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Growth orders of Cesàro and Abel means of uniformly continuous operator semi-groups and cosine functions

RYOTARO SATO

This paper is dedicated to the memory of Sen-Yen Shaw.

Abstract. It will be proved that if N is a bounded nilpotent operator on a Banach space X of order $k + 1$, where $k \geq 1$ is an integer, then the γ -th order Cesàro mean $C_t^\gamma := \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} T(s) ds$ and Abel mean $A_\lambda := \lambda \int_0^\infty e^{-\lambda s} T(s) ds$ of the uniformly continuous semigroup $(T(t))_{t \geq 0}$ of bounded linear operators on X generated by $iaI + N$, where $0 \neq a \in \mathbb{R}$, satisfy that (a) $\|C_t^\gamma\| \sim t^{k-\gamma}$ ($t \rightarrow \infty$) for all $0 < \gamma \leq k+1$; (b) $\|C_t^\gamma\| \sim t^{-1}$ ($t \rightarrow \infty$) for all $\gamma \geq k+1$; (c) $\|A_\lambda\| \sim \lambda$ ($\lambda \downarrow 0$). A similar result will be also proved for the uniformly continuous cosine function $(C(t))_{t \geq 0}$ of bounded linear operators on X generated by $(iaI + N)^2$.

Keywords: Cesàro mean, Abel mean, growth order, uniformly continuous operator semi-group and cosine function

Classification: 47D06, 47D09, 47A35

1. Introduction and results

Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup of bounded linear operators on a complex Banach space X . As for the γ -th order Cesàro mean C_t^γ of the semigroup $(T(t))_{t \geq 0}$, Chen-Sato-Shaw [1] studied the following question. Does there exist for any real $\delta > 0$ an example of $(T(t))_{t \geq 0}$ such that $\sup_{t>0} \|C_t^\gamma\| = \infty$ for all $0 < \gamma < \delta$, and $\sup_{t>0} \|C_t^\gamma\| < \infty$ for all $\gamma > \delta$? They proved in [1] the following result. Let $k \geq 1$ be an integer and N be a bounded nilpotent operator on X of order $k + 1$ (i.e., $N^k \neq 0$ and $N^{k+1} = 0$). Let $(T(t))_{t \geq 0}$ be the uniformly continuous semigroup of bounded linear operators on X generated by $A := iaI + N$, where $0 \neq a \in \mathbb{R}$, so that $T(t)$ has the form

$$(1) \quad T(t) := e^{tA} = e^{iat} e^{tN} = e^{iat} \sum_{n=0}^k \frac{t^n N^n}{n!}.$$

Then the γ -th order Cesàro mean C_t^γ and Abel mean A_λ of $(T(t))_{t \geq 0}$ satisfy that $\|C_t^\gamma\| \sim t^{k-\gamma}$ ($t \rightarrow \infty$) for all integers $\gamma = 1, 2, \dots, k + 1$; $\|C_t^\gamma\| \sim t^{-1}$ ($t \rightarrow \infty$) for all $\gamma \geq k + 1$; $\|A_\lambda\| \sim \lambda$ ($\lambda \downarrow 0$); $\sup_{t>0} \|C_t^\gamma\| = \infty$ for all $0 < \gamma < k$, and $\sup_{t>0} \|C_t^\gamma\| < \infty$ for all $\gamma \geq k$. Here $a(t) \sim b(t)$ ($t \rightarrow \infty$) [resp. ($t \downarrow 0$)] means that both the ratios $a(t)/b(t)$ and $b(t)/a(t)$ are bounded in some open interval (ϵ, ∞) [resp. $(0, \epsilon)$]. Thus they gave a partial solution to the question. It remains

still open for $\delta > 1$ which is not an integer. (If $0 < \delta < 1$, then the question has a positive answer. See Theorem 4.2 in [1].)

The aim of this article is to prove that the relation $\|C_t^\gamma\| \sim t^{k-\gamma}$ ($t \rightarrow \infty$) holds not only for all integers $\gamma = 1, 2, \dots, k + 1$ but also for all real numbers γ with $0 < \gamma \leq k + 1$. That is,

Theorem 1. *Let $(T(t))_{t \geq 0}$ be the above semigroup of operators. Let C_t^γ and A_λ denote the γ -th order Cesàro and Abel means of $(T(t))_{t \geq 0}$, respectively. Then*

- (a) $\|C_t^\gamma\| \sim t^{k-\gamma}$ ($t \rightarrow \infty$) for all $0 < \gamma \leq k + 1$;
- (b) $\|C_t^\gamma\| \sim t^{-1}$ ($t \rightarrow \infty$) for all $\gamma \geq k + 1$;
- (c) $\|A_\lambda\| \sim \lambda$ ($\lambda \downarrow 0$).

