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Martin’s Axiom and $\omega$-resolvability of Baire spaces

Fidel Casarrubias-Segura, Fernando Hernández-Hernández, Ángel Tamariz-Mascarúa

Abstract. We prove that, assuming MA, every crowded $T_0$ space $X$ is $\omega$-resolvable if it satisfies one of the following properties: (1) it contains a $\pi$-network of cardinality $< \mathfrak{c}$ constituted by infinite sets, (2) $\chi(X) < \mathfrak{c}$, (3) $X$ is a $T_2$ Baire space and $c(X) \leq \aleph_0$ and (4) $X$ is a $T_1$ Baire space and has a network $\mathcal{N}$ with cardinality $< \mathfrak{c}$ and such that the collection of the finite elements in it constitutes a $\sigma$-locally finite family.

Furthermore, we prove that the existence of a $T_1$ Baire irresolvable space is equivalent to the existence of a $T_1$ Baire $\omega$-irresolvable space, and each of these statements is equivalent to the existence of a $T_1$ almost-$\omega$-irresolvable space.

Finally, we prove that the minimum cardinality of a $\pi$-network with infinite elements of a space $\text{Seq}(u_t)$ is strictly greater than $\aleph_0$.

Keywords: Martin’s Axiom, Baire spaces, resolvable spaces, $\omega$-resolvable spaces, almost resolvable spaces, almost-$\omega$-resolvable spaces, infinite $\pi$-network

Classification: Primary 54E52, 54A35; Secondary 54D10, 54A10

1. Introduction

Every space in this article is $T_0$ and crowded (that is, without isolated points) and so it is infinite. A space $X$ is resolvable if it contains two dense disjoint subsets. A space which is not resolvable is called irresolvable. Resolvable and irresolvable spaces were studied extensively first by Hewitt [14]. Later, El’kin and Malykhin published a number of papers on these subjects and their connections with various topological problems. One of the problems considered by Malykhin in [22] refers to the existence of irresolvable spaces satisfying the Baire Category Theorem. Kunen, Szymański and Tall in [19] afterwards proved that there is such a space if and only if there is a space $X$ on which every real-valued function is continuous at some point. (The question about the existence of a –Hausdorff– space on which every real-valued function is continuous at some point was posed by M. Katětov in [16].) They also proved (see [18] as well):

1. if we assume $V = L$, there is no Baire irresolvable space,
2. the conditions “there is a measurable cardinal” and “there is a Baire irresolvable space” are equiconsistent.

Bolstein introduced in [5] the spaces $X$ in which it is possible to define a real-valued function $f$ with countable range and such that $f$ is discontinuous at every
point of $X$ (he called these spaces _almost resolvable_), and proved that every resolvable space satisfies this condition. It was proved in [12] that $X$ is almost resolvable iff there is a function $f : X \to \mathbb{R}$ such that $f$ is discontinuous at every point of $X$. _Almost-$\omega$-resolvable_ spaces were introduced in [26]; these are spaces in which it is possible to define a real-valued function $f$ with countable range, and such that $r \circ f$ is discontinuous in every point of $X$, for every real-valued finite-to-one function $r$. It was proved in that article that for a Tychonoff space $X$, the space of real continuous functions with the box topology, $C_{\square}(X)$, is discrete if and only if $X$ is almost-$\omega$-resolvable. It was also proved that the existence of a measurable cardinal is equiconsistent with the existence of a Tychonoff space without isolated points which is not almost-$\omega$-resolvable, and that, on the contrary, if $V = L$ then every crowded space is almost-$\omega$-resolvable. Later, it was pointed out in [2, Corollary 5.4] that a Baire space is resolvable if and only if it is almost resolvable; so,

1.1 Theorem. A Baire almost-$\omega$-resolvable space is resolvable.

It is unknown if every Baire almost-$\omega$-resolvable space is 3-resolvable. With respect to this problem we have the following theorems.

1.2 Theorem ([24]). G"odel’s axiom of constructibility, $V = L$, implies that every Baire space is $\omega$-resolvable.

1.3 Theorem ([2]). Every $T_1$ Baire space such that each of its dense subsets is almost-$\omega$-resolvable is $\omega$-resolvable.

These last two results transform our problem to that of finding subclasses of Baire spaces such that each of its crowded dense subsets is almost-$\omega$-resolvable, assuming axioms consistent with ZFC which contrast with $V = L$. Of course, a classic axiom with these characteristics is $\text{MA} + \neg \text{CH}$. This bet is strengthened by the following result due to V.I. Malykhin ([23, Theorem 1.2]):

1.4 Theorem [MA$_{\text{countable}}$]. Let a topology on a countable set $X$ have a $\pi$-network of cardinality less than $\mathfrak{c}$ consisting of infinite subsets. Then this topology is $\omega$-resolvable.

