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ON PERIODIC SOLUTIONS OF NON-AUTONOMOUS SECOND ORDER HAMILTONIAN SYSTEMS*

XINGYONG ZHANG, YINGGAO ZHOU, Changsha

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Abstract. The purpose of this paper is to study the existence of periodic solutions for the non-autonomous second order Hamiltonian system

\[
\begin{aligned}
\ddot{u}(t) &= \nabla F(t, u(t)), \text{ a.e. } t \in [0, T], \\
u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0.
\end{aligned}
\]

Some new existence theorems are obtained by the least action principle.

Keywords: periodic solution, critical point, non-autonomous second-order system, Sobolev’s inequality

MSC 2010: 34C25, 37J45, 58E50

1. Introduction

Consider the non-autonomous second order Hamiltonian system

\[
\begin{aligned}
\ddot{u}(t) &= \nabla F(t, u(t)), \text{ a.e. } t \in [0, T], \\
u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0,
\end{aligned}
\]

where \( T > 0, F: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \) satisfies the following assumption:

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(A) $F(t, x)$ is measurable in $t$ for every $x \in \mathbb{R}^N$ and continuously differentiable in $x$ for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0, T]; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The corresponding functional $\varphi$ on $H^1_T$ given by

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T F(t, u(t)) \, dt$$

is continuously differentiable and weakly lower semicontinuous on $H^1_T$, where

$$H^1_T = \{u: [0, T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous, } u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; \mathbb{R}^N)\}$$

is a Hilbert space with the usual scalar product and norm (see [4]). Moreover, one has

$$(\varphi'(u), v) = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (\nabla F(t, u(t)), v(t))] \, dt$$

for $u, v \in H^1_T$. It is well known that the solutions of problem (1.1) correspond to the critical points of $\varphi$ (see [4]).

For $u \in H^1_T$, let $\bar{u} = T^{-1} \int_0^T u(t) \, dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. Then one has

$$\|\tilde{u}\|_{\infty}^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 \, dt \quad \text{(Sobolev's inequality)}$$

(see [4], Proposition 1.3).

In many papers (see [1], [3]–[10]) it has been shown by the least action principle that problem (1.1) has at least one solution which minimizes $\varphi$ on $H^1_T$. When $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$, Mawhin-Willem [4] studied the existence of a solution which minimizes $\varphi$ on $H^1_T$ for problem (1.1). For non-convex potential cases, using the least action principle, the existence of a solution which minimizes $\varphi$ on $H^1_T$ has been investigated by many people (see [3], [6]–[10] and the references therein). Inspired and motivated by the results in [3] and [8]–[10], we consider problem (1.1) with the potential $F(t, x) = F_1(t, x) + F_2(t, x)$. In our Theorem 2.1, it is assumed that

$$F_1(t, x) \geq G(x)|f(t)|,$$
where \( G(x) \) is subconvex and \( \nabla F_2(t, x) \) has sublinear growth. In Theorem 2.2, it is assumed that \( F_1(t, x) \) satisfies (1.2) and \( F_2(t, x) \) has subquadratic growth. In Theorem 2.3, it is assumed that \( F_1(t, x) \) satisfies (1.2) and

\[
F_2(t, x) \geq (h(t), x) + g(t),
\]

where \( h(t) \in L^1(0, T; \mathbb{R}^N) \) and \( g(t) \in L^1(0, T; \mathbb{R}) \). In Theorem 2.4, it is assumed that \( F_1(t, x) \) is subconvex with subquadratic growth and \( F_2(t, x) \) satisfies (1.3). In Theorem 2.5, it is assumed that \( F_1(t, x) \to +\infty \) uniformly for a.e. \( t \in [0, T] \), as \( |x| \to \infty \) and \( F_2(t, x) \) satisfies (1.3). By using the least action principle, we obtain that system (1.1) has at least one solution. Theorems 2.1–2.4 develop and generalize the corresponding results in [8] and [10] and Theorem 2.5 is a new result.

2. Main results and proofs

We first recall a definition due to Wu-Tang [9].

A function \( G: \mathbb{R}^N \to \mathbb{R} \) is called \((\lambda, \mu)\)-subconvex if

\[
G(\lambda(x + y)) \leq \mu(G(x) + G(y))
\]

for some \( \lambda, \mu > 0 \) and all \( x, y \in \mathbb{R}^N \). A function is called \( \gamma \)-subadditive if it is \((1, \gamma)\)-subconvex. A function is called subadditive if it is \((1, 1)\)-subadditive. The convex and subadditive functions are special cases of subconvex functions.