We also consider the uniformly continuous cosine function $(C(t))_{t \geq 0}$ of bounded linear operators on X generated by $B := A^2$ (cf. [4]). Thus $C(t)$ has the form

$$\begin{aligned}
 (2) \quad C(t) &= \frac{1}{2}(e^{tA} + e^{-tA}) = \sum_{n=0}^k \left(\frac{e^{iat}t^n + e^{-iat}(-t)^n}{2} \right) \frac{N^n}{n!} \\
 &= \sum'_{0 \leq n \leq k} \frac{t^n \cos at}{n!} N^n + \sum''_{0 \leq n \leq k} i \frac{t^n \sin at}{n!} N^n,
 \end{aligned}$$

where $\sum'_{0 \leq n \leq k}$ [resp. $\sum''_{0 \leq n \leq k}$] means that the summation is taken for all n such that $0 \leq n \leq k$, and n is even [resp. odd]. In this case the γ -th order Cesàro mean C_t^γ and Abel mean A_λ of $(C(t))_{t \geq 0}$ are defined as $C_t^\gamma := \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} C(s) ds$ and $A_\lambda := \lambda \int_0^\infty e^{-\lambda s} C(s) ds$, respectively. It was proved in [1] that $\sup_{t>0} \|C_t^\gamma\| = \infty$ for all $0 < \gamma < k$, and $\sup_{t>0} \|C_t^\gamma\| < \infty$ for all $\gamma \geq k$. The next theorem improves the result considerably.

Theorem 2. *Let $(C(t))_{t \geq 0}$ be the above cosine function of operators. Let C_t^γ and A_λ denote the γ -th order Cesàro and Abel means of $(C(t))_{t \geq 0}$, respectively. Then*

- (a) $\|C_t^\gamma\| = O(t^{k-\gamma})$ ($t \rightarrow \infty$), $\|C_t^\gamma\| \neq o(t^{k-\gamma})$ ($t \rightarrow \infty$), and $\|C_t^\gamma\| \not\sim t^{k-\gamma}$ ($t \rightarrow \infty$) for all $0 < \gamma < k + 2$;
- (b) $\|C_t^\gamma\| \sim t^{-2}$ ($t \rightarrow \infty$) for all $\gamma \geq k + 2$;
- (c) $\|A_\lambda\| \sim \lambda^2$ ($\lambda \downarrow 0$).

For related topics the author would like to refer the reader to [2] and [3] (see also [5]).

2. Lemmas

For a complex-valued continuous function u on $[0, \infty)$, we let

$$c_t^\gamma(u) := \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} u(s) ds \quad (\gamma, t > 0).$$

Let u_n be the function defined by $u_n(t) := t^n e^{it}$ for $t \geq 0$, where $n \geq 0$ is an integer. Then we have

$$c_t^\gamma(u_n) = \frac{\gamma}{t^\gamma} \int_0^t (t-s)^{\gamma-1} s^n e^{is} ds = \frac{\gamma}{t^\gamma} e^{it} \int_0^t s^{\gamma-1} (t-s)^n e^{-is} ds,$$

so that, by letting

$$(3) \quad U_n(\gamma, t) := \int_0^t s^{\gamma-1} (t-s)^n e^{-is} ds \quad (\gamma, t > 0),$$

we have

$$(4) \quad c_t^\gamma(u_n) = \frac{\gamma}{t^\gamma} e^{it} U_n(\gamma, t) \quad (\gamma, t > 0).$$

Integration by parts gives

$$(5) \quad U_0(\gamma, t) = it^{\gamma-1} e^{-it} - i(\gamma-1)U_0(\gamma-1, t) \quad (\gamma > 1, t > 0).$$

Lemma 1. *Let $n \geq 0$ be an integer. Then $U_n(\gamma, t) \sim t^n$ ($t \rightarrow \infty$) for all $0 < \gamma < 1$.*