It was proved in [2] that every space with countable tightness, every space with $\pi$-weight $\leq \aleph_1$ and every $\sigma$-space are hereditarily almost-$\omega$-resolvable. So, by Theorem 1.3, every $T_1$ Baire space with either countable tightness or $\pi$-weight $\leq \aleph_1$ or $\sigma$ is $\omega$-resolvable.

In this article we are going to continue the study of almost-$\omega$-resolvable and Baire resolvable spaces, and we will solve some problems related to these posed in [2]. Section 2 is devoted to establishing basic definitions and results. In Section 3 we prove that under MA every space with either $\pi$-weight $< \mathfrak{c}$ or $\chi(X) < \mathfrak{c}$ is $\omega$-resolvable. Furthermore, we are going to see in Section 4 that under SH every $T_2$ Baire space with countable cellularity is $\omega$-resolvable. Section 5 is devoted to
analyse almost-$\omega$-irresolvable spaces. We prove in this section that there is a $T_1$ Baire irresolvable space iff there is a $T_1$ Baire $\omega$-irresolvable space, iff there is a $T_1$ almost-$\omega$-irresolvable space. Finally in Section 6, we prove that the minimum cardinality of a $\pi$-network with infinite elements of a space Seq($u_t$) is strictly greater than $\aleph_0$. Moreover, we propose several problems related to our matter through the article.

2. Basic definitions and preliminaries

A space $X$ is resolvable if it is the union of two disjoint dense subsets. We say that $X$ is irresolvable if it is not resolvable. For a cardinal number $\kappa > 1$, we say that $X$ is $\kappa$-resolvable if $X$ is the union of $\kappa$ pairwise disjoint dense subsets.

The dispersion character $\Delta(X)$ of a space $X$ is the minimum of the cardinalities of non-empty open subsets of $X$. If $X$ is $\Delta(X)$-resolvable, then we say that $X$ is maximally resolvable. A space $X$ is hereditarily irresolvable if every subspace of $X$ is irresolvable. And $X$ is open-hereditarily irresolvable if every open subspace of $X$ is irresolvable.

We call a space $(X, t)$ maximal if $(X, t')$ contains at least one isolated point when $t'$ strictly contains the topology $t$. And a space $X$ is submaximal if every dense subset of $X$ is open. Moreover, maximal spaces are submaximal, and these are hereditarily irresolvable spaces, which in turn are open-hereditarily irresolvable.

It is possible to prove that a space $X$ is almost resolvable if and only if $X$ is the union of a countable collection of subsets each of them with an empty interior [5].

It was proved in [26] that the following formulation can be given as a definition of almost-$\omega$-resolvable space: A space $X$ is called almost-$\omega$-resolvable if $X$ is the union of a countable collection $\{X_n : n < \omega\}$ of subsets in such a way that for each $m < \omega$, $\text{int}(\bigcup_{i<m} X_i) = \emptyset$. In particular, every almost-$\omega$-resolvable space is almost-resolvable, every $\omega$-resolvable space is almost-$\omega$-resolvable, every almost-resolvable space is infinite, and every $T_1$ separable space is almost-$\omega$-resolvable.

We are going to say that a space $X$ is hereditarily almost-$\omega$-resolvable if each crowded subspace of $X$ is almost-$\omega$-resolvable, and $X$ is dense-hereditarily almost-$\omega$-resolvable if each crowded dense subspace of $X$ is almost-$\omega$-resolvable.

Let $X$ be a $\kappa$-resolvable (resp., almost-resolvable, almost-$\omega$-resolvable) space. A $\kappa$-resolution (resp., an almost resolution, an almost-$\omega$-resolution) for $X$ is a partition $\{V_\alpha : \alpha < \kappa\}$ (resp., a partition $\{V_n : n < \omega\}$) of $X$ such that each $V_\alpha$ is a dense subset of $X$ (resp., $\text{int}(V_n) = \emptyset$ for every $n < \omega$, $\text{int}(\bigcup_{i=n}^{\kappa} V_i) = \emptyset$ for every $n < \omega$).

Finally, a space $X$ is almost-$\omega$-irresolvable (resp., $\kappa$-irresolvable) if $X$ is not almost-$\omega$-resolvable (resp., $X$ is not $\kappa$-resolvable). The hereditary version of almost-$\omega$-irresolvability or $\kappa$-irresolvability is that which states that every crowded subspace of $X$ is not almost-$\omega$-resolvable, and, respectively, is not $\kappa$-resolvable.
2.1 Example. There are non-$T_0$ topological spaces which are almost resolvable but not almost-$\omega$-resolvable. In fact, let $X$ be an infinite set and $x, y \in X$ with $x \neq y$. We define a collection $T$ of subsets of $X$ as follows: $A \in T$ if either $A$ is the empty set or $x, y \in A$. The family $T$ is a topology in $X$ and $(X, T)$ satisfies the required conditions.

