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ON HÖLDER REGULARITY FOR VECTOR-VALUED MINIMIZERS OF QUASILINEAR FUNCTIONALS

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Abstract. We discuss the interior Hölder everywhere regularity for minimizers of quasilinear functionals of the type

\[ A(u; \Omega) = \int_{\Omega} A_{ij}^{\alpha\beta}(x, u) D_{\alpha}u^i D_{\beta}u^j \, dx \]

whose gradients belong to the Morrey space \( L^{2,n-2}(\Omega, \mathbb{R}^N) \).

Keywords: quasilinear functional, minimizer, regularity, Campanato-Morrey space

MSC 2010: 35J60

1. Introduction

In this paper we study the interior everywhere regularity of functions minimizing variational integrals

\[ A(u; \Omega) = \int_{\Omega} A_{ij}^{\alpha\beta}(x, u) D_{\alpha}u^i D_{\beta}u^j \, dx \]

where \( u: \Omega \to \mathbb{R}^N, N > 1, \Omega \subset \mathbb{R}^n, n \geq 3 \) is a bounded open set, \( x = (x_1, \ldots, x_n) \in \Omega, u(x) = (u^1(x), \ldots, u^N(x)) \), \( Du = \{D_{\alpha}u^i\}, D_{\alpha} = \partial / \partial x_{\alpha}, \alpha = 1, \ldots, n, i = 1, \ldots, N \).

Throughout the whole text we use the summation convention over repeated indices. We call a function \( u \in W^{1,2}(\Omega, \mathbb{R}^N) \) a minimizer of the functional \( A(u; \Omega) \) if

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and only if \( A(u; \Omega) \leq A(v; \Omega) \) for every \( v \in W^{1,2}(\Omega, \mathbb{R}^N) \) with \( u - v \in W_{0}^{1,2}(\Omega, \mathbb{R}^N) \). For more information see [6], [9].

On the functional \( A \) we assume the following conditions:

(i) \( A^{\alpha \beta}_{ij} = A^{\beta \alpha}_{ji} \), \( A^{\alpha \beta}_{ij} \) are continuous functions in \( u \in \mathbb{R}^N \) for every \( x \in \Omega \) and there exists \( M > 0 \) such that \( |A^{\alpha \beta}_{ij}(x, u)| \leq M \), \( \forall x \in \Omega \), \( \forall u \in \mathbb{R}^N \).

(ii) (ellipticity) There exists \( \nu > 0 \) such that

\[
A^{\alpha \beta}_{ij}(x, u) x^i u^j \geq \nu |\xi|^2, \quad \forall x \in \Omega, \quad \forall u \in \mathbb{R}^N, \quad \forall \xi \in \mathbb{R}^n.
\]

(iii) (oscillation of coefficients) There exists a real function \( \omega \) continuous on \( [0, \infty) \) which is bounded, nondecreasing, concave, \( \omega(0) = 0 \) and such that for all \( x \in \Omega \) and \( u, v \in \mathbb{R}^N \)

\[
|A^{\alpha \beta}_{ij}(x, u) - A^{\alpha \beta}_{ij}(x, v)| \leq \omega(|u - v|).
\]

We set \( \omega_\infty = \lim_{t \to \infty} \omega(t) \leq 2M \).

(iv) For all \( u \in \mathbb{R}^N \), \( A^{\alpha \beta}_{ij}(\cdot, u) \in \text{VMO}(\Omega) \) (uniformly with respect to \( u \in \mathbb{R}^N \)).

It is well known (see [6], p. 169) that (iii) implies absolute continuity of \( \omega \) on \([0, \infty) \). In what follows, by pointwise derivative \( \omega' \) of \( \omega \) we will understand the right derivative which is finite on \((0, \infty)\). Considering the assumption (iv) it is worth recalling that since \( C^0 \) is a proper subset of \( \text{VMO} \), the continuity of coefficients \( A^{\alpha \beta}_{ij} = A^{\alpha \beta}_{ij}(x, u) \) with respect to \( x \) is not supposed.