PROOF: (i) First we consider the case $n = 0$. Suppose $0 < \gamma < 1$. Then, since the function $s \mapsto s^{\gamma-1}$ is decreasing on $(0, \infty)$ and since

$$U_0(\gamma, t) = \int_0^t s^{\gamma-1} e^{-is} ds = \int_0^t s^{\gamma-1} \cos s ds - i \int_0^t s^{\gamma-1} \sin s ds,$$

it follows easily that

$$\begin{aligned} 0 < \int_0^{2\pi} s^{\gamma-1} \sin s ds &\leq \inf_{t>2\pi} \int_0^t s^{\gamma-1} \sin s ds \leq \sup_{t>2\pi} |U_0(\gamma, t)| \\ &\leq \int_0^{\pi/2} s^{\gamma-1} \cos s ds + \int_0^\pi s^{\gamma-1} \sin s ds \leq 2 \int_0^\pi s^{\gamma-1} ds = \frac{2\pi^\gamma}{\gamma}. \end{aligned}$$

Hence $U_0(\gamma, t) \sim t^0$ ($t \rightarrow \infty$). To use an induction argument we need to consider the case $\gamma \geq 1$. First we have $U_0(1, t) = i(e^{-it} - 1)$. Suppose $1 < \gamma \leq 2$. Then, since $|U_0(\gamma-1, t)| \leq 2\pi^{\gamma-1}/(\gamma-1)$ for all $t > \pi$, it follows from (5) that $U_0(\gamma, t) = it^{\gamma-1} e^{-it}(1 + o(1))$ ($t \rightarrow \infty$). We then apply an induction argument to $\gamma > 2$. Suppose $U_0(\gamma, t) = it^{\gamma-1} e^{-it}(1 + o(1))$ ($t \rightarrow \infty$) for all γ with $n < \gamma \leq n+1$, where $n \geq 1$ is an integer, and suppose $n+1 < \beta \leq n+2$. Then, since $U_0(\beta, t) = it^{\beta-1} e^{-it} - i(\beta-1)U_0(\beta-1, t)$ by (5), it follows that $U_0(\beta, t) = it^{\beta-1} e^{-it}(1 + o(1))$ ($t \rightarrow \infty$). Consequently, $U_0(\gamma, t) = it^{\gamma-1} e^{-it}(1 + o(1))$ ($t \rightarrow \infty$) for all $\gamma > 1$, which will be used below.

(ii) We consider the case $n \geq 1$. Suppose the lemma holds for $n - 1$. We note that

$$\begin{aligned}
 (6) \quad U_n(\gamma, t) &= \int_0^t s^{\gamma-1}(t-s)^n e^{-is} ds \\
 &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} t^k \int_0^t s^{\gamma+n-k-1} e^{-is} ds = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} t^k U_0(\gamma+n-k, t) \\
 &= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k \left(it^{\gamma+n-k-1} e^{-it} - i(\gamma+n-k-1)U_0(\gamma+n-k-1, t) \right) \\
 &\quad + \binom{n}{n} (-1)^0 t^n U_0(\gamma, t) \quad (\text{by (5)}) \\
 &= -it^{\gamma+n-1} e^{-it} - i \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (\gamma-1+n-k)U_0(\gamma-1+n-k, t) \\
 &\quad + \binom{n}{n} (-1)^0 t^n U_0(\gamma, t),
 \end{aligned}$$

where the last equality comes from the fact that $0 = (1-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k$.

Suppose $0 < \gamma < 1$. If $0 \leq k \leq n - 2$, then, since $\gamma - 1 + n - k > 1$, the result obtained in (i) shows that

$$t^k U_0(\gamma - 1 + n - k, t) \sim t^{\gamma+n-2} = o(t^{n-1}) \quad (t \rightarrow \infty).$$

Similarly, if $k = n - 1$, then

$$t^{n-1} U_0(\gamma - 1 + n - (n - 1), t) = t^{n-1} U_0(\gamma, t) \sim t^{n-1} \quad (t \rightarrow \infty).$$

Thus, from (6) and the fact that $t^n U_0(\gamma, t) \sim t^n (t \rightarrow \infty)$, it follows that $U_n(\gamma, t) \sim t^n (t \rightarrow \infty)$. This completes the proof. \square

