), \(\tilde{A}\) is coercive:

\[
\lim_{\|(u, \tilde{w})\|_{X_T} \to \infty} \frac{[\tilde{A}(u, \tilde{w}), (u, \tilde{w})]}{\|u\| + \|\tilde{w}\|} = +\infty.
\]

Since \(\tilde{A} : X_T \to (X_T)^*\) is bounded, demicontinuous, pseudomonotone with respect to \(D(L)\) and coercive, we obtain the existence of a solution \((u, \tilde{w})\) of (2.6), (2.7) and thus the existence of a solution \((u, w)\) of (2.4), (2.5). (See, e.g. [3], [10].)

**Example.** Conditions \((A₁)\)–\((A₅)\) are satisfied if e.g.

\[
a_i(t, x, \zeta_i, \zeta; u, w) = b(H(u))\zeta_i|\zeta|^{p_1-2}, \quad i = 1, 2, \ldots, n,
\]

\[
a_0(t, x, \zeta_i, \zeta; u, w) = b(H(u))\zeta_0|\zeta_0|^{p_1-2} + b_0(F_0(u)) + b_1(F_1(w))
\]

where \(b, b_0, b_1\) are continuous functions satisfying with some positive constants \(c_3, c_4, c_5\) the inequalities

\[
b(\theta) \geq \frac{c_3}{1 + |\theta|^\sigma} \quad (0 \leq \sigma < p_1 - 1),
\]

\[
|b_0(\theta)| \leq c_4(|\theta|^{\lambda-1} + 1) \quad \text{where } 1 \leq \lambda < p_1 - \sigma,
\]

\[
|b_1(\theta)| \leq c_5(|\theta|^{\mu_1-1} + 1) \quad \text{where } 0 \leq \mu_1 < 2 - 2/p_1
\]

and

\[
H : L^{p_1}(0, T; W^{1-\delta, p_1}(\Omega)) \to C(\bar{Q_T}),
\]

\[
F_0 : L^{p_1}(0, T; W^{1-\delta, p_1}(\Omega)) \to L^{p_1}(Q_T), \quad F_1 : L^2(Q_T) \to \mathbb{R}
\]

are linear continuous operators. If \(b\) is between two positive constants, \(H\) may be the same as \(F_0\).

Conditions (F₁)–(F₃) are satisfied if e.g.

\[
F(t, x, \eta; u, w) = \beta(\eta)\gamma_1(H_1(u)) + \gamma_2(H_2(u))\delta(G(w)) + \gamma_3(H_3(u))
\]
where $\beta, \delta$ are globally Lipschitz functions, $\gamma_1, \gamma_2$ are continuous and bounded, $\gamma_3$ is continuous and satisfies

$$|\gamma_3(\theta)| \leq c_5|\theta|^{3/2} + c_6$$

with some constants $c_5, c_6$, and

$$G: L^2(Q_T) \rightarrow L^2(Q_T), \quad H_1, H_2: L^{p_1}(0, T; W^{1-\delta, p_1}(\Omega)) \rightarrow L^{p_1}(Q_T),$$

$$H_3: L^{p_1}(0, T; W^{1-\delta, p_1}(\Omega)) \rightarrow L^2(Q_T)$$

are continuous linear operators such that $G$ satisfies for all $w \in L^2(Q_T)$

$$\int_\Omega |G(w)(\tau, x)|^2 \, dx \leq K_1 \int_{Q_T} |w(s, x)|^2 \, ds \, dx + K_2 \int_\Omega |w(\gamma(s), x)|^2 \, dx$$

where $\gamma \in C^1$, $\gamma' > 0$, $\gamma(s) \leq s$.

3. **Case $g \neq 0$**

Now we shall consider equations (1.1), (1.2) with a bounded, continuous function $g$. This problem will be transformed to the case $g = 0$, considered in Theorem 2.1. Let $f = \int g$, $f(0) = 0$, $p_1 = p > 2$.