2.2 Example. It was proved in Theorem 4.4 of [19] that the existence of an $\omega_1$-complete ideal $I$ over $\omega_1$ which has a dense set of size $\omega_1$ implies the existence of a $T_2$ Baire strongly irresolvable topology $T$ on $\omega_1$. On the other hand, it was observed in [26, Corollary 4.9] that every Baire irresolvable space is not almost resolvable. Therefore, $(\omega_1, T)$ is not almost resolvable.

2.3 Example. If there is a measurable cardinal $\kappa$, then there is a resolvable Baire space $X$ which is not almost-$\omega$-resolvable and $\Delta(X) = \kappa$. Indeed, let $\kappa$ be a non-countable Ulam-measurable cardinal, and let $p$ be a free ultrafilter on $\kappa$ $\omega_1$-complete. Let $X = \kappa \cup \{p\}$. We define a topology $t$ for $X$ as follows: $A \in t$ if and only if $p \in A$ and $A \cap \kappa \in p$. This space is a Baire resolvable non-almost-$\omega$-resolvable space with $\Delta(X) = \alpha$. Now, let $T$ be equal to $\{A \subseteq X : A \cap \kappa \in p\}$; $T$ is a topology in $X$ too, and $(X, T)$ is $T_1$ submaximal, Baire with $\Delta(X) = \alpha$, but it is not almost resolvable.

Related to the last examples we have:

2.4 Question. Is there a $T_2$ resolvable Baire space which is not almost-$\omega$-resolvable?

2.5 Examples. In ZFC, there are almost-$\omega$-resolvable spaces which are not resolvable. Indeed, the union of Tychonoff crowded topologies in $\mathbb{Q}$ generates a Tychonoff crowded topology. By Zorn’s Lemma, we can consider a maximal Tychonoff topology $T$ in $\mathbb{Q}$. The space $(\mathbb{Q}, T)$ is countable (so, almost-$\omega$-resolvable) hereditarily irresolvable ([14, Theorems 15 and 26], [8, Example 3.3]). $(\mathbb{Q}, T)$ is Tychonoff.

In [1], the authors construct by transfinite recursion a “concrete” (in the sense that we can say how its open sets look) example of a countable dense subset $X$ of the space $2^\mathbb{N}$ which is irresolvable. Since $X$ is countable, it is almost-$\omega$-resolvable.

2.6 Example. For every cardinal number $\kappa$, there exists a Tychonoff space $X$ which is almost-$\omega$-resolvable, hereditarily irresolvable and $\Delta(X) \geq \kappa$. In fact, let $\lambda$ be a cardinal number such that $\kappa \leq \lambda$ and $\text{cof}(\lambda) = \aleph_0$. Let $H$, $G$ and $\tau$ be the topological groups and the topology in $G$, respectively, defined in [11, pp. 33 and 34], with $|H| = \lambda$. L. Feng proved there that $(H, \tau|_H)$ is an irresolvable card-homogeneous (every open subset of $H$ has the same cardinality as $H$) Tychonoff space, and each subset $S \subseteq H$ with cardinality strictly less than $\lambda$ is a nowhere dense subset of $H$. Let $(\lambda_n)_{n<\omega}$ be a sequence of cardinal numbers such that $\lambda_n < \lambda_{n+1}$ for every $n < \omega$ and $\sup\{\lambda_n : n < \omega\} = \lambda$. We take subsets $H_n$ of $H$ with the properties $H_n \subseteq H_{n+1}$ and $|H_n| = \lambda_n$ for each $n < \omega$, and
H = \bigcup_{n<\omega} H_n. We have that each H_n is nowhere dense in H; so \{H_n : n < \omega\} is an almost-\omega-resolvable sequence on H. That is, H is almost-\omega-resolvable. By the Hewitt Decomposition Theorem (see [14, Theorem 28]), there exists a non-empty open subset U of H which is hereditarily irresolvable. Besides, \(\Delta(U) = \Delta(H) \geq \kappa\) and U is almost-\omega-resolvable.

2.7 Examples. The first example of a Hausdorff maximal group was constructed by Malykhin in [21] under Martin’s Axiom. Malykhin also constructed in [23], in the BK model \(M_{\omega_1}\) (see [3]) a topological group topology \(T'\) in the infinite countable Boolean group \(\Omega\) of all finite subsets of \(\omega\) with symmetric difference as the group operation, such that \((\Omega, T')\) is \(T_2\), irresolvable and its weight is \(\omega_1\) (compare with Corollary 3.6 below). Moreover, in \(M_{\omega_1}\), \(\omega_1 < \tau\). Moreover, he constructed in \(M_{\omega_1}\) a countable irresolvable dense subset in \(2^{\omega_1}\). This space has of course weight \(\omega_1\).

