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VARIATIONAL INCLUSIONS FOR A STURM-LIOUVILLE TYPE DIFFERENTIAL INCLUSION

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Abstract. We establish several variational inclusions for solutions of a nonconvex Sturm-Liouville type differential inclusion on a separable Banach space.

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1. Introduction

In control theory, mainly, if we want to obtain necessary optimality conditions, it is essential to have several "differentiability" properties of solutions with respect to initial conditions. One of the most powerful results in the theory of differential equations, the classical Bendixson-Picard-Lindelöf theorem, states that the maximal flow of a differential equation is differentiable with respect to initial conditions, and its derivatives verify the variational equation. This result has been generalized in various ways to differential inclusions by considering several variational inclusions and proving the corresponding theorems that extend the Bendixson-Picard-Lindelöf theorem.

The present paper is concerned with second-order differential inclusions of the form

$$(p(t)x'(t))' \in F(t, x(t)) \quad \text{a.e. } ([0,T]), \tag{1.1}$$

with initial conditions

$$x(0) = x_0, \quad x'(0) = x_1, \tag{1.2}$$

where $F\colon [0,T]\times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, x_1 \in X$ and $p(\cdot)\colon [0,T] \to (0,\infty)$ is continuous.

Even if we deal with an initial value problem instead of a boundary value problem, the differential inclusion (1.1)–(1.2) may be regarded as an extension to the set-valued framework of the classical Sturm-Liouville differential equation. Several existence results for problem (1.1)–(1.2) may be found in [2], [3], [7].

The aim of this note is to extend the results concerning the differentiability of solutions of differential inclusions with respect to initial conditions to the solutions of problem (1.1). The results we extend, known as the contingent, the intermediate (quasitangent) and the circatangent variational inclusion, are obtained in the "classical case" of first-order differential inclusions. For these results and for a complete discussion on this topic we refer to [1].

The proofs of our results follow by an approach similar to the classical case of differential inclusions ([1], [6]) and use a recent result ([2]) concerning the existence of solutions of problem (1.1).

The paper is organized as follows: in Section 2 we present preliminary results to be used in the next section and in Section 3 we prove our main results.

2. Preliminaries

In this short section we recall some basic notation and concepts concerning differential inclusions.

Let Y be a normed space, $X \subset Y$ and $x \in \overline{X}$ (the closure of X).

From the multitude of the tangent cones in literature (e.g. [1]) we recall only the contingent, the quasitangent and Clarke's tangent cones, defined, respectively, by

$$\begin{aligned}
K_xX &= \{v \in Y;\ \exists s_m \to 0+,\ \exists v_m \to v\colon\ x + s_m v_m \in X\},\\
Q_xX &= \{v \in Y;\ \forall s_m \to 0+,\ \exists v_m \to v\colon\ x + s_m v_m \in X\},\\
C_xX &= \Big\{v \in Y;\ \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X\colon\ \frac{y_m - x_m}{s_m} \to v\Big\}.
\end{aligned}$$

These cones are related as follows: $C_xX \subset Q_xX \subset K_xX$.

Corresponding to each type of the tangent cone, say τ_xX, one may introduce a *set-valued directional derivative* of a multifunction $G(\cdot)\colon X \subset Y \to \mathcal{P}(Y)$ (in particular of a single-valued mapping) at a point $(x,y) \in \operatorname{Graph}(G)$ as follows:

$$\tau_yG(x;v) = \{w \in Y;\quad (v,w) \in \tau_{(x,y)} \operatorname{Graph}(G)\},\quad v \in \tau_xX.$$

Let us denote by I the interval $[0,T]$, $T > 0$ and let X be a real separable Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot,\cdot)$. Denote by B the closed unit ball in X.

Consider a set-valued map $F\colon I\times X\to\mathcal{P}(X)$, $x_0,x_1\in X$ and a continuous mapping $p(\cdot)\colon I\to(0,\infty)$ that define the Cauchy problem (1.1).

