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ON HÖLDER REGULARITY FOR VECTOR-VALUED MINIMIZERS OF QUASILINEAR FUNCTIONALS

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Abstract. We discuss the interior Hölder everywhere regularity for minimizers of quasilinear functionals of the type

$$\mathcal{A}(u;\Omega)=\int_\Omega A_{ij}^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j \,\mathrm{d}x$$

whose gradients belong to the Morrey space $L^{2,n-2}(\Omega,\mathbb{R}^{nN})$.

Keywords: quasilinear functional, minimizer, regularity, Campanato-Morrey space

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1. Introduction

In this paper we study the interior everywhere regularity of functions minimizing variational integrals

$$\mathcal{A}(u;\Omega)=\int_\Omega A_{ij}^{\alpha\beta}(x,u)D_\alpha u^i D_\beta u^j \,\mathrm{d}x \tag{1.1}$$

where $u\colon \Omega\to\mathbb{R}^N$, $N>1$, $\Omega\subset\mathbb{R}^n$, $n\geqslant 3$ is a bounded open set, $x=(x_1,\dots,x_n)\in\Omega$, $u(x)=(u^1(x),\dots,u^N(x))$, $Du=\{D_\alpha u^i\}$, $D_\alpha=\partial/\partial x_\alpha$, $\alpha=1,\dots,n$, $i=1,\dots,N$.

Throughout the whole text we use the summation convention over repeated indices. We call a function $u\in W^{1,2}(\Omega,\mathbb{R}^N)$ a minimizer of the functional $\mathcal{A}(u;\Omega)$ if

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and only if $\mathcal{A}(u;\Omega) \leqslant \mathcal{A}(v;\Omega)$ for every $v \in W^{1,2}(\Omega,\mathbb{R}^N)$ with $u - v \in W_0^{1,2}(\Omega,\mathbb{R}^N)$. For more information see [6], [9].

On the functional $\mathcal{A}$ we assume the following conditions:

(i) $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, $A_{ij}^{\alpha\beta}$ are continuous functions in $u \in \mathbb{R}^N$ for every $x \in \Omega$ and there exists $M > 0$ such that $|A_{ij}^{\alpha\beta}(x,u)| \leqslant M$, $\forall\, x \in \Omega$, $\forall\, u \in \mathbb{R}^N$.

(ii) (ellipticity) There exists $\nu > 0$ such that

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j \geqslant \nu|\xi|^2, \quad \forall\, x \in \Omega, \quad \forall\, u \in \mathbb{R}^N, \quad \forall\, \xi \in \mathbb{R}^{nN}. \tag{1.2}$$

(iii) (oscillation of coefficients) There exists a real function ω continuous on $[0,\infty)$ which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and u, $v \in \mathbb{R}^N$

$$|A_{ij}^{\alpha\beta}(x,u) - A_{ij}^{\alpha\beta}(x,v)| \leqslant \omega\left(|u-v|\right). \tag{1.3}$$

We set $\omega_\infty = \lim\limits_{t\to\infty} \omega(t) \leqslant 2M$.

(iv) For all $u \in \mathbb{R}^N$, $A_{ij}^{\alpha\beta}(\cdot,u) \in \mathrm{VMO}(\Omega)$ (uniformly with respect to $u \in \mathbb{R}^N$).

It is well known (see [6], p. 169) that (iii) implies absolute continuity of ω on $[0,\infty)$. In what follows, by pointwise derivative ω' of ω we will understand the right derivative which is finite on $(0,\infty)$. Considering the assumption (iv) it is worth recalling that since C^0 is a proper subset of VMO, the continuity of coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x,u)$ with respect to x is not supposed.

In this paper we deal with the case $n \geqslant 3$ because for $n = 2$ higher integrability of the gradient of minimizer (see Preliminaries, Lemma 2.4) and the Sobolev imbedding theorem imply that u is locally Hölder continuous in Ω. From many examples (see [4], [6], [9], [10], [12], [14]) for $n \geqslant 3$ it is known that the minimizer u of the functional (1.1) need not be continuous or bounded even in the case of smooth coefficients $A_{ij}^{\alpha\beta}$. For this reason the so called partial regularity for minimizers of the functional (1.1) was studied by many authors ([7], [8], [5]). In our paper (which is motivated by [3]) we concentrate on conditions that imply an everywhere regularity result. More precisely, we state conditions which imply that the minimizer u with gradient $Du \in L^{2,n-2}(\Omega,\mathbb{R}^{nN})$ belongs to $C^{0,\gamma}(\Omega,\mathbb{R}^N)$. The condition $Du \in L^{2,n-2}(\Omega,\mathbb{R}^{nN})$ seems to be natural with respect to the paper [2].