**Theorem 2.1.** Suppose that \( F(t, x) = F_1(t, x) + F_2(t, x) \), where \( F_1 \) and \( F_2 \) satisfy assumption (A) and the following conditions:

(i) there exist \( M > 0 \), \( f \in L^1(0, T; \mathbb{R}) \) and \( G: \mathbb{R}^N \to \mathbb{R} \) which is continuous and \((\lambda, \mu)\)-subconvex for some \( \lambda > \frac{1}{2} \) and \( 0 < \mu < 2\lambda^2 \), such that

\[
F_1(t, x) \geq G(x)|f(t)|
\]

for all \( |x| \geq M \) and a.e. \( t \in [0, T] \);

(ii) there exist \( p_1, p_2 \in L^1(0, T; \mathbb{R}^+) \) and \( \alpha \in [0, 1) \) such that

\[
|\nabla F_2(t, x)| \leq p_1(t)|x|^{\alpha} + p_2(t)
\]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \);

(iii)

\[
\frac{1}{|x|^{2\alpha}} \left[ \frac{G(\lambda x)}{\mu} \int_0^T |f(t)| \, dt + \int_0^T F_2(t, x) \, dt \right] \to +\infty \quad \text{as} \quad |x| \to \infty.
\]

Then problem (1.1) has at least one solution which minimizes \( \varphi \) on \( H_1^T \).
Proof. Let $\beta = \log_{2\lambda} 2\mu$. Then $\beta < 2$. In a way similar to Wu-Tang [9], by the $(\lambda, \mu)$-subconvexity and continuity of $G(\cdot)$, one can obtain that there exists a constant $a_0 > 0$ such that

$$|f(t)|G(x) \leq a_0(2\mu|x|^\beta + 1)|f(t)|$$

for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}^N$. Thus by assumption (A) and condition (i), we have

$$F_1(t, x) \geq G(x)|f(t)| + p(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$ and for $p \in L^1(0, T)$ given by

$$p(t) = -\max_{0 \leq |x| \leq M} a(|x|) b(t) - a_0(2\mu M^\beta + 1)|f(t)|.$$

It follows from (i) and Sobolev’s inequality that

$$\left| \int_0^T F_1(t, u(t)) \, dt \right| \leq \int_0^T G(u(t))|f(t)| \, dt + \int_0^T p(t) \, dt \leq \int_0^T G(\lambda \bar{u})|f(t)| \, dt - \int_0^T G(-\tilde{u}(t))|f(t)| \, dt + \int_0^T p(t) \, dt \leq \frac{1}{\mu} \int_0^T G(\lambda \bar{u})|f(t)| \, dt - a_0(2\mu \|\tilde{u}\|^\beta_\infty + 1) \int_0^T |f(t)| \, dt + \int_0^T p(t) \, dt \leq \frac{1}{\mu} \int_0^T G(\lambda \bar{u})|f(t)| \, dt \leq C_1 \left( \int_0^T |\bar{u}(t)|^2 \, dt \right)^{\beta/2} - C_2 + C_3$$

for all $u \in H^1_T$ and some constants $C_1, C_2, C_3$. It follows from assumption (ii) and Sobolev’s inequality that

$$\left| \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] \, dt \right| \leq \int_0^T \int_0^1 (\nabla F_2(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t)) \, ds \, dt \leq \int_0^T \int_0^1 p_1(t)|\bar{u} + s\tilde{u}(t)|^\alpha |\tilde{u}(t)| \, ds \, dt + \int_0^T p_2(t)|\tilde{u}(t)| \, dt.$$
\[\begin{align*}
\leq & \frac{2}{3} \|u\|_{\infty}^2 + \frac{T}{2} \|\dot{u}\|_{\infty}^2 \int_0^T p_1(t) \, dt + \|\dot{u}\|_{\infty} \int_0^T p_2(t) \, dt \\
\leq & \frac{3}{T} \|\dot{u}\|_{\infty}^2 + \frac{T}{3} \|\dot{u}\|_{\infty}^{2\alpha} \left( \int_0^T p_1(t) \, dt \right)^2 + \frac{2}{3} \|u\|_{\infty}^\alpha \int_0^T p_1(t) \, dt \\
& \quad + \|\dot{u}\|_{\infty} \int_0^T p_2(t) \, dt \\
\leq & \frac{1}{4} \int_0^T \|\dot{u}(t)\|^2 \, dt + C_4 \|\dot{u}\|_{\infty}^{2\alpha} + C_5 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{\alpha+1/2} \\
& \quad + C_6 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{1/2}
\end{align*}\]