A continuous mapping $x(\cdot)\in C(I,X)$ is called a solution of problem (1.1) if there exists a (Bochner) integrable function $f(\cdot)\in L^1(I,X)$ such that

$$f(t)\in F(t,x(t))\quad \text{a.e. } (I), \tag{2.1}$$

$$x(t)=x_0+p(0)x_1\int_0^t\frac{1}{p(s)}\,\mathrm{d}s+\int_0^t\frac{1}{p(s)}\int_0^s f(u)\,\mathrm{d}u\,\mathrm{d}s\quad \forall t\in I. \tag{2.2}$$

Note that, if we denote $G(t,u):=\int_u^t\frac{1}{p(s)}$, $t\in I$, then (2.2) may be rewritten as

$$x(t)=x_0+p(0)x_1G(t,0)+\int_0^t G(t,u)f(u)\,\mathrm{d}u\quad \forall t\in I. \tag{2.3}$$

We shall call $(x(\cdot),f(\cdot))$ a *trajectory-selection pair* of (1.1) if (2.1) and (2.2) are satisfied.

We shall use the following notation for the solution sets of (1.1)

$$\mathcal{S}(x_0,x_1)=\{(x(\cdot),f(\cdot));\ (x(\cdot),f(\cdot)) \text{ is a trajectory-selection pair of (1.1)}\}. \tag{2.4}$$

In what follows $y_0,y_1\in X$, $g(\cdot)\in L^1(I,X)$, and $y(\cdot)$ is a solution of the Cauchy problem

$$(p(t)y'(t))'=g(t),\quad y(0)=y_0,\quad y'(0)=y_1. \tag{2.5}$$

Hypothesis 2.1. i) *$F(\cdot,\cdot)\colon I\times X\to\mathcal{P}(X)$ has nonempty closed values and for every $x\in X$, $F(\cdot,x)$ is measurable.*

ii) *There exist $\beta>0$ and $L(\cdot)\in L^1(I,(0,\infty))$ such that for almost all $t\in I$, $F(t,\cdot)$ is $L(t)$-Lipschitz on $y(t)+\beta B$ in the sense that*

$$d_H(F(t,x_1),F(t,x_2))\leqslant L(t)|x_1-x_2|\quad \forall\ x_1,x_2\in y(t)+\beta B,$$

where $d_H(A,C)$ is the Hausdorff distance between $A,C\subset X$:

$$d_H(A,C)=\max\{d^*(A,C),d^*(C,A)\},\quad d^*(A,C)=\sup\{d(a,C);a\in A\}.$$

iii) *The function $t\to\gamma(t):=d(g(t),F(t,y(t))$ is integrable on I.*

Set $m(t)=\exp\bigl(MT\int_0^t L(u)\,\mathrm{d}u\bigr)$, $t\in I$ and $M:=\sup\limits_{t\in I}1/p(t)$. Note that $|G(t,u)|\leqslant M(t-u)$ $\forall t,\ u\in I,\ u\leqslant t$.

On $C(I,X) \times L^1(I,X)$ we consider the norm

$$|(x,f)|_{C\times L} = |x|_C + |f|_1 \quad \forall\ (x,f) \in C(I,X) \times L^1(I,X),$$

where, as usual, $|x|_C = \sup_{t\in I} |x(t)|$, $x \in C(I,X)$ and $|f|_1 = \int_0^T |f(t)|\,\mathrm{d}t$, $f \in L^1(I,X)$.

The next result (see [2]) is an extension of Filippov's theorem concerning the existence of solutions to a Lipschitzian differential inclusion (see [6]), to second-order differential inclusions of the form (1.1).