Now we can state the following result:

Theorem 1.1. *Let $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ be a minimizer of the functional* (1.1) *such that $Du \in L^{2,n-2}(\Omega,\mathbb{R}^{nN})$ and let the hypotheses* (i), (ii), (iii), (iv) *be satisfied. Assume that there exists $p > 1$ such that*

$$Q_p := \min\left\{ \sup_{t\in(0,\infty)} \frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p/(p-1)})(t), \int_0^\infty t^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p/(p-1)})(t)\,\mathrm{d}t \right\} < \infty$$

and let $\gamma \in (0,1)$. *Then the inequality*

$$(Q_p \|Du\|_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})})^{1-1/p} \leqslant \nu C \tag{1.4}$$

implies that $u \in C^{0,\gamma}(\Omega,\mathbb{R}^N)$.

Here

$$C = \frac{2}{3c(n,N,p,M/\nu)(2^{n+3}L)^{\frac{1}{2}n/(1-\gamma)}},$$

where L *is from Lemma* 2.3.

2. Preliminaries

If $x \in \mathbb{R}^n$ and r is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n \colon |y-x| < r\}$, $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by

$$u_{x,r} = \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} u(y)\,\mathrm{d}y = \fint_{\Omega_r(x)} u(y)\,\mathrm{d}y$$

the mean value of the function $u \in L^1(\Omega,\mathbb{R}^N)$ over the set $\Omega_r(x)$, where $|\Omega_r(x)|$ is the n-dimensional Lebesgue measure of $\Omega_r(x)$.

Beside the standard space $C_0^\infty(\Omega,\mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$, $W_0^{k,p}(\Omega,\mathbb{R}^N)$ we use Morrey spaces $L^{q,\lambda}(\Omega,\mathbb{R}^N)$ (for more detail see e.g. [11]).

For $f \in L^1(\Omega)$, $0 < a < \infty$ we set

$$\mathcal{M}_a(f,\Omega) := \sup_{x\in\Omega, r<a} \fint_{\Omega_r(x)} |f(y) - f_{x,r}|\,\mathrm{d}y.$$

Definition 2.1 (see [13]). A function $f \in L^1(\Omega)$ is said to belong to $\mathrm{BMO}(\Omega)$ if

$$\mathcal{M}_{\operatorname{diam}\Omega}(f,\Omega) < \infty;$$

a function $f \in L^1(\Omega)$ is said to belong to $\mathrm{VMO}(\Omega)$ if

$$\lim_{a\to 0} \mathcal{M}_a(f,\Omega) = 0.$$

In the proof of the theorem we will use the following results.

Lemma 2.1 ([15], p. 37)**.** *Let* $\psi\colon [0,\infty) \to [0,\infty]$ *be a non decreasing function which is absolutely continuous on every closed interval of finite length,* $\psi(0)=0$*. If* $w \geqslant 0$ *is measurable and* $E(t) = \{y \in \mathbb{R}^n \colon\ w(y) > t\}$ *then*

$$\int_{\mathbb{R}^n} \psi \circ w \,\mathrm{d}y = \int_0^\infty \mu\left(E(t)\right) \psi'(t)\,\mathrm{d}t.$$

Proposition 2.1 (see [1], [6], [11])**.** *For a bounded domain* $\Omega \subset \mathbb{R}^n$ *with a Lipschitz boundary, for* $q \in [1,\infty)$ *and* $0 < \lambda < \mu \leqslant n$ *we have*

(a) $L^{q,\mu}(\Omega,\mathbb{R}^N) \subsetneq L^{q,\lambda}(\Omega,\mathbb{R}^N)$;

(b) $L^{q,n}(\Omega,\mathbb{R}^N)$ *is isomorphic to the* $L^\infty(\Omega,\mathbb{R}^N)$;

(c) *if* $u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ *and* $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$, $\lambda \in (n-2,n)$ *then* $u \in C^{0,\alpha}(\Omega,\mathbb{R}^N)$, $\alpha = (\lambda+2-n)/2$.