for all \(u \in H_T^1\) and some positive constants \(C_4, C_5, C_6\). It follows from (2.1) and (2.2) that

\[\varphi(u) = \frac{1}{2} \int_0^T \|\dot{u}(t)\|^2 \, dt + \int_0^T F_1(t, u(t)) \, dt + \int_0^T \left[ F_2(t, u(t)) - F_2(t, \bar{u}) \right] \, dt\]

\[\geq \frac{1}{2} \int_0^T \|\dot{u}(t)\|^2 \, dt + \frac{1}{\mu} \int_0^T G(\lambda \bar{u}) |f(t)| \, dt - C_1 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{\beta/2} - C_2 + C_3\]

\[- \frac{1}{4} \int_0^T \|\dot{u}(t)\|^2 \, dt - C_4 \|\dot{u}\|_{\infty}^{2\alpha} - C_5 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{\alpha+1/2}\]

\[- C_6 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{1/2} + \int_0^T F_2(t, \bar{u}) \, dt\]

\[= \frac{1}{4} \int_0^T \|\dot{u}(t)\|^2 \, dt - C_1 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{\beta/2} - C_2 + C_3\]

\[- C_5 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{\alpha+1/2} - C_6 \left( \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{1/2}\]

\[+ \|\dot{u}\|_{\infty}^{2\alpha} \left( \frac{\int_0^T G(\lambda \bar{u}) |f(t)| \, dt}{\mu \|\dot{u}\|_{\infty}^{2\alpha}} + \frac{\int_0^T F_2(t, \bar{u}) \, dt}{\|\dot{u}\|_{\infty}^{2\alpha}} - C_4 \right)\]

for all \(u \in H_T^1\), which implies that

\[\varphi(u) \to +\infty\]

as \(\|u\| \to \infty\) by (iii), because \(\alpha < 1, \beta < 2,\) and

\[\|u\| \to \infty \iff \left( \|\dot{u}\|^2 + \int_0^T \|\dot{u}(t)\|^2 \, dt \right)^{1/2} \to \infty.\]

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □
Theorem 2.2. Suppose that \( F(t, x) = F_1(t, x) + F_2(t, x) \), where \( F_1 \) and \( F_2 \) satisfy assumption (A) and the following conditions:

(i) there exist \( M > 0 \), \( f \in L^1(0, T; \mathbb{R}) \) satisfying \( \int_0^T |f(t)| \, dt \neq 0 \) and \( G: \mathbb{R}^N \to \mathbb{R} \) which is continuous and \((\lambda, \mu)-\text{subconvex} \) for some \( \lambda > \frac{1}{2} \) and \( 0 < \mu < 2\lambda^2 \) such that

\[
F_1(t, x) \geq G(x)|f(t)|
\]

for all \( |x| \geq M \) and a.e. \( t \in [0, T] \);

(ii) there exist \( \delta \in [0, 2) \), \( k_1 \in L^1(0, T; \mathbb{R}^+) \), and \( k_2 \in L^1(0, T; \mathbb{R}) \) such that

\[
|F_2(t, x)| \leq k_1(t)|x|^{\delta} + k_2(t)
\]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \);

(iii) \[
\frac{G(x)}{|x|^\delta} \to +\infty \quad \text{as} \quad |x| \to \infty.
\]

Then problem (1.1) has at least one solution which minimizes \( \varphi \) on \( H^1_T \).