Theorem 2.1. *Consider $\delta \geqslant 0$, assume that Hypothesis* 2.1 *is satisfied and set $\eta(t) = m(t)(\delta + MT\int_0^t \gamma(s)\,\mathrm{d}s)$.*

If $\eta(T) \leqslant \beta$, then for any $x_0, x_1 \in X$ with $(|x_0 - y_0| + MTp(0)|x_1 - y_1|) \leqslant \delta$ and any $\varepsilon > 0$ there exists $(x(\cdot), f(\cdot)) \in \mathcal{S}(x_0, x_1)$ such that

$$\begin{aligned}
&|x(t) - y(t)| \leqslant \eta(t) + \varepsilon MTtm(t) \quad \forall t \in I,\\
&|f(t) - g(t)| \leqslant L(t)(\eta(t) + \varepsilon MTtm(t)) + \gamma(t) + \varepsilon \quad \text{a.e. } (I).
\end{aligned}$$

3. Main results

Let $(y(\cdot), g(\cdot))$ be a trajectory-selection pair of problem (1.1). We wish to "linearize" (1.1) along $(y(\cdot), g(\cdot))$ by replacing it by several second-order variational inclusions.

Consider, first, the quasitangent variational inclusion

$$\text{(3.1)} \qquad \begin{cases} (p(t)w'(t))' \in Q_{g(t)}(F(t,\cdot))(y(t); w(t)) \quad \text{a.e. } (I)\\ w(0) = u, \quad w'(0) = v, \end{cases}$$

where $u, v \in X$.

Theorem 3.1. *Consider the solution map $\mathcal{S}(\cdot,\cdot)$ as a set valued map from $X \times X$ into $C(I,X) \times L^1(I,X)$ and assume that Hypothesis* 2.1 *is satisfied.*

Then for any $u, v \in X$ and any trajectory-selection pair (w, π) of the linearized inclusion (3.1) *one has*

$$(w, \pi) \in Q_{(y,g)}\mathcal{S}((y(0), y'(0); (u,v)).$$

Proof. Let $u, v \in X$ and let $(w, \pi) \in C(I, X) \times L^1(I, X)$ be a trajectory-selection pair of (3.1). By the definition of the quasitangent derivative and from the Lipschitzianity of $F(t, \cdot)$ for almost all $t \in I$ we have

$$\lim_{h \to 0+} d\Big(\pi(t), \frac{F(t, y(t) + hw(t)) - g(t)}{h}\Big) = 0. \tag{3.2}$$

Moreover, since $g(t) \in F(t, y(t))$ a.e. (I), from Hypothesis 2.1, for all small enough $h > 0$ and for almost all $t \in I$ one has

$$d(g(t) + h\pi(t), F(t, y(t) + hw(t))) \leqslant h(|\pi(t)| + L(t)|w(t)|).$$

By standard arguments (e.g., Lemmas 1.4 and 1.5 in [6]) the function $t \to d(g(t) + h\pi(t), F(t, y(t) + hw(t)))$ is measurable. Therefore, using the Lebesgue dominated convergence theorem we infer

$$\int_0^T d(g(t) + h\pi(t), F(t, y(t) + hw(t))) = o(h), \tag{3.3}$$

where $\lim\limits_{h \to 0+} o(h)/h = 0$.

We apply Theorem 2.1 with $\varepsilon = h^2$ and by (3.3) we deduce the existence of $M \geqslant 0$ and of trajectory-selection pairs $(y_h(\cdot), g_h(\cdot))$ of the second-order differential inclusion (1.1) satisfying

$$\begin{gathered} |y_h - y - hw|_C + |g_h - g - h\pi|_1 \leqslant M(o(h) + h^2), \\ y_h(0) = y(0) + hu, \quad y_h'(0) = y'(0) + hv, \end{gathered}$$

which implies

$$\begin{aligned} \lim_{h \to 0+} \frac{y_h - y}{h} &= w \quad \text{in} \quad C(I, X), \\ \lim_{h \to 0+} \frac{g_h - g}{h_n} &= \pi \quad \text{in} \quad L^1(I, X). \end{aligned}$$