Next, let v_n and w_n be the functions on $[0, \infty)$ defined by $v_n(t) := t^n \cos t$ and $w_n(t) := t^n \sin t$, where $n \geq 0$ is an integer. Then we have

$$\begin{aligned}
 \mathfrak{c}_t^\gamma(v_n) &= \frac{\gamma}{t^\gamma} \int_0^t (t-s)^{\gamma-1} s^n \cos s ds = \frac{\gamma}{t^\gamma} \int_0^t s^{\gamma-1} (t-s)^n \cos(t-s) ds \\
 &= \frac{\gamma}{t^\gamma} \left(\cos t \int_0^t s^{\gamma-1} (t-s)^n \cos s ds + \sin t \int_0^t s^{\gamma-1} (t-s)^n \sin s ds \right)
 \end{aligned}$$

so that, letting

$$(7) \quad F_n(\gamma, t) := \int_0^t s^{\gamma-1} (t-s)^n \cos s ds,$$

$$(8) \quad G_n(\gamma, t) := \int_0^t s^{\gamma-1} (t-s)^n \sin s ds,$$

$$(9) \quad V_n(\gamma, t) := F_n(\gamma, t) \cos t + G_n(\gamma, t) \sin t,$$

$$(10) \quad W_n(\gamma, t) := F_n(\gamma, t) \sin t - G_n(\gamma, t) \cos t,$$

we have

$$(11) \quad c_t^\gamma(v_n) = \frac{\gamma}{t^\gamma} V_n(\gamma, t) \quad (\gamma, t > 0);$$

and similarly

$$(12) \quad c_t^\gamma(w_n) = \frac{\gamma}{t^\gamma} \int_0^t (t-s)^{\gamma-1} s^n \sin s \, ds = \frac{\gamma}{t^\gamma} W_n(\gamma, t) \quad (\gamma, t > 0).$$

Lemma 2. $F_0(\gamma, t)$ and $G_0(\gamma, t)$ satisfy that

$$(13) \quad \begin{cases} \lim_{t \rightarrow \infty} F_0(\gamma, t) = \int_0^\infty s^{\gamma-1} \cos s \, ds > 0, \\ 0 < \inf_{t > \pi} G_0(\gamma, t) \leq \sup_{t > \pi} G_0(\gamma, t) < \infty \end{cases} \quad \text{for all } 0 < \gamma < 1;$$

$$(14) \quad F_0(1, t) = \sin t, \quad G_0(1, t) = 1 - \cos t;$$

$$(15) \quad \begin{cases} F_0(\gamma, t) = t^{\gamma-1}(\sin t + o(1)) \quad (t \rightarrow \infty), \\ G_0(\gamma, t) = t^{\gamma-1}(-\cos t + o(1)) \quad (t \rightarrow \infty) \end{cases} \quad \text{for all } \gamma > 1.$$

PROOF: Suppose $0 < \gamma < 1$. Then, since the function $s \mapsto s^{\gamma-1}$ is decreasing and convex on $(0, \infty)$, we have as in the proof of Lemma 1

$$\lim_{t \rightarrow \infty} F_0(\gamma, t) = \lim_{n \rightarrow \infty} \sum_{j=0}^n \int_{2j\pi}^{2(j+1)\pi} s^{\gamma-1} \cos s \, ds \geq \int_0^{2\pi} s^{\gamma-1} \cos s \, ds > 0.$$

Similarly

$$\int_0^\pi s^{\gamma-1} \sin s \, ds \geq G_0(\gamma, t) \geq \int_0^{2\pi} s^{\gamma-1} \sin s \, ds > 0 \quad (t > \pi).$$

The proof of (14) is direct.

Next suppose $\gamma > 1$. Then integration by parts gives

$$(16) \quad \begin{aligned} F_0(\gamma, t) &= \int_0^t s^{\gamma-1} \cos s \, ds = t^{\gamma-1} \sin t - (\gamma - 1) \int_0^t s^{\gamma-2} \sin s \, ds \\ &= t^{\gamma-1} \sin t - (\gamma - 1)G_0(\gamma - 1, t); \end{aligned}$$

and similarly

$$(17) \quad G_0(\gamma, t) = \int_0^t s^{\gamma-1} \sin s \, ds = -t^{\gamma-1} \cos t + (\gamma - 1)F_0(\gamma - 1, t).$$