In this paper we deal with the case \( n \geq 3 \) because for \( n = 2 \) higher integrability of the gradient of minimizer (see Preliminaries, Lemma 2.4) and the Sobolev imbedding theorem imply that \( u \) is locally Hölder continuous in \( \Omega \). From many examples (see [4], [6], [9], [10], [12], [14]) for \( n \geq 3 \) it is known that the minimizer \( u \) of the functional (1.1) need not be continuous or bounded even in the case of smooth coefficients \( A^{\alpha \beta}_{ij} \). For this reason the so called partial regularity for minimizers of the functional (1.1) was studied by many authors ([7], [8], [5]). In our paper (which is motivated by [3]) we concentrate on conditions that imply an everywhere regularity result. More precisely, we state conditions which imply that the minimizer \( u \) with gradient \( Du \in L^{2,n-2}(\Omega, \mathbb{R}^n) \) belongs to \( C^{0,\gamma}(\Omega, \mathbb{R}^N) \). The condition \( Du \in L^{2,n-2}(\Omega, \mathbb{R}^n) \) seems to be natural with respect to the paper [2].

Now we can state the following result:

**Theorem 1.1.** Let \( u \in W^{1,2}(\Omega, \mathbb{R}^N) \) be a minimizer of the functional (1.1) such that \( Du \in L^{2,n-2}(\Omega, \mathbb{R}^n) \) and let the hypotheses (i), (ii), (iii), (iv) be satisfied. Assume that there exists \( p > 1 \) such that

\[
Q_p := \min \left\{ \sup_{t \in (0, \infty)} \frac{d}{dt} (\omega^{p/(p-1)})(t), \int_{0}^{\infty} t^{-1} \frac{d}{dt} (\omega^{p/(p-1)})(t) dt \right\} < \infty
\]

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and let $\gamma \in (0, 1)$. Then the inequality

\begin{equation}
(Q_p \|Du\|_{L^{2,n-2}(\Omega, \mathbb{R}^n)})^{1-1/p} \leq \nu C
\end{equation}

implies that $u \in C^{0,\gamma}(\Omega, \mathbb{R}^N)$.

Here

$$C = \frac{2}{3c(n, N, p, M/\nu)(2^{n+3}L)^{\frac{1}{2n}(1-\gamma)},}$$

where $L$ is from Lemma 2.3.

2. Preliminaries

If $x \in \mathbb{R}^n$ and $r$ is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n: |y - x| < r\}$, $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by

$$u_{x,r} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} u(y) \, dy = \int_{\Omega_r(x)} u(y) \, dy$$

the mean value of the function $u \in L^1(\Omega, \mathbb{R}^N)$ over the set $\Omega_r(x)$, where $|\Omega_r(x)|$ is the $n$-dimensional Lebesgue measure of $\Omega_r(x)$.

Beside the standard space $C^\infty(\Omega, \mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\Omega, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W^{k,p}_0(\Omega, \mathbb{R}^N)$ we use Morrey spaces $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ (for more detail see e.g. [11]).

For $f \in L^1(\Omega)$, $0 < a < \infty$ we set

$$\mathcal{M}_a(f, \Omega) := \sup_{x \in \Omega, r < a} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{x,r}| \, dy.$$  

Definition 2.1 (see [13]). A function $f \in L^1(\Omega)$ is said to belong to $\text{BMO}(\Omega)$ if

$$\mathcal{M}_{\text{diam} \Omega}(f, \Omega) < \infty;$$

a function $f \in L^1(\Omega)$ is said to belong to $\text{VMO}(\Omega)$ if

$$\lim_{a \to 0} \mathcal{M}_a(f, \Omega) = 0.$$

In the proof of the theorem we will use the following results.
Lemma 2.1 ([15], p.37). Let $\psi: [0, \infty) \to [0, \infty]$ be a non decreasing function which is absolutely continuous on every closed interval of finite length, $\psi(0) = 0$. If $w \geq 0$ is measurable and $E(t) = \{ y \in \mathbb{R}^n : w(y) > t \}$ then
\[
\int_{\mathbb{R}^n} \psi \circ w \, dy = \int_0^\infty \mu(E(t)) \psi'(t) \, dt.
\]

Proposition 2.1 (see [1], [6], [11]). For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary, for $q \in [1, \infty)$ and $0 < \lambda < \mu \leq n$ we have
(a) $L^{q,\mu}(\Omega, \mathbb{R}^N) \subsetneq L^{q,\lambda}(\Omega, \mathbb{R}^N)$;
(b) $L^{q,n}(\Omega, \mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega, \mathbb{R}^N)$;
(c) if $u \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N)$ and $Du \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN})$, $\lambda \in (n-2,n)$ then $u \in C^{0,\alpha}(\Omega, \mathbb{R}^N)$, $\alpha = (\lambda + 2 - n)/2$.