On the other hand, the class of resolvable spaces includes spaces with well known properties:

2.8 Theorem. (1) If \(X\) has a \(\pi\)-network \(\mathcal{N}\) such that \(|\mathcal{N}| \leq \Delta(X)\) and each \(N \in \mathcal{N}\) satisfies \(|N| \geq \Delta(X)\), then \(X\) is maximally resolvable [9].

(2) Hausdorff \(k\)-spaces are maximally resolvable [25].

(3) Countably compact regular \(T_1\) spaces are \(\omega\)-resolvable [7].

(4) Arc connected spaces are \(\omega\)-resolvable.

(5) Every biradial space is maximally resolvable [29].

(6) Every homogeneous space containing a non-trivial convergent sequence is \(\omega\)-resolvable [28].

(7) If \(G\) is an uncountable \(\aleph_0\)-bounded topological group, then \(G\) is \(\aleph_1\)-resolvable [29].

(8) \(T_1\) Baire spaces with either countable tightness or \(\pi\)-weight \(\leq \aleph_1\) are \(\omega\)-resolvable [2].

The following basic results will be very helpful (see, for example, [6]).

2.9 Propositions. (1) If \(X\) is the union of \(\kappa\)-resolvable (resp., almost-resolvable, almost-\(\omega\)-resolvable) subspaces, then \(X\) has the same property.

(2) Every open and every regular closed subset of a \(\kappa\)-resolvable (resp., almost resolvable, almost-\(\omega\)-resolvable) space shares this property.

(3) Let \(X\) be a space which contains a dense subset which is \(\kappa\)-resolvable (resp., almost resolvable, almost-\(\omega\)-resolvable). Then, \(X\) satisfies this property too.

The following results are easy to prove and are well known.

2.10 Proposition. Let \(Y\) be a \(\kappa\)-resolvable (resp., almost-resolvable, almost-\(\omega\)-resolvable) space. If \(f : X \to Y\) is a continuous and onto function, and for each
open subset $A$ of $X$ the interior of $f[A]$ is not empty, then $X$ is $\kappa$-resolvable (resp., almost-resolvable, almost-$\omega$-resolvable).

2.11 Proposition. Let $f : X \to Y$ be continuous and bijective. If $X$ is $\kappa$-resolvable (resp., almost-resolvable, almost-$\omega$-resolvable) and $Y$ is an arbitrary topological space, then $X \times Y$ is $\kappa$-resolvable (resp., almost resolvable, almost-$\omega$-resolvable), so is $Y$.

2.12 Proposition. (1) If $X$ is $\kappa$-resolvable (resp., almost resolvable, almost-$\omega$-resolvable) and $Y$ is an arbitrary topological space, then $X \times Y$ is $\kappa$-resolvable (resp., almost resolvable, almost-$\omega$-resolvable). (2) [2] If $X$ and $Y$ are almost resolvable, then $X \times Y$ is resolvable. (3) (O. Masaveu) If $X$ is the product space $\prod_{\alpha < \kappa} X_\alpha$ where $\kappa \geq \omega$ and each $X_\alpha$ has more than one point, then $X$ is $2^\kappa$-resolvable.

The following lemmas will be useful later.

2.13 Proposition. If $X$ is a crowded space such that $\text{cof}(|X|) = \aleph_0$ and every open subset of $X$ has cardinality $|X|$, then $X$ is almost-$\omega$-resolvable.

2.14 Proposition. If $X$ has tightness equal to $\kappa$, then each point $x \in X$ is contained in a crowded subset of $X$ of cardinality $\leq \kappa$.

Proof: Let $x_0 \in X$ be an arbitrary fixed point. Since $X$ is crowded, $x_0 \in \text{cl}_X [X \setminus \{x_0\}]$; so there is a subset $F_1 \subseteq X \setminus \{x_0\}$ of cardinality $\leq \kappa$ such that $x_0 \in \text{cl}_X F_1$. If $F_0 \cup F_1$ is crowded, where $F_0 = \{x_0\}$, then we have finished. Otherwise, for each isolated point $x$ of $F_0 \cup F_1$, there is a subset $F_x^2 \subseteq X \setminus (\{x_0\} \cup F_1)$ of cardinality $\leq \kappa$ such that $x \in \text{cl}_X F_x^2$. Let $F_2 = \bigcup_{x \in G_1} F_x^2$ where $G_1$ is the set of isolated points of $F_0 \cup F_1$. Again, there are two possible situations: either $F_0 \cup F_1 \cup F_2$ is a crowded subspace of cardinality $\leq \kappa$ containing $x_0$, or $G_2 = \{x \in F_2 : x$ is an isolated point of $F_0 \cup F_1 \cup F_2\}$ is not empty. In this last case, for each $x \in G_2$ we take a subset $F_x^3 \subseteq X \setminus (F_0 \cup F_1 \cup F_2)$ of cardinality $\leq \kappa$ for which $x \in \text{cl}_X F_x^3$. We write $F_3 = \bigcup_{x \in G_2} F_x^3$. Continuing this process if necessary, we obtain either a finite sequence $F_0, \ldots, F_n$ of subsets of $X$ such that $x_0 \in F = \bigcup_{0 \leq i \leq n} F_n$ and $F$ has cardinality $\leq \kappa$ and is crowded, or we have to go further: $x_0 \in F = \bigcup_{n < \omega} F_n$. In this last case too, $F$ has cardinality $\leq \kappa$ and is crowded. \hfill $\square$