Thus, if $1 < \gamma \leq 2$, then, by the results for $0 < \gamma - 1 \leq 1$, we see that $F_0(\gamma, t) = t^{\gamma-1}(\sin t + o(1))$ ($t \rightarrow \infty$), and $G_0(\gamma, t) = t^{\gamma-1}(-\cos t + o(1))$ ($t \rightarrow \infty$). We can repeat this process to prove (15) for all $\gamma > 1$. \square

Lemma 3. *Let $n \geq 1$. Then $F_n(\gamma, t)$ and $G_n(\gamma, t)$ satisfy that*

$$(18) \quad \begin{cases} F_n(\gamma, t) = t^n(F_0(\gamma, t) + o(1)) & (t \rightarrow \infty), \\ G_n(\gamma, t) = t^n(G_0(\gamma, t) + o(1)) & (t \rightarrow \infty) \end{cases} \quad \text{for all } 0 < \gamma < 1;$$

$$(19) \quad \begin{cases} F_n(1, t) = O(t^{n-1}) & (t \rightarrow \infty), \\ G_n(1, t) = t^n(1 + o(1)) & (t \rightarrow \infty); \end{cases}$$

$$(20) \quad \begin{cases} F_n(\gamma, t) = -(\gamma - 1)G_n(\gamma - 1, t) + nG_{n-1}(\gamma, t), \\ G_n(\gamma, t) = (\gamma - 1)F_n(\gamma - 1, t) - nF_{n-1}(\gamma, t) \end{cases} \quad \text{for all } \gamma > 1.$$

PROOF: It follows from (7) that

$$\begin{aligned} (21) \quad F_n(\gamma, t) &= \int_0^t s^{\gamma-1}(t-s)^n \cos s \, ds \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} t^k \int_0^t s^{\gamma+n-k-1} \cos s \, ds \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} t^k F_0(\gamma + n - k, t) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k \left(t^{\gamma+n-k-1} \sin t - (\gamma + n - k - 1)G_0(\gamma + n - k - 1, t) \right) \\ &\quad + t^n F_0(\gamma, t) \quad (\text{by (16)}) \\ &= -t^{\gamma+n-1} \sin t - \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (\gamma - 1 + n - k)G_0(\gamma - 1 + n - k, t) \\ &\quad + t^n F_0(\gamma, t). \end{aligned}$$

Similarly

$$\begin{aligned} (22) \quad G_n(\gamma, t) &= \int_0^t s^{\gamma-1}(t-s)^n \sin s \, ds \\ &= t^{\gamma+n-1} \cos t + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (\gamma - 1 + n - k)F_0(\gamma - 1 + n - k, t) \\ &\quad + t^n G_0(\gamma, t). \end{aligned}$$

First suppose $0 < \gamma < 1$. Then by Lemma 2

$$t^k F_0(\gamma - 1 + n - k, t) = \begin{cases} O(t^{\gamma+n-2}) & (t \rightarrow \infty) & \text{for } 0 \leq k \leq n - 2, \\ O(t^{n-1}) & (t \rightarrow \infty) & \text{for } k = n - 1; \end{cases}$$

and

$$t^k G_0(\gamma - 1 + n - k, t) = \begin{cases} O(t^{\gamma+n-2}) & (t \rightarrow \infty) & \text{for } 0 \leq k \leq n - 2, \\ O(t^{n-1}) & (t \rightarrow \infty) & \text{for } k = n - 1. \end{cases}$$

Thus (18) follows from (21) and (22).

Next suppose $\gamma = 1$. Then by (21) and (14)

$$\begin{aligned} F_n(1, t) &= -t^n \sin t - \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (n - k) G_0(n - k, t) + t^n F_0(1, t) \\ &= - \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (n - k) G_0(n - k, t); \end{aligned}$$

and similarly by (22) and (14)

$$G_n(1, t) = t^n + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k (n - k) F_0(n - k, t).$$

Here it follows from (14) and (15) that

$$\begin{cases} t^k F_0(n - k, t) = O(t^{n-1}) & (t \rightarrow \infty) \\ t^k G_0(n - k, t) = O(t^{n-1}) & (t \rightarrow \infty) \end{cases} \quad \text{for all } 0 \leq k \leq n - 1,$$

whence (19) follows.