Therefore

$$\lim_{h \to 0+} d_{C \times L}\Big((w, \pi), \frac{\mathcal{S}((y(0) + hu, y'(0) + hv)) - (y, g)}{h}\Big) = 0$$

and the proof is complete. $\square$

We consider next the variational inclusion defined by the Clarke directional derivative of the set-valued map $F(t, \cdot)$, i.e., the so called circatangent variational inclusion

$$\begin{cases} (p(t)w'(t))' \in C_{g(t)}(F(t, \cdot))(y(t); w(t)) \quad \text{a.e. } (I) \\ w(0) = u, \quad w'(0) = v, \end{cases} \tag{3.4}$$

Theorem 3.2. *Consider the solution map $\mathcal{S}(\cdot,\cdot)$ as a set valued map from $X\times X$ into $C(I,X)\times L^1(I,X)$ and assume that Hypothesis* 2.1 *is satisfied.*

Then for any $u,v\in X$ and any trajectory-selection pair (w,π) of the linearized inclusion (3.4) *one has*

$$(w,\pi)\in C_{(y,g)}\mathcal{S}((y(0),y'(0);(u,v)).$$

P r o o f. Let $u,v\in X$, let $(w,\pi)\in C(I,X)\times L^1(I,X)$ be a trajectory-selection pair of (3.4), let (y_n,g_n) be a sequence of trajectory-selection pairs of (1.1) that converges to $(y,g)\in C(I,X)\times L^1(I,X)$ and let $h_n\to 0+$. Then there exists a subsequence $g_j(\cdot):=g_{n_j}(\cdot)$ such that

$$\lim_{j\to\infty} g_j(t)=g(t)\quad \text{a.e. } (I). \tag{3.5}$$

Denote $\lambda_j:=h_{n_j}$. From (3.4) and from the definition of the Clarke directional derivative, for almost all $t\in I$ we have

$$\lim_{j\to\infty} d\Big(\pi(t),\frac{F(t,y_j(t)+\lambda_j w(t))-g_j(t)}{\lambda_j}\Big)=0. \tag{3.6}$$

Since $g_j(t)\in F(t,y_j(t))$ a.e. (I), for almost all $t\in I$, we get

$$d(g_j(t)+\lambda_j\pi(t),F(t,y_j(t)+\lambda_j w(t)))\leqslant \lambda_j(|\pi(t)|+L(t)|w(t)|).$$

The last inequality together with the Lebesgue dominated convergence theorem implies

$$\int_0^T d(g_j(t)+\lambda_j\pi(t),F(t,y_j(t)+\lambda_j w(t)))=o(\lambda_j), \tag{3.7}$$

where $\lim\limits_{j\to\infty} o(\lambda_j)/\lambda_j=0$.

We apply Theorem 2.1 with $\varepsilon=\lambda_j^2$ and by (3.7) we deduce the existence of $M\geqslant 0$ and of trajectory-selections pairs $(\overline{y}_j(\cdot),\overline{g}_j(\cdot))$ of the second-order differential inclusion (1.1) satisfying

$$\begin{gathered}|\overline{y}_j-y_j-\lambda_j w|_C+|\overline{g}_j-g_j-\lambda_j\pi|_1\leqslant M(o(\lambda_j)+\lambda_j^2),\\ \overline{y}_j(0)=y(0)+\lambda_j u,\quad \overline{y}_j'(0)=y'(0)+\lambda_j v.\end{gathered}$$

It follows that

$$\begin{aligned}\lim_{j\to\infty}\frac{\overline{y}_j-y}{\lambda_j}&=w\quad\text{in } C(I,X),\\ \lim_{j\to\infty}\frac{\overline{g}_j-g}{\lambda_j}&=\pi\quad\text{in } L^1(I,X),\end{aligned}$$

which completes the proof. $\square$

Finally, we consider the contingent variational inclusion

$$\text{(3.8)} \qquad \begin{cases} (p(t)w'(t))' \in \overline{co} K_{g(t)}(F(t,\cdot))(y(t);w(t)) & \text{a.e. } (I) \\ w(0)=u, \quad w'(0)=v. \end{cases}$$