3. Martin’s Axiom, $\pi$-netweight and $\omega$-resolvable spaces

First, in this section we are going to present, by using Martin’s Axiom, a generalization of Theorem 1.4. As usual, if $I$ and $J$ are two sets, $\text{Fn}(I, J)$ stands for the collection of the finite functions with domain contained in $I$ and range contained in $J$. We define a partial order $\leq$ in $\text{Fn}(I, J)$ by letting $p \leq q$ iff $p \supseteq q$. The partial order set ($\text{Fn}(I, J), \leq$) is ccc if and only if $|J| \leq \aleph_0$ (Lemma 5.4, p. 205 in [17]).

Let $(X, \tau)$ be a topological space. A collection $\mathcal{N} \subseteq \mathcal{P}(X)$ is a $\pi$-network of $X$ if each element $U \in \tau \setminus \{\emptyset\}$ contains an element of $\mathcal{N}$. 

3.1 Definitions. Let $\kappa$ be an infinite cardinal.

(1) A space $X$ is almost-$\kappa$-resolvable if $X$ can be partitioned as $X = \bigcup_{\alpha < \kappa'} X_\alpha$ where $\omega \leq \kappa' \leq \kappa$, $X_\alpha \neq \emptyset$, and $X_\alpha \cap X_\xi = \emptyset$ if $\alpha \neq \xi$, such that every non-empty open subset of $X$ has a non-empty intersection with an infinite collection of elements in $\{X_\alpha : \alpha < \kappa\}$.

(2) Let $\mathcal{X} = \{X_\alpha : \alpha < \kappa\}$ be a partition of $X$. A collection $\mathcal{N} = \{N_\xi : \xi < \tau\}$ of infinite subsets of $\kappa$ is a $\pi$-network of $X$ if for each open set $U$ of $X$, $\{\alpha < \kappa : X_\alpha \cap U \neq \emptyset\} \supseteq N_\xi$ for a $\xi < \tau$.

(3) A space $X$ is called precisely almost-$\kappa$-resolvable if $X$ contains a resolution with a $\pi$-network $\mathcal{N}$ such that $|\mathcal{N}| \leq \kappa$.

The following well known result is due to K. Kuratowski.

3.2 Lemma (The disjoint refinement lemma). Let $\{A_\xi : \xi < \kappa\}$ be a collection of sets such that, for each $\xi < \kappa$, $|A_\xi| \geq \kappa$. Then, there is a collection $\{B_\xi : \xi < \kappa\}$ of sets satisfying:

1. $B_\xi \subseteq A_\xi$ for all $\xi < \kappa$,
2. $|B_\xi| = \kappa$ for all $\xi < \kappa$,
3. $B_\xi \cap B_\zeta = \emptyset$ for $\xi, \zeta < \kappa$ with $\xi \neq \zeta$.

3.3 Proposition. A space $X$ is precisely almost-$\omega$-resolvable if and only if $X$ is $\omega$-resolvable.

Proof: Let $X$ be a precisely almost-$\omega$-resolvable space. Let $\mathcal{X} = \{X_\xi : \xi < \tau\}$ be a precise partition of $X$, and $\mathcal{M} = \{M_n : n < \omega\}$ be a $\pi$-network of $\mathcal{X}$. Because of Lemma 3.2, there are infinite and pairwise disjoint sets $T_0, T_1, \ldots, T_n, \ldots$ such that $T_i \subseteq M_i$ for all $i < \omega$.

For each $n < \omega$, we faithfully enumerate $T_n$: $\{k_i^n : i < \omega\}$. Now we define for each $i < \omega$, $D_i = \bigcup_{j < \omega} X_{k_i^j}$. Each $D_n$ is dense in $X$ and $D_i \cap D_j = \emptyset$ if $i \neq j$.