Lemma 2.2 (see [1])**.** *Let* A, d *be positive constants,* $\beta \in (0,n)$*. Then there exist* ε_0, C *positive such that for any nonnegative, nondecreasing function* φ *defined on* $[0,2d]$ *and satisfying the inequality*

$$\varphi(\sigma) \leqslant \Big(A\Big(\frac{\sigma}{R}\Big)^n + K\Big)\varphi(2R) \qquad \forall\, 0<\sigma<R\leqslant d \tag{2.1}$$

with $K \in (0,\varepsilon_0]$ *we have*

$$\varphi(\sigma) \leqslant C\sigma^\beta (2d)^{-\beta}\varphi(2d), \qquad \forall \sigma\colon\ 0<\sigma\leqslant d. \tag{2.2}$$

Lemma 2.3 (see e.g. [1], [6])**.** *Let* $u \in W^{1,2}(\Omega,\mathbb{R}^N)$ *be a weak solution to the system*

$$-D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = 0, \quad i = 1,\dots,N$$

where $A^{\alpha\beta}_{ij}$ *are constants satisfying* (i) *and* (ii)*. Then there exists a constant* $L = L(n, M/\nu) \geqslant 1$ *such that for every weak solution* $v \in W^{1,2}(\Omega,\mathbb{R}^N)$*, for every* $x \in \Omega$ *and* $0 < \sigma \leqslant R \leqslant \operatorname{dist}(x,\partial\Omega)$ *the estimate*

$$\int_{B_\sigma(x)} |Du(y)|^2 \,\mathrm{d}y \leqslant L\Big(\frac{\sigma}{R}\Big)^n \int_{B_R(x)} |Du(y)|^2\,\mathrm{d}y$$

holds.

Lemma 2.4 (see [6], [9]). *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of the functional* (1.1) *under the assumptions* (i) *and* (ii). *Then $Du \in L^{2p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ for some $p > 1$ and there exists a constant $c = c(n, p, M/\nu)$ such that for all balls $B_{2R}(x) \subset \Omega$*

$$\left(\fint_{B_R(x)} |Du|^{2p} \,\mathrm{d}y \right)^{1/2p} \leqslant c \left(\fint_{B_{2R}(x)} |Du|^2 \,\mathrm{d}y \right)^{1/2}$$

holds.

Let x_0 be any fixed point of Ω, $0 < R \leqslant \operatorname{dist}(x_0, \partial\Omega)$. We set

$$(A^{\alpha\beta}_{ij}(u_{x_0,R}))_{x_0,R} = \fint_{B_R(x_0)} A^{\alpha\beta}_{ij}(y, u_{x_0,R}) \,\mathrm{d}y.$$

If v is a solution to the system

$$\begin{cases} D_\alpha((A^{\alpha\beta}_{ij}(u_{x_0,R}))_{x_0,R} D_\beta v^j) = 0 \text{ in } B_R(x_0), \\ v - u \in W^{1,2}_0(B_R(x_0), \mathbb{R}^N) \end{cases} \tag{2.3}$$

then the next lemma is true.

Lemma 2.5 (see [6], [9]). *Let $v \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ be a solution to the problem* (2.3) *with $u \in W^{1,2p}(B_R(x_0), \mathbb{R}^N)$, $p \geqslant 1$. Then*

$$\int_{B_R(x)} |Dv|^{2p} \,\mathrm{d}y \leqslant c(M/\nu) \int_{B_R(x)} |Du|^{2p} \,\mathrm{d}y$$

holds.

Remark 2.1. Revising proofs of Lemmas 2.4 and 2.5 one can see that the constants from the above estimates depend increasingly on M/ν.

3. Proof of theorem

We set $\varphi(r) = \varphi(x_0, r) = \int_{B_r(x_0)} |Du(y)|^2 \,\mathrm{d}y$ for $B_r(x_0) \subset \Omega$. Now let x_0 be any fixed point of Ω, dist $(x_0, \partial\Omega) \geqslant 2d > 0$, $R \leqslant d$ and let v be a minimizer of the frozen functional

$$\mathcal{A}^0(v; B_R(x_0)) = \int_{B_R(x_0)} (A^{\alpha\beta}_{ij}(u_R))_R D_\alpha v^i D_\beta v^j \,\mathrm{d}x$$

among all functions in $W^{1,2}(B_R(x_0), \mathbb{R}^N)$ taking the values u on $\partial B_R(x_0)$.