Theorem 3.3. *Consider the solution map $\mathcal{S}(\cdot,\cdot)$ as a set valued map from $X\times X$ into $C(I,X)\times L^\infty(I,X)$, with $L^\infty(I,X)$ supplied with the weak-* topology and assume that Hypothesis* 2.1 *is satisfied.*

Then for any $u,v\in X$ one has

$$K_{(y,g)}\mathcal{S}((y(0),y'(0);(u,v))\subset\{(w,\pi);(w,\pi) \text{ is a trajectory-selection pair of } (3.8)\}.$$

Proof. Let $u,v\in X$ and let $(w,\pi)\in K_{(y,g)}\mathcal{S}((y(0),y'(0);(u,v))$. According to the definition of the contingent derivative there exist $h_n\to 0+, u_n\to u, v_n\to v$, $w_n(\cdot)\to w(\cdot)$ in $C(I,X)$, $\pi_n(\cdot)\to\pi(\cdot)$ in the weak-* topology of $L^\infty(I,X)$ and $c>0$ such that

$$\text{(3.9)} \qquad \begin{aligned} &|\pi_n(t)|\leqslant c \quad \text{a.e. } (I),\\ &g(t)+h_n\pi_n(t)\in F(t,y(t)+h_nw_n(t)) \quad \text{a.e. } (I),\\ &w_n(0)=u_n, w_n'(0)=v_n. \end{aligned}$$

Therefore,

$$\text{(3.10)} \qquad \begin{aligned} &w_n(\cdot) \quad \text{converges pointwise to} \quad w(\cdot),\\ &\pi_n(\cdot) \quad \text{converges weakly in } L^1(I,X) \text{ to } \quad \pi(\cdot). \end{aligned}$$

We apply Mazur's theorem (e.g., [4]) and find that there exists

$$v_m(t)=\sum_{p=m}^{\infty}a_m^p\pi_p(t),$$

$v_m(\cdot)\to\pi(\cdot)$ (strongly) in $L^1(I,X)$, where $a_m^p\geqslant 0$, $\sum\limits_{p=m}^{\infty}a_m^p=1$ and for any m, $a_m^p\neq 0$, for a finite number of p.

Therefore, a subsequence (again denoted) by $v_m(\cdot)$ converges to $\pi(\cdot)$ a.e. From (3.9) for any p and for almost all $t\in I$ one obtains

$$w_p'(t)\in\frac{1}{h_p}(F(t,y(t)+h_pw_p(t))-g(t))\cap cB.$$

Let $t \in I$ be such that $v_m(t) \to \pi(t)$ and $g(t) \in F(t, y(t))$. Fix $n \geqslant 1$ and $\varepsilon > 0$. By (3.9) there exists m such that $h_p \leqslant 1/n$ and $|w_p(t) - w(t)| \leqslant 1/n$ for any $p \geqslant m$.

If we denote

$$\varphi(z,h) := \frac{1}{h}(F(t, y(t) + hz) - g(t)) \cap cB$$

then

$$v_m(t) \in co\Big(\bigcup_{\substack{h\in(0,1/n],\\ z\in B(w(t),1/n)}} \varphi(z,h)\Big)$$

and if $m \to \infty$, we get

$$\pi(t) \in \overline{co}\Big(\bigcup_{\substack{h\in(0,1/n],\\ z\in B(w(t),1/n)}} \varphi(z,h)\Big).$$

Since $\varphi(z,h) \subset cB$, we infer that

$$\pi(t) \in \overline{co} \bigcap_{\varepsilon>0, n\geqslant 1} \Big(\bigcup_{\substack{h\in(0,1/n],\\ z\in B(w(t),1/n)}} \varphi(z,h) + \varepsilon B\Big).$$

On the other hand,

$$\bigcap_{\varepsilon>0, n\geqslant 1} \Big(\bigcup_{\substack{h\in(0,1/n],\\ z\in B(w(t),1/n)}} \varphi(z,h) + \varepsilon B\Big) \subset K_{g(t)}F(t,\cdot)(y(t); w(t))$$

and the proof is complete. □

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