Lemma 2.2 (see [1]). Let $A$, $d$ be positive constants, $\beta \in (0,n)$. Then there exist $\varepsilon_0$, $C$ positive such that for any nonnegative, nondecreasing function $\varphi$ defined on $[0,2d]$ and satisfying the inequality
\[
(2.1) \quad \varphi(\sigma) \leq \left(A \left(\frac{\sigma}{R}\right)^n + K\right)\varphi(2R) \quad \forall 0 < \sigma < R \leq d
\]
with $K \in (0,\varepsilon_0]$ we have
\[
(2.2) \quad \varphi(\sigma) \leq C\sigma^\beta(2d)^{-\beta}\varphi(2d), \quad \forall \sigma : 0 < \sigma \leq d.
\]

Lemma 2.3 (see e.g. [1], [6]). Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to the system
\[-D_\alpha(A^{\alpha\beta}_{ij}D_\beta u^j) = 0, \quad i = 1, \ldots, N\]
where $A^{\alpha\beta}_{ij}$ are constants satisfying (i) and (ii). Then there exists a constant $L = L(n, M/\nu) \geq 1$ such that for every weak solution $v \in W^{1,2}(\Omega, \mathbb{R}^N)$, for every $x \in \Omega$ and $0 < \sigma \leq R \leq \text{dist}(x, \partial \Omega)$ the estimate
\[
\int_{B_\sigma(x)} |Du(y)|^2 \, dy \leq L \left(\frac{\sigma}{R}\right)^n \int_{B_R(x)} |Du(y)|^2 \, dy
\]
holds.
Lemma 2.4 (see [6], [9]). Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of the functional (1.1) under the assumptions (i) and (ii). Then $Du \in L^{2p}_{\text{loc}}(\Omega, \mathbb{R}^{nN})$ for some $p > 1$ and there exists a constant $c = c(n, p, M/\nu)$ such that for all balls $B_{2R}(x) \subset \Omega$

$$\left( \int_{B_{R}(x)} |Du|^{2p} \, dy \right)^{1/2p} \leq c \left( \int_{B_{2R}(x)} |Du|^{2} \, dy \right)^{1/2}$$

holds.

Let $x_0$ be any fixed point of $\Omega$, $0 < R \leq \text{dist}(x_0, \partial\Omega)$. We set

$$(A^\alpha_{ij}(u_{x_0,R}))_{x_0,R} = \int_{B_{R}(x_0)} A^\alpha_{ij}(y, u_{x_0,R}) \, dy.$$ 

If $v$ is a solution to the system

$$(2.3) \begin{cases} D_\alpha((A^\alpha_{ij}(u_{x_0,R}))_{x_0,R}D_\beta v^j) = 0 \text{ in } B_R(x_0), \\ v - u \in W^{1,2}_{0}(B_R(x_0), \mathbb{R}^N) \end{cases}$$

then the next lemma is true.

Lemma 2.5 (see [6], [9]). Let $v \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ be a solution to the problem (2.3) with $u \in W^{1,2p}(B_R(x_0), \mathbb{R}^N)$, $p \geq 1$. Then

$$\int_{B_R(x)} |Dv|^{2p} \, dy \leq c(M/\nu) \int_{B_R(x)} |Du|^{2p} \, dy$$

holds.

Remark 2.1. Revising proofs of Lemmas 2.4 and 2.5 one can see that the constants from the above estimates depend increasingly on $M/\nu$.

3. PROOF OF THEOREM

We set $\varphi(r) = \varphi(x_0, r) = \int_{B_r(x_0)} |Du(y)|^2 \, dy$ for $B_r(x_0) \subset \Omega$. Now let $x_0$ be any fixed point of $\Omega$, dist $(x_0, \partial\Omega) \geq 2d > 0$, $R \leq d$ and let $v$ be a minimizer of the frozen functional

$$\mathcal{A}^0(v; B_R(x_0)) = \int_{B_R(x_0)} (A^\alpha_{ij}(u_R))_{R}D_\alpha v^i D_\beta v^j \, dx$$

among all functions in $W^{1,2}(B_R(x_0), \mathbb{R}^N)$ taking the values $u$ on $\partial B_R(x_0)$.
From the Euler equation for \( v \) and from Lemma (2.3) we have