Moreover, if $X$ is $\omega$-resolvable and $\mathcal{D} = \{D_n : n < \omega\}$ is a collection of pairwise disjoint dense subsets of $X$, then $\mathcal{D}$ is a precise partition of $X$ and $\mathcal{M} = \{\omega\}$ is a $\pi$-network of $\mathcal{D}$.

When we assume Martin’s Axiom, we can generalize Proposition 3.3:

3.4 Theorem. Let $\mathcal{X} = \{X_\alpha : \alpha < \tau\}$ be an almost-$\tau$-resolvable partition of $X$. Let $\mathcal{N} = \{N_\xi : \xi < \kappa\}$ be a $\pi$-network of $\mathcal{X}$ such that $\kappa < c$. If we assume Martin’s Axiom, then $X$ is $\omega$-resolvable. In particular, MA implies that $\omega$-resolvability and almost-$\kappa$-resolvability precise coincide when $\kappa < c$.

Proof: In this case, we put $\mathbb{P} = (\text{Fn}(\kappa, \omega), \leq)$ where $\leq$ is defined at the beginning of this section. For each $k \in \omega$ and $N \in \mathcal{N}$, we take the set $D_N^k = \{p \in \mathbb{P} : \exists \xi \in N \text{ such that } p(\xi) = k\}$.
It happens that each $D^k_N$ is dense in $\mathbb{P}$. In fact, let $q$ be an arbitrary element in $\mathbb{P}$. We can take $\xi \in N \setminus \text{dom}(q)$ because $N$ is infinite. The function $p = q \cup \{(\xi, k)\}$ belongs to $D^k_N$ and is less than $q$.

The partially ordered set $\mathbb{P}$ is ccc and $\mathcal{D} = \{D^k_N : k < \omega, N \in \mathcal{N}\}$ has cardinality strictly less than $\mathfrak{c}$. So, there exists a $\mathcal{D}$-generic filter $G$ in $\mathbb{P}$. Take $f = \bigcup G$. Then $f : \kappa \to \omega$ is onto and $\kappa = \bigcup_{n<\omega} Y_n$ where $Y_n = f^{-1}[[n]]$.

Now, for each $n < \omega$, we consider the set $X_n = \bigcup_{\alpha \in Y_n} X_\alpha$. It is easy to prove that $\{X_n : n < \omega\}$ is a partition of $\bigcup_{n<\omega} X_n$. Moreover, each $X_n$ is a dense subset of $X$. Indeed, let $n_0$ be a natural number. We are going to prove that $X_{n_0}$ is dense. Let $U$ be an open set of $X$. Because of the properties of $\mathcal{N}$, there is $N_0 \in \mathcal{N}$ such that $\{\alpha < \tau : X_\alpha \cap U \neq \emptyset\} \supseteq N_0$. We take $p \in D^0_{N_0} \cap G$. It happens that there is a $\xi \in N_0$ such that $p(\xi) = n_0$. Hence, $f(\xi) = n_0$. This means that $\xi \in f^{-1}[[n_0]] = Y_{n_0}$. By definition, $X_\xi$ must have a non-empty intersection with $U$, and therefore $U \cap X_{n_0} = U \cap \bigcup_{\alpha \in Y_{n_0}} X_\alpha \neq \emptyset$. \hfill \Box

Assume that $\{x_\xi : \xi < \tau\}$ is a faithful enumeration of a space $X$. If $X$ possesses a $\pi$-network $\mathcal{N}$ with infinite elements, the collection $\{M_N : N \in \mathcal{N}\}$ where $M_N = \{\xi < \tau : x_\xi \in N\}$, is a $\pi$-network of the partition $\{\{x_\xi\} : \xi < \tau\}$. So the following result is a corollary of Theorem 3.4.

**3.5 Theorem.** Let $X$ be a crowded topological space with a $\pi$-network $\mathcal{N}$ with cardinality $\kappa < \mathfrak{c}$ and such that each element in $\mathcal{N}$ is infinite. If we assume Martin’s Axiom, then $X$ is an $\omega$-resolvable space.

Recall that every biradial space is maximally resolvable. Moreover, every space with $\pi w(X) \leq \Delta(X)$ is maximally resolvable (see [4]). With respect to these ideas we have:

**3.6 Corollary [MA].** Every crowded space $X$ with $\pi$-weight $< \mathfrak{c}$ is $\omega$-resolvable. In particular, every space with weight $< \mathfrak{c}$ is hereditarily $\omega$-resolvable.