Finally suppose $\gamma > 1$. Then by (21) and (16)

$$\begin{aligned} F_n(\gamma, t) &= -t^{\gamma+n-1} \sin t - (\gamma - 1) \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n-k} t^k G_0(\gamma - 1 + n - k, t) \\ &\quad + n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} G_0(\gamma - 1 + n - k, t) + t^n F_0(\gamma, t) \\ &= -(\gamma - 1) \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} t^k G_0(\gamma - 1 + n - k, t) \\ &\quad + n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} t^k G_0(\gamma + n - 1 - k, t) \\ &= -(\gamma - 1) G_n(\gamma - 1, t) + n G_{n-1}(\gamma, t); \end{aligned}$$

and similarly by (22) and (17)

$$G_n(\gamma, t) = (\gamma - 1)F_n(\gamma - 1, t) - nF_{n-1}(\gamma, t).$$

This proves (20), and hence the proof is complete. □

Lemma 4. $V_0(\gamma, t)$ and $W_0(\gamma, t)$ satisfy that

$$(23) \quad \begin{cases} V_0(\gamma, t) = O(t^0) \ (t \rightarrow \infty), \ V_0(\gamma, t) \neq o(t^0) \ (t \rightarrow \infty), \\ V_0(\gamma, t) \not\sim t^0 \ (t \rightarrow \infty), \ \text{and} \\ W_0(\gamma, t) = O(t^0) \ (t \rightarrow \infty), \ W_0(\gamma, t) \neq o(t^0) \ (t \rightarrow \infty), \\ W_0(\gamma, t) \not\sim t^0 \ (t \rightarrow \infty) \end{cases}$$

for all $0 < \gamma < 1$;

$$(24) \quad V_0(1, t) = \sin t, \quad W_0(1, t) = 1 - \cos t;$$

$$(25) \quad \begin{cases} V_0(\gamma, t) = O(t^0) \ (t \rightarrow \infty), \ V_0(\gamma, t) \neq o(t^0) \ (t \rightarrow \infty), \\ V_0(\gamma, t) \not\sim t^0 \ (t \rightarrow \infty), \ \text{and} \\ W_0(\gamma, t) = t^{\gamma-1}(1 + o(1)) \ (t \rightarrow \infty) \end{cases}$$

for all $1 < \gamma < 2$.

PROOF: (23) and (24) follow directly from Lemma 2 together with the definitions of $V_0(\gamma, t)$ and $W_0(\gamma, t)$ (cf. (9), (10)). Suppose $\gamma > 1$. Then by (16) and (17)

$$(26) \quad \begin{aligned} V_0(\gamma, t) &= F_0(\gamma, t) \cos t + G_0(\gamma, t) \sin t \\ &= (t^{\gamma-1} \sin t - (\gamma - 1)G_0(\gamma - 1, t)) \cos t \\ &\quad + (-t^{\gamma-1} \cos t + (\gamma - 1)F_0(\gamma - 1, t)) \sin t \\ &= -(\gamma - 1)G_0(\gamma - 1, t) \cos t + (\gamma - 1)F_0(\gamma - 1, t) \sin t \\ &= (\gamma - 1)W_0(\gamma - 1, t); \end{aligned}$$

and similarly

$$(27) \quad W_0(\gamma, t) = t^{\gamma-1} - (\gamma - 1)V_0(\gamma - 1, t).$$

Thus (25) follows from (23). This completes the proof. □

Lemma 5. Let $n \geq 1$. Then $V_n(\gamma, t)$ and $W_n(\gamma, t)$ satisfy that

$$(28) \quad \begin{cases} V_n(\gamma, t) = t^n(V_0(\gamma, t) + o(1)) \ (t \rightarrow \infty), \\ W_n(\gamma, t) = t^n(W_0(\gamma, t) + o(1)) \ (t \rightarrow \infty) \end{cases} \quad \text{for all } 0 < \gamma < 1;$$

$$(29) \quad \begin{cases} V_n(1, t) = t^n(\sin t + o(1)) \ (t \rightarrow \infty), \\ W_n(1, t) = -t^n(\cos t + o(1)) \ (t \rightarrow \infty); \end{cases}$$

$$(30) \quad \begin{cases} V_n(\gamma, t) = (\gamma - 1)W_n(\gamma - 1, t) - nW_{n-1}(\gamma, t), \\ W_n(\gamma, t) = -(\gamma - 1)V_n(\gamma - 1, t) + nV_{n-1}(\gamma, t) \end{cases} \quad \text{for all } \gamma > 1.$$