From the Euler equation for v and from Lemma (2.3) we have

$$\int_{B_\sigma(x_0)} |Dv|^2\,\mathrm{d}x \leqslant L\Big(\frac{\sigma}{R}\Big)^n \int_{B_R(x_0)} |Dv|^2\,\mathrm{d}x, \quad \forall\ 0<\sigma\leqslant R. \tag{3.1}$$

Put $w=u-v$. It is clear that $w\in W_0^{1,2}(B_R(x_0),\mathbb{R}^N)$. Using (3.1), by standard arguments we obtain

$$\int_{B_\sigma(x_0)} |Du|^2\,\mathrm{d}x \leqslant 2\Big(1+2L\Big(\frac{\sigma}{R}\Big)^n\Big)\int_{B_R(x_0)} |Dw|^2\,\mathrm{d}x + 4L\Big(\frac{\sigma}{R}\Big)^n \int_{B_R(x_0)} |Du|^2\,\mathrm{d}x. \tag{3.2}$$

In the sequel we will estimate the first integral on the right hand side of (3.2). From [8] (see Lemma 2.1) we have

$$\begin{aligned}\int_{B_R(x_0)} |Dw|^2\,\mathrm{d}x &\leqslant \frac{2}{\nu}(\mathcal{A}^0(u;B_R(x_0))-\mathcal{A}^0(v;B_R(x_0)))\\
&\leqslant \frac{2}{\nu}\Big\{\int_{B_R(x_0)}((A_{ij}^{\alpha\beta}(u_R))_R - A_{ij}^{\alpha\beta}(x,u_R))D_\alpha u^i D_\beta u^j\,\mathrm{d}x\\
&\qquad+\int_{B_R(x_0)}(A_{ij}^{\alpha\beta}(x,u_R)-A_{ij}^{\alpha\beta}(x,u))D_\alpha u^i D_\beta u^j\,\mathrm{d}x\\
&\qquad+\int_{B_R(x_0)}(A_{ij}^{\alpha\beta}(x,u_R)-(A_{ij}^{\alpha\beta}(u_R))_R)D_\alpha v^i D_\beta v^j\,\mathrm{d}x\\
&\qquad+\int_{B_R(x_0)}(A_{ij}^{\alpha\beta}(x,v)-A_{ij}^{\alpha\beta}(x,u_R))D_\alpha v^i D_\beta v^j\,\mathrm{d}x\\
&\qquad+\mathcal{A}(u;B_R(x_0))-\mathcal{A}(v;B_R(x_0))\Big\}\\
&=\frac{2}{\nu}\{\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathcal{A}(u;B_R(x_0))-\mathcal{A}(v;B_R(x_0))\}\\
&\leqslant\frac{2}{\nu}(\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}).\end{aligned} \tag{3.3}$$

Notice that $\mathcal{A}(u;B_R(x_0))-\mathcal{A}(v;B_R(x_0))\leqslant 0$, since u is a minimizer.

Now we will estimate the terms I, II, III and IV from (3.3). We will denote $(A_{ij}^{\alpha\beta})=:A$. Using the Hölder inequality and higher integrability of the gradient of minima ($p>1$, $p'=p/(p-1)$) we obtain

$$\begin{aligned}|I| &\leqslant \int_{B_R(x_0)} |(A(u_R))_R - A(x,u_R)|\,|Du|^2\,\mathrm{d}x\\
&\leqslant cR^{n/p}\Big(\int_{B_R(x_0)}|(A(u_R))_R - A(x,u_R)|^{p'}\,\mathrm{d}x\Big)^{1/p'}\Big(\fint_{B_R(x_0)}|Du|^{2p}\,\mathrm{d}x\Big)^{1/p}\\
&\leqslant c(n,N,p,M/\nu)R^{n/p}\Big(\int_{B_R(x_0)}|(A(u_R))_R - A(x,u_R)|^{p'}\,\mathrm{d}x\Big)^{1/p'}\fint_{B_{2R}(x_0)}|Du|^2\,\mathrm{d}x.\end{aligned}$$

Taking into account the assumptions (i), (iv) and Definition 2.1 we obtain

$$|\mathrm{I}| \leqslant c(n, N, p, M/\nu)(2M)^{1/p} \left(\mathcal{M}_R\left(A(\cdot, u_R)\right)\right)^{1/p'} \varphi(2R). \tag{3.4}$$

A similarity of the terms I and III enables us to write (by means of Lemma 2.5, see [2] for details) the inequality

$$|\mathrm{III}| \leqslant c(n, N, p, M/\nu)(2M)^{1/p} \left(\mathcal{M}_R\left(A(\cdot, u_R)\right)\right)^{1/p'} \varphi(2R). \tag{3.5}$$