**Proof:** Let $\mathcal{N}$ be a $\pi$-base of $X$ of cardinality $< \mathfrak{c}$. Since $X$ is crowded and each element of $\mathcal{N}$ is open in $X$, then $|\mathcal{N}| \geq \aleph_0$ for each $N \in \mathcal{N}$. On the other hand, $\mathcal{N}$ is a $\pi$-network in $X$, so the conclusion follows. \hfill \Box

It is easy to see that if $X$ has $\pi$-character and density $\leq \kappa$, then $X$ has a $\pi$-base of cardinality $\leq \kappa$.

**3.7 Proposition [MA].** If $X$ is a space with density and $\pi$-character $< \mathfrak{c}$, then every dense subset of $X$ is $\omega$-resolvable.

**Proof:** The space $X$ has a $\pi$-base $\mathcal{B}$ of cardinality $< \mathfrak{c}$. Let $H$ be an arbitrary dense subset of $X$. It happens now that $\mathcal{M} = \{N \cap H : N \in \mathcal{N}\}$ is a $\pi$-base of $H$ and has cardinality $< \mathfrak{c}$. So, by Corollary 3.6, $H$ is $\omega$-resolvable. \hfill \Box

For every space $X$, $\max\{t(X), \pi \chi(X)\} \leq \chi(X)$, so, as a consequence of the last result, and related to Theorems 2.8(2) and 2.8(8), we have:
3.8 Theorem [MA]. If $X$ is a space such that $\chi(x, X) < c$ for each $x \in X$, then $X$ is hereditarily $\omega$-resolvable.

Proof: Let $Y$ be a crowded subspace of $X$. The character of $Y$ is strictly less than $c$; thus, the tightness of $Y$ is $< c$. Hence, each point $y$ in $Y$ is contained in a crowded subspace $Y_y$ of $Y$ of cardinality $< c$ (Proposition 2.14). The density and character of each $Y_y$ is strictly less than $c$. By Proposition 3.7, $Y_y$ is $\omega$-resolvable. Then $Y$ is $\omega$-resolvable (see Proposition 2.9(1)). □

The following result is a generalization of Theorems 3.5 and 3.8, which answers, affirmatively, a question posed by the referee. A collection $\mathcal{N} \subseteq \mathcal{P}(X)$ is a $\pi$-network of $X$ at the point $x \in X$ if every open set of $X$ containing $x$ contains an element of $\mathcal{N}$. For each point $x \in X$, we define $\pi nw^*(x, X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a $\pi$-network of $X$ at $x$ and each element in $\mathcal{N}$ is infinite$\}$. Of course, for each $x \in X$, $\pi nw^*(x, X) < \chi(x, X)$. Since MA implies that $c$ is a regular cardinal, we have that, by Theorem 3.5, MA implies that every space $X$ containing a dense subset $Y$ of cardinality $\leq \kappa < c$ and such that for every $y \in Y$, $\pi nw^*(y, X) < c$, is $\omega$-resolvable. This result can be ameliorated. Indeed, by using a similar proof to that of Proposition 2.14, if $X$ is a space with $\pi nw^*(x, X) < c$ for each $x \in X$, then each point $x \in X$ is contained in a crowded subspace $X_x$ of $X$ of cardinality $< c$ and having, for each $y \in X_x$, $\pi nw^*(y, X_x) < c$. So:

3.9 Corollary [MA]. Let $X$ be a space such that for every $x \in X$, $\pi nw^*(x, X) < c$. Then $X$ is $\omega$-resolvable.

We obtain another result with a slightly different mood of that of the previous corollary by defining for each point $x \in X$ the number $R(x, X) = \min\{|\Lambda| : \Lambda$ is a directed partially ordered set and there is a net $(x_\alpha)_{\alpha \in \Lambda}$ in $X \setminus \{x\}$ such that $(x_\alpha)_{\alpha \in \Lambda}$ converges to $x$ in $X\}$. Indeed, following a similar argumentation to that given in the previous paragraph of Corollary 3.9, we obtain:

3.10 Corollary [MA]. Let $X$ be a space such that for every $x \in X$, $R(x, X) < c$. Then $X$ is $\omega$-resolvable.

In Proposition 4.5 of [2] it was proved that every $T_2$ $\sigma$-space is almost-$\omega$-resolvable. When $X$ has a countable network, we can repeat that proof assuming only the weaker condition $T_0$. So every space with countable network is almost-$\omega$-resolvable. With respect to $\sigma$-spaces, Proposition 4.5 in [2] and Martin’s Axiom, Proposition 3.11 allows us to say something else which is, in some sense, stronger that Theorem 3.5:

3.11 Proposition [MA]. Let $\kappa$ be an infinite cardinal $< c$. Let $X$ be a space with a network $\mathcal{N}$ such that for each finite subcollection $\mathcal{N}'$ of $\mathcal{N}$, $\bigcap \mathcal{N}'$ is infinite or empty, and for each $x \in X$, $|\{N \in \mathcal{N} : x \in N\}| \leq \kappa$. Then, $X$ is hereditarily $\omega$-resolvable.
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**Proof:** The space $X$ is the condensation of a crowded space $Y$ (where $Y$ is $X$ with the topology generated by $\mathcal{N}$ as a base) which has character strictly less than $\mathfrak{c}$ (see Proposition 2.11).