\[(3.1) \quad \int_{B_\sigma(x_0)} |Dv|^2 \, dx \leq L \left( \frac{\sigma}{R} \right)^n \int_{B_R(x_0)} |Du|^2 \, dx, \quad \forall \ 0 < \sigma \leq R. \]

\[\text{Put } w = u - v. \text{ It is clear that } w \in W^{1,2}_0(B_R(x_0), \mathbb{R}^N). \text{ Using (3.1), by standard arguments we obtain} \]

\[(3.2) \quad \int_{B_\sigma(x_0)} |Du|^2 \, dx \leq 2 \left(1 + 2L \left( \frac{\sigma}{R} \right)^n \right) \int_{B_R(x_0)} |Dw|^2 \, dx + 4L \left( \frac{\sigma}{R} \right)^n \int_{B_R(x_0)} |Du|^2 \, dx. \]

In the sequel we will estimate the first integral on the right hand side of (3.2). From [8] (see Lemma 2.1) we have

\[(3.3) \quad \int_{B_R(x_0)} |Dw|^2 \, dx \leq \frac{2}{\nu} (\mathcal{A}^0(u; B_R(x_0)) - \mathcal{A}^0(v; B_R(x_0))) \]

\[\leq \frac{2}{\nu} \left\{ \int_{B_R(x_0)} ((A_{ij}^{\alpha \beta}(u_R))_R - A_{ij}^{\alpha \beta}(x, u_R)) D_{ij} u \, dx \right. \]

\[\left. + \int_{B_R(x_0)} (A_{ij}^{\alpha \beta}(x, u_R) - A_{ij}^{\alpha \beta}(x, u)) D_{ij} u \, dx \right. \]

\[+ \int_{B_R(x_0)} (A_{ij}^{\alpha \beta}(x, u_R) - (A_{ij}^{\alpha \beta}(u_R))_R) D_{ij} v \, dx \]

\[+ \int_{B_R(x_0)} (A_{ij}^{\alpha \beta}(x, v) - A_{ij}^{\alpha \beta}(x, u_R)) D_{ij} v \, dx \]

\[+ \mathcal{A}(u; B_R(x_0)) - \mathcal{A}(v; B_R(x_0)) \}

\[= \frac{2}{\nu} \{ I + II + III + IV + \mathcal{A}(u; B_R(x_0)) - \mathcal{A}(v; B_R(x_0)) \}

\[\leq \frac{2}{\nu} (I + II + III + IV). \]

Notice that \( \mathcal{A}(u; B_R(x_0)) - \mathcal{A}(v; B_R(x_0)) \leq 0 \), since \( u \) is a minimizer.

Now we will estimate the terms I, II, III and IV from (3.3). We will denote \((A_{ij}^{\alpha \beta})_R = A\). Using the Hölder inequality and higher integrability of the gradient of minima \((p > 1, p' = p/(p - 1))\) we obtain

\[|I| \leq \int_{B_R(x_0)} |(A(u_R))_R - A(x, u_R)||Du|^2 \, dx \]

\[\leq cR^{n/p} \left( \int_{B_R(x_0)} |(A(u_R))_R - A(x, u_R)|^{p'} \, dx \right)^{1/p'} \left( \int_{B_R(x_0)} |Du|^{2p} \, dx \right)^{1/p} \]

\[\leq c(n, p, M/\nu) R^{n/p} \left( \int_{B_R(x_0)} |(A(u_R))_R - A(x, u_R)|^{p'} \, dx \right)^{1/p'} \int_{B_{2R}(x_0)} |Du|^2 \, dx. \]
Taking into account the assumptions (i), (iv) and Definition 2.1 we obtain

\[(3.4) \quad |I| \leq c(n, N, p, M/\nu) (2M)^{1/p} (\mathcal{M}_R (A(\cdot, u_R)))^{1/p'} \varphi(2R).\]

A similarity of the terms I and III enables us to write (by means of Lemma 2.5, see [2] for details) the inequality

\[(3.5) \quad |III| \leq c(n, N, p, M/\nu) (2M)^{1/p} (\mathcal{M}_R (A(\cdot, u_R)))^{1/p'} \varphi(2R).\]