Using the Hölder inequality, property (iii) and Lemma 2.4 we get

$$|\mathrm{II}| \leqslant c(n, N, p, M/\nu)\left(\frac{1}{R^n}\int_{B_R(x_0)} \omega^{p'}(|u-u_R|)\,\mathrm{d}x\right)^{1/p'} \varphi(2R).$$

Taking in Lemma 2.1 $\psi(t) = \omega^{p'}(t)$, $w = |u-u_R|$ on $B_R(x_0)$ and $w = 0$ out of $B_R(x_0)$, we have $E_R(t) = \{y \in B_R\colon\ |u-u_R| > t\}$ and so we get

$$\int_{B_R(x_0)} \omega^{p'}\left(|u-u_R|\right)\,\mathrm{d}x = \int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\right]\mu\left(E_R(t)\right)\,\mathrm{d}t.$$

Now under the assumptions of Theorem 1.1 if we suppose

$$Q_p = \int_0^\infty t^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\,\mathrm{d}t < \infty,$$

then (taking into account that $\mu(E_R(t)) \leqslant t^{-1}\int_0^t \mu(E_R(s))\,\mathrm{d}s$) we have

$$\begin{aligned}\int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\right]\mu(E_R(t))\,\mathrm{d}t &\leqslant \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\left(\frac{1}{t}\int_0^t \mu(E_R(s))\,\mathrm{d}s\right)\mathrm{d}t \\ &\leqslant Q_p \int_{B_R(x_0)} |u-u_R|\,\mathrm{d}x.\end{aligned}$$

On the other hand, if we suppose $Q_p = \sup\limits_{t\in(0,\infty)} (\mathrm{d}/\mathrm{d}t)(\omega^{p'})(t) < \infty$ then

$$\int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}t}(\omega^{p'})(t)\right]\mu(E_R(t))\,\mathrm{d}t \leqslant Q_p \int_{B_R(x_0)} |u-u_R|\,\mathrm{d}x$$

holds as well. So in both the cases we have

$$\int_{B_R(x_0)} \omega^{p'}\left(|u-u_R|\right)\,\mathrm{d}x \leqslant Q_p \int_{B_R(x_0)} |u-u_R|\,\mathrm{d}x.$$

Using the Poincaré inequality and the assumption about Du we finally get

$$|\mathrm{II}| \leqslant c(n, N, p, M/\nu) Q_p^{1/p'} \|Du\|^{1/p'}_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})} \varphi(2R). \tag{3.6}$$

Combining the last arguments with Lemma 2.4 and Lemma 2.5 we can conclude in a similar way

$$|\mathrm{IV}| \leqslant c(n, N, p, M/\nu) Q_p^{1/p'} \|Du\|^{1/p'}_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})} \varphi(2R). \tag{3.7}$$

Estimates (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7) lead to the following inequality

$$\begin{aligned}\varphi(\sigma) &= \int_{B_\sigma(x_0)} |Du|^2 \,\mathrm{d}x \\ &\leqslant \Big\{ 4L\Big(\frac{\sigma}{R}\Big)^n + \frac{8}{\nu}\Big(1 + 2L\Big(\frac{\sigma}{R}\Big)^n\Big) \\ &\quad \times c[(2M)^{1/p} \left(\mathcal{M}_R\left(A(\cdot, u_R)\right)\right)^{1/p'} + \big(Q_p \|Du\|_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})}\big)^{1/p'}] \Big\} \varphi(2R)\end{aligned}$$

where $c = c(n, N, p, M/\nu)$.

Now we can use Lemma 2.2 in the following manner:

We take $\gamma \in (0, 1)$ and set

$$A = 4L, \qquad \varepsilon_0 = \frac{1}{2(2^{n+3}L)^{(n-2+2\gamma)/2(1-\gamma)}}$$

and

$$K = \frac{8}{\nu}\,(1 + 2L)\, c[(2M)^{1/p} \left(\mathcal{M}_R\left(A(\cdot, u_R)\right)\right)^{1/p'} + \big(Q_p \|Du\|_{L^{2,n-2}(\Omega,\mathbb{R}^{nN})}\big)^{1/p'}].$$

Then the assumption (1.4) and a suitable small $d > 0$ (remember the condition (iv) and Definition 2.1) imply that $K < \varepsilon_0$ and hence

$$\varphi(\sigma) \leqslant c\sigma^{n-2+2\gamma}.$$

The result is then a consequence of Proposition 2.1.(c)

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