PROOF: Suppose $0 < \gamma < 1$. Then

$$\begin{aligned} V_n(\gamma, t) &= F_n(\gamma, t) \cos t + G_n(\gamma, t) \sin t \\ &= t^n(F_0(\gamma, t) + o(1)) \cos t + t^n(G_0(\gamma, t) + o(1)) \sin t \quad (\text{by (18)}) \\ &= t^n(F_0(\gamma, t) \cos t + G_0(\gamma, t) \sin t + o(1)) \\ &= t^n(V_0(\gamma, t) + o(1)) \quad (t \rightarrow \infty); \end{aligned}$$

and similarly $W_n(\gamma, t) = t^n(W_0(\gamma, t) + o(1)) \quad (t \rightarrow \infty)$. (29) follows easily from (9), (10) and (19). Finally suppose $\gamma > 1$. Then by (20)

$$\begin{aligned} V_n(\gamma, t) &= F_n(\gamma, t) \cos t + G_n(\gamma, t) \sin t \\ &= \left(-(\gamma - 1)G_n(\gamma - 1, t) + nG_{n-1}(\gamma, t) \right) \cos t \\ &\quad + \left((\gamma - 1)F_n(\gamma - 1, t) - nF_{n-1}(\gamma, t) \right) \sin t \\ &= (\gamma - 1) \left(F_n(\gamma - 1, t) \sin t - G_n(\gamma - 1, t) \cos t \right) \\ &\quad - n \left(F_{n-1}(\gamma, t) \sin t - G_{n-1}(\gamma, t) \cos t \right) \\ &= (\gamma - 1)W_n(\gamma - 1, t) - nW_{n-1}(\gamma, t), \end{aligned}$$

so that the first half of (30) follows. The second half follows similarly. □

As an immediate consequence of Lemmas 4 and 5 (see especially (30)) we have the following

Lemma 6. *Let $n \geq 1$. Then $V_n(\gamma, t)$ and $W_n(\gamma, t)$ satisfy that*

$$(31) \quad \begin{cases} V_n(\gamma, t) = O(t^n) \quad (t \rightarrow \infty), \quad V_n(\gamma, t) \neq o(t^n) \quad (t \rightarrow \infty), \\ V_n(\gamma, t) \not\sim t^n \quad (t \rightarrow \infty), \quad \text{and} \\ W_n(\gamma, t) = O(t^n) \quad (t \rightarrow \infty), \quad W_n(\gamma, t) \neq o(t^n) \quad (t \rightarrow \infty), \\ W_n(\gamma, t) \not\sim t^n \quad (t \rightarrow \infty) \end{cases}$$

for all $0 < \gamma < 2$.

3. Proofs of the theorems

PROOF OF THEOREM 1: We may assume without loss of generality that $a = 1$. By (1), (4) and Lemma 1, if $0 \leq \gamma < 1$, then $\|C_t^\gamma\| \sim t^{k-\gamma} \quad (t \rightarrow \infty)$, where $C_t^0 := T(t)$. Further, since $0 \in \rho(A)$, we may apply [1, Theorem 3.3] to infer that $\|C_t^{\gamma+1}\| \sim \|C_t^\gamma - I\|t^{-1} \quad (t \rightarrow \infty)$. By this and an induction argument, if $0 \leq \gamma < k + 1$, then $\|C_t^\gamma\| \sim t^{k-\gamma} \quad (t \rightarrow \infty)$. The fact that $\|C_t^{k+1}\| \sim t^{-1} \quad (t \rightarrow \infty)$

has been proved in [1, Theorem 3.4]. Next suppose $k + 1 < \gamma \leq k + 2$. Then, since $\lim_{t \rightarrow \infty} \|C_t^{\gamma-1}\| = 0$, the equation

$$(32) \quad AC_t^\gamma = \gamma t^{-1}(C_t^{\gamma-1} - I)$$

(cf. [3]) yields that

$$(33) \quad \lim_{t \rightarrow \infty} \frac{t}{\gamma} C_t^\gamma = \lim_{t \rightarrow \infty} A^{-1}(C_t^{\gamma-1} - I) = -A^{-1}.$$