Using the Hölder inequality, property (iii) and Lemma 2.4 we get

\[(3.6) \quad |II| \leq c(n, N, p, M/\nu) \left( \frac{1}{R^n} \int_{B_R(x_0)} \omega^{p'}(|u - u_R|) \, dx \right)^{1/p'} \varphi(2R).\]

Taking in Lemma 2.1 \( \psi(t) = \omega^{p'}(t), \ w = |u - u_R| \) on \( B_R(x_0) \) and \( w = 0 \) out of \( B_R(x_0) \), we have \( E_R(t) = \{ y \in B_R : |u - u_R| > t \} \) and so we get

\[\int_{B_R(x_0)} \omega^{p'}(|u - u_R|) \, dx = \int_0^\infty \left[ \frac{d}{dt} (\omega^{p'})(t) \right] \mu(E_R(t)) \, dt.\]

Now under the assumptions of Theorem 1.1 if we suppose

\[Q_p = \int_0^\infty t^{-1} \frac{d}{dt} (\omega^{p'})(t) \, dt < \infty,\]

then (taking into account that \( \mu(E_R(t)) \leq t^{-1} \int_0^t \mu(E_R(s)) \, ds \)) we have

\[\int_0^\infty \left[ \frac{d}{dt} (\omega^{p'})(t) \right] \mu(E_R(t)) \, dt \leq \int_0^\infty \frac{d}{dt} (\omega^{p'})(t) \left( \frac{1}{t} \int_0^t \mu(E_R(s)) \, ds \right) \, dt \]

\[\leq Q_p \int_{B_R(x_0)} |u - u_R| \, dx.\]

On the other hand, if we suppose \( Q_p = \sup_{t \in (0, \infty)} (d/dt)(\omega^{p'})(t) < \infty \) then

\[\int_0^\infty \left[ \frac{d}{dt} (\omega^{p'})(t) \right] \mu(E_R(t)) \, dt \leq Q_p \int_{B_R(x_0)} |u - u_R| \, dx\]

holds as well. So in both the cases we have

\[\int_{B_R(x_0)} \omega^{p'}(|u - u_R|) \, dx \leq Q_p \int_{B_R(x_0)} |u - u_R| \, dx.\]
Using the Poincaré inequality and the assumption about $Du$ we finally get

$$
(3.6) \quad |I| \leq c(n, N, p, M/\nu)Q_p^{1/p'} \|Du\|^{1/p'}_{L^2, n-2(\Omega, \mathbb{R}^n)} \varphi(2R).
$$

Combining the last arguments with Lemma 2.4 and Lemma 2.5 we can conclude in a similar way

$$
(3.7) \quad |IV| \leq c(n, N, p, M/\nu)Q_p^{1/p'} \|Du\|^{1/p'}_{L^2, n-2(\Omega, \mathbb{R}^n)} \varphi(2R).
$$

Estimates (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7) lead to the following inequality

$$
\varphi(\sigma) = \int_{B_{\sigma}(x_0)} |Du|^2 \, dx
\leq \left\{ 4L\left( \frac{\sigma}{R} \right)^n + \frac{8}{\nu} \left( 1 + 2L \left( \frac{\sigma}{R} \right)^n \right) \times c\left[ (2M)^{1/p} (M_R (A(\cdot, u_R)))^{1/p'} + (Q_p \|Du\|_{L^2, n-2(\Omega, \mathbb{R}^n)})^{1/p'} \right]^\frac{1}{p'} \right\} \varphi(2R)
$$

where $c = c(n, N, p, M/\nu)$.

Now we can use Lemma 2.2 in the following manner:

We take $\gamma \in (0, 1)$ and set

$$
A = 4L, \quad \varepsilon_0 = \frac{1}{2(2n+3L)(n-2+2\gamma)/2(1-\gamma)}
$$

and

$$
K = \frac{8}{\nu} (1 + 2L) c\left[ (2M)^{1/p} (M_R (A(\cdot, u_R)))^{1/p'} + (Q_p \|Du\|_{L^2, n-2(\Omega, \mathbb{R}^n)})^{1/p'} \right].
$$

Then the assumption (1.4) and a suitable small $d > 0$ (remember the condition (iv) and Definition 2.1) imply that $K < \varepsilon_0$ and hence

$$
\varphi(\sigma) \leq c\sigma^{n-2+2\gamma}.
$$

The result is then a consequence of Proposition 2.1.(c)

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