Proof. By condition (ii) and Sobolev’s inequality, one has

\[
\left| \int_0^T F_2(t, u(t)) \, dt \right| \leq \int_0^T [k_1(t)|u(t)|^{\delta} + k_2(t)] \, dt
\]

\[
\leq 2^{\delta}(|\bar{u}|^{\delta} + \|\bar{u}\|^{\delta}_{\infty}) \int_0^T k_1(t) \, dt + \int_0^T k_2(t) \, dt
\]

\[
\leq D_1 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{\delta/2} + D_2|\bar{u}|^{\delta} + D_3
\]

for all \( u \in H^1_T \) and some constants \( D_1, D_2, \) and \( D_3 \). It follows from (2.1) and (2.3) that

\[
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T F(t, u(t)) \, dt
\]

\[
\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt - C_1 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{\beta/2} - C_2 + C_3
\]

\[
- D_1 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{\delta/2}
\]

\[
- D_3 + |\bar{u}|^{\delta} \left( \frac{G(\lambda \bar{u})}{\mu|\bar{u}|^{\delta}} \int_0^T |f(t)| \, dt - D_2 \right)
\]

for all \( u \in H^1_T \), which implies that

\[
\varphi(u) \to +\infty
\]
as $\|u\| \to \infty$ by (iii), because $\delta < 2$, $\beta < 2$, and

$$\|u\| \to \infty \iff \left( \|\tilde{u}\|^2 + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} \to \infty.$$ 

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

**Theorem 2.3.** Suppose that $F(t, x) = F_1(t, x) + F_2(t, x)$, where $F_1$ and $F_2$ satisfy assumption (A) and the following conditions:

(i) there exist $M > 0$, $f \in L^1(0, T; \mathbb{R})$ satisfying $\int_0^T |f(t)| \, dt \neq 0$, and $G: \mathbb{R}^N \to \mathbb{R}$ which is continuous and $(\lambda, \mu)$-subconvex for some $\lambda > \frac{1}{2}$ and $0 < \mu < 2\lambda^2$ such that

$$F_1(t, x) \geq G(x)|f(t)|$$

for all $|x| \geq M$ and a.e. $t \in [0, T]$;

(ii) there exist $g(t) \in L^1(0, T; \mathbb{R})$ and $h(t) \in L^1(0, T; \mathbb{R}^N)$ such that

$$F_2(t, x) \geq (h(t), x) + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$;

(iii) $$\frac{G(x)}{|x|} \to +\infty \text{ as } |x| \to \infty.$$ 

Then problem (1.1) has at least one solution which minimizes $\varphi$ on $H^1_T$.

**Proof.** By condition (ii) and Sobolev’s inequality, one has

$$\int_0^T F_2(t, u(t)) \, dt \geq \int_0^T [(h(t), \tilde{u} + \check{u}(t)) + g(t)] \, dt$$

$$\geq - \|\tilde{u}\| \int_0^T |h(t)| \, dt - |\tilde{u}| \int_0^T |h(t)| \, dt + \int_0^T g(t) \, dt$$

$$\geq - D_4 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} - D_5 \|\tilde{u}\| + D_6$$

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for all $u \in H^1_T$ and some constants $D_4$, $D_5$, and $D_6$. It follows from (2.1) and (2.4) that

$$
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T F(t, u(t)) \, dt \\
\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \frac{1}{\mu} \int_0^T G(\lambda \bar{u})|f(t)| \, dt - C_1 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{\beta/2} \\
- C_2 + C_3 - D_4 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} - D_5 |\bar{u}| + D_6 \\
= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt - C_1 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{\beta/2} - C_2 + C_3 \\
- D_4 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} + D_6 + |\bar{u}| \left( \frac{1}{\mu |\bar{u}|} \int_0^T G(\lambda \bar{u})|f(t)| \, dt - D_5 \right)
$$

for all $u \in H^1_T$, which implies that

$$
\varphi(u) \to +\infty
$$

as $\|u\| \to \infty$ by (iii), because $\beta < 2$ and

$$
\|u\| \to \infty \iff \left( |\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} \to \infty.
$$

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

Remark 2.1. In [8], the case that $G$ is subadditive is considered. Our theorems generalize that result to the case that $G$ is $(\lambda, \mu)$-subconvex by modifying some conditions. Moreover, the restriction about $F_2(t, x)$ is also modified. Especially, our Theorem 2.1 generalizes the restriction about $|\nabla F_2(t, x)|$ in [8].