Define

$$\tilde{X}^T = L^p(0, T; W^{1,p}(\Omega)) \times L^p(0, T; W^{1,p}(\Omega))$$

and an operator $A_1: \tilde{X}_T \rightarrow (X_T^1)^*$ for $(u, w) \in \tilde{X}_T$, $v \in X^1_T$ by

$$[A_1(u, w), v] = \int_{Q_T} \left\{ \sum_{i=1}^n a_i(t, x, u, Du + g(w)Dw; u, w)D_i v \right\} \, dt \, dx$$

$$+ \int_{Q_T} a_0(t, x, u(t, x), Du(t, x) + g(w(t, x))Dw(t, x); u, w) v \, dt \, dx.$$ 

Further, assume

(F4) $F$ has the form $F(t, x; u, w) = F_1(t, x, [h(u)](t, x), w(t, x))$ where $F_1$ is continuously differentiable with respect to the last three variables, the partial derivatives are bounded and either $h(u) = u$ or $h: L^p(Q_T) \rightarrow L^p(0, T; W^{1,p}(\Omega))$ is a continuous linear operator such that $h(u) \in L^p(0, T; C^1(\Omega))$ for all $u \in L^p(Q_T)$ and the following estimate holds for any $\tau \in [0, T]$ with a suitable constant:

$$\int_\Omega |[h(u)](\tau, x)|^2 \, dx \leq \text{const} \int_{Q_T} |u(s, x)|^2 \, ds \, dx.$$ 

Further, there exists a constant $c_0 > 0$ such that

$$F_1(t, x, \zeta_0, \eta)\eta < 0 \quad \text{if} \quad |\eta| \geq c_0.$$
**Theorem 3.1.** Assume that \((A_1)-(A_5)\) and \((F_1)-(F_4)\) are satisfied with \(p_1 = p > 2, \delta = 1, \sigma < p - 2\) such that for the operators \(g_1, k_1, g_2, k_2\) in \((A_2)-(A_4)\) we have

\[
g_1(u, w)^q \leq \text{const} \ g_2(u, w), \quad k_1(u, w)^q \leq \text{const} \ k_2(u, w) \quad \text{if} \quad w(t, x) \leq c_0 \text{ a.e.}
\]

Further, let \(g\) be a bounded, continuous function. Then for any \(G \in (X_1^T)^*\) there exists \((u, w) \in \tilde{X}^T\) such that \(u + f(w) \in L^p(0, T; V_1),\)

\[
D_t u + D_t[f(w)] \in (X_1^T)^*, \quad D_t w \in L^2(Q_T),
\]

(3.1)

\[
D_t u + A_1(u, w) = G, \quad u(0) = 0,
\]

(3.2)

\[
D_t w = F(t, x; u, w) \quad \text{for a.e.} \quad (t, x) \in Q_T, \quad w(0) = w.
\]

**Sketch of the proof.** Instead of \(u\) introduce a new unknown function \(\tilde{u}\) by

\[
\tilde{u}(t, x) = u(t, x) + f(w(t, x)) \quad \text{(where} \quad f = \int g, \quad f(0) = 0)\]

By using the formulas

\[
D_t \tilde{u} = D_t u + f'(w)D_t w, \quad D \tilde{u} = Du + f'(w)D w
\]

we obtain that \((u, w) \in \tilde{X}^T\) is a solution of (3.1), (3.2) if and only if \((\tilde{u}, w) \in \tilde{X}^T\) satisfies

\[
D_t \tilde{u} + \tilde{A}_1(\tilde{u}, w) = G, \quad \tilde{u}(0) = 0,
\]

(3.5)

\[
D_t w = F(t, x; \tilde{u} - f(w), w), \quad w(0) = 0
\]

(3.6)

where

\[
[\tilde{A}_1(\tilde{u}, w), v] = \int_{Q_T} \left\{ \sum_{i=1}^n a_i(t, x, \tilde{u}(t, x) - f(w(t, x)), D\tilde{u}(t, x); \tilde{u} - f(w), w)D_i v \right\} dt \; dx
\]

\[
+ \int_{Q_T} \left\{ a_0(t, x, \tilde{u} - f(w), D\tilde{u}; \tilde{u} - f(w), w) - f'(w)F(t, x; \tilde{u} - f(w), w) \right\} v \; dt \; dx.
\]

One can show that by Theorem 2.1 there is a solution \((\tilde{u}, w) \in X^T\) of (3.5), (3.6) (such that \(D_t w \in L^2(Q_T)\)). Then one proves that \(w \in L^p(0, T; W^{1, p}(\Omega))\), hence \((\tilde{u}, w) \in \tilde{X}^T\) and thus with \(u = \tilde{u} - f(w), (u, w)\) satisfies (3.1), (3.2).
References


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