This argument can be repeated, and hence (33) holds for all $\gamma > k + 1$. Finally

$$\lim_{\lambda \downarrow 0} \lambda^{-1} A_\lambda = \lim_{\lambda \downarrow 0} \int_0^\infty e^{-\lambda t} T(t) dt = \lim_{\lambda \downarrow 0} (\lambda I - A)^{-1} = -A^{-1}.$$

Hence in particular $\|A_\lambda\| \sim \lambda$ ($\lambda \downarrow 0$), and $\|C_t^\gamma\| \sim t^{-1}$ ($t \rightarrow \infty$) for all $\gamma > k + 1$. This completes the proof. \square

PROOF OF THEOREM 2: This is similar to the above proof. We may assume that $a = 1$. By (2), (11), (12), and Lemmas 4 and 6, if $0 \leq \gamma < 2$, then (a) in Theorem 2 holds. We then use the equation

$$(34) \quad BC_t^\gamma = \frac{\gamma(\gamma - 1)}{t^2}(C_t^{\gamma-2} - I) \quad (\gamma \geq 2, t > 0),$$

where $C_t^0 := C(t)$ (cf. [3]). Since $0 \in \rho(B) = \rho(A^2)$, it follows that $\|C_t^{\gamma+2}\| \sim \|C_t^\gamma - I\|t^{-2}$ ($t \rightarrow \infty$), and so if $0 \leq \gamma < k + 2$, then (a) in Theorem 2 holds. Next, let $x \in X$ be such that $\|x\| = 1$ and $Nx = 0$. Then, since $\lim_{t \rightarrow \infty} C_t^k x = \lim_{t \rightarrow \infty} (k/t^k) (\int_0^t (t-s)^{k-1} \cos s ds) x = 0$, it follows that $\liminf_{t \rightarrow \infty} \|C_t^k - I\| \geq \liminf_{t \rightarrow \infty} \|C_t^k x - x\| = \|x\| = 1$, whence $\|C_t^{k+2}\| \sim t^{-2}$ ($t \rightarrow \infty$). Now suppose $k + 2 < \gamma \leq k + 4$. Then $\lim_{t \rightarrow \infty} \|C_t^{\gamma-2}\| = 0$, and so

$$(35) \quad \lim_{t \rightarrow \infty} \frac{t^2}{\gamma(\gamma - 1)} C_t^\gamma = \lim_{t \rightarrow \infty} B^{-1}(C_t^{\gamma-2} - I) = -B^{-1}.$$

This argument can be repeated, and hence (35) holds for all $\gamma > k + 2$. Finally

$$\lim_{\lambda \downarrow 0} \lambda^{-2} A_\lambda = \lim_{t \rightarrow \infty} \lambda^{-1} \int_0^\infty e^{-\lambda t} C(t) dt = \lim_{\lambda \downarrow 0} (\lambda^2 I - B)^{-1} = -B^{-1}$$

(cf. [4]). Hence in particular $\|A_\lambda\| \sim \lambda^2$ ($\lambda \downarrow 0$), and $\|C_t^\gamma\| \sim t^{-2}$ ($t \rightarrow \infty$) for all $\gamma > k + 2$. This completes the proof. \square

Remark. $\lim_{t \rightarrow \infty} (t/(k + 1))C_t^{k+1}$ does not exist in Theorem 1. To see this we write $AC_t^k = (k/t)(C_t^{k-1} - I) =: (k/t)C_t^{k-1} + D_t^1$, where $\lim_{t \rightarrow \infty} \|D_t^1\| = 0$, and finally $A^k C_t^k =: (k!/t^k)T(t) + D_t^k$, where $\lim_{t \rightarrow \infty} \|D_t^k\| = 0$. Since

$$\frac{k!}{t^k} T(t) = k! e^{iat} \sum_{n=0}^k \frac{t^{n-k} N^n}{n!} =: e^{iat}(N^k + E_t^k), \quad \text{where} \quad \lim_{t \rightarrow \infty} \|E_t^k\| = 0,$$

it follows that if $N^k x \neq 0$, then $\lim_{t \rightarrow \infty} C_t^k x$ does not exist. Hence $\lim_{t \rightarrow \infty} (t/(k+1))C_t^{k+1}$ does not exist by (32). Similarly $\lim_{t \rightarrow \infty} (t^2/(k+2)(k+1))C_t^{k+2}$ does not exist in Theorem 2.

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