Theorem 2.4. Suppose that $F(t, x) = F_1(t, x) + F_2(t, x)$, where $F_1$ and $F_2$ satisfy assumption (A) and the following conditions:

(i) $F_1(t, x)$ is $(\lambda, \mu)$-subconvex for a.e. $t \in [0, T]$ and there exist $\delta \in [0, 2)$, $\theta \in L^1(0, T; \mathbb{R}^+)$ and $\omega \in L^1(0, T; \mathbb{R})$ such that

$$
F_1(t, x) \leq \theta(t)|x|^\delta + \omega(t)
$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$;

(ii) there exist $q(t) \in L^1(0, T; \mathbb{R})$ and $h(t) \in L^1(0, T; \mathbb{R}^N)$ such that

$$
F_2(t, x) \geq (h(t), x) + q(t)
$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$;
Then problem (1.1) has at least one solution which minimizes $\varphi$ on $H^1_T$.

Proof. By the $(\lambda, \mu)$-subconvexity of $F_1(t, \cdot)$, one has

\[
\int_0^T F_1(t, u(t)) \, dt \geq \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) \, dt - \int_0^T F_1(t, -\bar{u}(t)) \, dt
\]

\[
\geq \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) \, dt - \int_0^T \left[ \theta(t)|\bar{u}(t)|^\delta + \omega(t) \right] \, dt
\]

\[
\geq \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) \, dt - \|\bar{u}\|_\infty \int_0^T \theta(t) \, dt - \int_0^T \omega(t) \, dt
\]

\[
\geq \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) \, dt - E_1 \left( \int_0^T |\bar{u}(t)|^2 \, dt \right)^{\delta/2} - E_2
\]

for all $u \in H^1_T$ and some constants $E_1, E_2$. By condition (ii), one has

\[
\int_0^T F_2(t, u(t)) \, dt \geq \int_0^T [(h(t), \bar{u} + \bar{u}(t)) + q(t)] \, dt
\]

\[
= \int_0^T (h(t), \bar{u}) \, dt + \int_0^T (h(t), \bar{u}(t)) \, dt + \int_0^T q(t) \, dt
\]

\[
\geq - |\bar{u}| \int_0^T |h(t)| \, dt - \|\bar{u}\|_\infty \int_0^T |h(t)| \, dt + \int_0^T q(t) \, dt
\]

\[
\geq - E_3 \left( \int_0^T |\bar{u}(t)|^2 \, dt \right)^{1/2} - E_4 |\bar{u}| + E_5
\]

for all $u \in H^1_T$ and some constants $E_3, E_4, E_5$. It follows from (2.5) and (2.6) that

\[
\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T F(t, u(t)) \, dt
\]

\[
\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt - E_1 \left( \int_0^T |\bar{u}(t)|^2 \, dt \right)^{\delta/2} - E_2 - E_3 \left( \int_0^T |\bar{u}(t)|^2 \, dt \right)^{1/2}
\]

\[
+ E_5 + |\bar{u}| \left( \frac{\int_0^T F_1(t, \lambda \bar{u}) \, dt}{\mu |\bar{u}|} - E_4 \right)
\]

for all $u \in H^1_T$, which implies that

\[
\varphi(u) \to +\infty
\]
as \( \|u\| \to \infty \) by (iii), because \( \delta < 2 \) and
\[
\|u\| \to \infty \iff \left( |\ddot{u}|^2 + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} \to \infty.
\]

By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

**Corollary 2.1.** Suppose that \( F(t, x) = F_1(t, x) + F_2(t, x) \), where \( F_1 \) and \( F_2 \) satisfy assumption (A) and the following conditions:

(i) \( F_1(t, x) \) is \((\lambda, \mu)\)-subconvex for a.e. \( t \in [0, T] \), where \( \lambda > \frac{1}{2} \) and \( \mu < 2\lambda^2 \);

(ii) there exist \( q(t) \in L^1(0, T; \mathbb{R}) \) and \( h(t) \in L^1(0, T; \mathbb{R}^N) \) such that
\[ F_2(t, x) \geq (h(t), x) + q(t) \]

for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \);

(iii) \[ \frac{1}{|x|} \int_0^T F_1(t, x) \, dt \to +\infty \quad \text{as} \quad |x| \to \infty. \]

Then problem (1.1) has at least one solution which minimizes \( \varphi \) on \( H_T^1 \).

**Proof.** Let \( \beta = \log_{2\lambda} 2\mu \). Then \( \beta < 2 \). In a way similar to Wu-Tang [9], by the \((\lambda, \mu)\)-subconvexity of \( F_1(t, \cdot) \) and assumption (A) one can prove that
\[ F_1(t, x) \leq c_0(2\mu|x|^\beta + 1)b(t) \]

for a.e. \( t \in [0, T] \) and all \( x \in \mathbb{R}^N \), where \( \beta < 2 \), \( c_0 = \max_{0 \leq s \leq 1} a(s) \). Thus by Theorem 2.4, the proof is completed. □

**Remark 2.2.** In [10], the case with \( \int_0^T h(t) \, dt = 0 \) is considered. Our Theorem 2.4 and Corollary 2.1 prove the conclusion holds as \( \int_0^T h(t) \, dt = 0 \) is omitted by modifying some conditions.

**Lemma A (see [7]).** Assume that \( F \) satisfies assumption (A) and
\[ F(t, x) \to +\infty \quad \text{as} \quad |x| \to \infty \]

uniformly for a.e. \( t \in [0, T] \). Then there exist \( \eta(t) \in L^1(0, T; \mathbb{R}) \) and a subadditive function \( G: \mathbb{R}^N \to \mathbb{R} \) such that
\[ G(x) + \eta(t) \leq F(t, x) \]
for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \) and
\[
G(x) \to +\infty \quad \text{as} \quad |x| \to \infty
\]
and
\[
0 \leq G(x) \leq |x| + 1
\]
for all \( x \in \mathbb{R}^N \).

**Theorem 2.5.** Suppose that \( F(t, x) = F_1(t, x) + F_2(t, x) \), where \( F_1 \) and \( F_2 \) satisfy assumption (A) and the following conditions:

(i) \( F_1(t, x) \to +\infty \) as \( |x| \to \infty \) uniformly for a.e. \( t \in [0, T] \);

(ii) there exist \( v(t) \in L^1(0, T; \mathbb{R}) \) and \( h(t) \in L^1(0, T; \mathbb{R}^N) \) with \( \int_0^T h(t) \, dt = 0 \) such that
\[
F_2(t, x) \geq (h(t), x) + v(t)
\]
for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \).

Then problem (1.1) has at least one solution which minimizes \( \varphi \) on \( H^1_T \).

**Proof.** By condition (ii) and Sobolev’s inequality one has
\begin{align*}
\int_0^T F_2(t, u(t)) \, dt &\geq \int_0^T [(h(t), \bar{u} + \tilde{u}(t)) + v(t)] \, dt \\
&\geq - \|\tilde{u}\|_\infty \int_0^T |h(t)| \, dt + \int_0^T v(t) \, dt \\
&\geq - H_1 \left( \int_0^T |\tilde{u}(t)|^2 \, dt \right)^{1/2} + H_2
\end{align*}
for all \( u \in H^1_T \) and some constants \( H_1 \) and \( H_2 \). By Lemma A, (2.7), and Sobolev’s inequality one has
\begin{align*}
\varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T F_1(t, u(t)) \, dt + \int_0^T F_2(t, u(t)) \, dt \\
&\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T G(u(t)) \, dt + \int_0^T \eta(t) \, dt - H_1 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} + H_2 \\
&\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T G(\bar{u}) \, dt - \int_0^T G(-\bar{u}(t)) \, dt \\
&\quad + \int_0^T \eta(t) \, dt - H_1 \left( \int_0^T |\tilde{u}(t)|^2 \, dt \right)^{1/2} + H_2 \\
&\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + TG(\bar{u}) - T(\|\bar{u}\|_\infty + 1) + H_3 - H_1 \left( \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} + H_2
\end{align*}

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for all \( u \in H^1_T \) and some constant \( H_3 \). From the coercivity of \( G \) we obtain
\[
\varphi(u) \to +\infty \quad \text{as} \quad \|u\| \to \infty,
\]
because
\[
\|u\| \to \infty \iff \left( |\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{1/2} \to \infty.
\]
By Theorem 1.1 and Corollary 1.1 in Mawhin-Willem [4], the proof is completed. □

Remark 2.3. In [2], A. Fonda and J.-P. Gossez obtained an abstract theorem in which it is necessary to seek a functional \( \hat{b} \). However, we find that in general it is difficult to find the functional \( \hat{b} \) satisfying the conditions of the theorem. It is therefore not very suitable for practical use. In our conclusions, for the second order Hamiltonian systems, we start from the property of \( F \) itself to seek suitable restrictive conditions so that the necessity of finding \( \hat{b} \) is avoided. This is easier.

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References


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