Next, we obtain a result that we can locate between Theorem 3.5 which deals with $\pi$-networks and Corollary 3.6 which speaks of bases. First a definition and some remarks. A space $X$ is called $\sigma$-locally finite if $X$ can be written as $\bigcup_{n<\omega} X_n$ where, for each $n < \omega$, the collection $\{\{x\} : x \in X_n\}$ is locally finite in $X$. It can be proved that a $\sigma$-locally finite crowded space is hereditarily almost-$\omega$-resolvable.

**3.12 Theorem** [MA]. Let $X$ be a crowded topological space with a network $\mathcal{N}$ with cardinality $\kappa < \mathfrak{c}$ and such that $\mathcal{N}_0 = \{N \in \mathcal{N} : |N| < \aleph_0\}$ is $\sigma$-locally finite in $\bigcup \mathcal{N}_0$. Then $X$ can be written as $Y_0 \cup Y_1$ where $Y_0$ is a (possibly empty) regular closed $\omega$-resolvable subspace and $Y_1$ is an open (possibly empty) almost-$\omega$-resolvable, hereditarily $\omega$-irresolvable space. Besides, if $Y_1$ is not void, it contains a non-empty open subset which is hereditarily almost-$\omega$-resolvable. Moreover, if $X$ is a $T_1$ Baire space, then $X$ must be $\omega$-resolvable.

**Proof:** Let $\mathcal{M}$ be the collection of all subspaces of $X$ which are $\omega$-resolvable. Take $Y_0 = \text{cl}_X \bigcup \mathcal{M}$ and $Y_1 = X \setminus Y_0$. Of course $Y_0$ is closed and $\omega$-resolvable. Now, if $Y_1$ is empty, we have already finished; if the contrary happens, $Y_1$ is hereditarily $\omega$-irresolvable and the collection $\mathcal{N}' = \{N \in \mathcal{N} : N \subseteq Y_1\}$ is a network in $Y_1$ with cardinality $\kappa < \mathfrak{c}$ and such that $\mathcal{N}'_0 = \{N \in \mathcal{N}' : |N| < \aleph_0\}$ is $\sigma$-locally finite in $\bigcup \mathcal{N}'_0$. Of course $\mathcal{N}'_0$ is not empty, because otherwise, by Theorem 3.5, $Y_1$ would be $\omega$-resolvable, but this is not possible. Let $Z$ be the subspace $\bigcup_{N \in \mathcal{N}'_0} N$ of $X$. The space $Z$ is $\sigma$-locally finite. Since $Y_1$ is hereditarily $\omega$-irresolvable, $Z$ is a dense subset of $Y_1$. Then, $Y_1$ is almost-$\omega$-resolvable. Furthermore, there must exist a non-empty open subset $U$ of $Y_1$ such that each element of $\mathcal{N}'$ contained in $U$ is finite because otherwise $Y_1$ would be $\omega$-resolvable (again by Theorem 3.5). So, $\text{int} Z$ is a non-empty open subset which is hereditarily almost-$\omega$-resolvable.

Assume now that $X$ is $T_1$ and satisfies all the conditions of our proposition including the Baire property. In this case $Y_1$ must be empty because if this is not the case, the subspace $\text{int} Z$ of $Y_1$ would be a $T_1$ Baire hereditarily almost-$\omega$-resolvable space. But this means, by Theorem 1.3, that $\text{int} Z$ is $\omega$-resolvable, which is not possible.

If we consider in the previous theorem $\pi$-networks instead of networks, we still get something interesting.

**3.13 Proposition** [MA]. Let $X$ be a crowded topological space with a $\pi$-network $\mathcal{N}$ with cardinality $\kappa < \mathfrak{c}$ and such that $\mathcal{N}_0 = \{N \in \mathcal{N} : |N| < \aleph_0\}$ is $\sigma$-locally finite. Then $X$ is equal to $X_0 \cup X_1$ where $X_0 \cap X_1 = \emptyset$, $X_0$ is a regular closed (possibly empty) almost-$\omega$-resolvable space and $X_1$ is an open (possibly empty) $\omega$-resolvable subspace. In particular, $X$ is, in this case, almost-$\omega$-resolvable.
Proof: Let $Y$ be the subspace $\bigcup_{N \in \mathcal{N}_0} N$. The space $Y$ is $\sigma$-locally finite. If $Y$ is empty, we obtain our result by Theorem 3.5. If $Y$ is crowded, then it is almost-$\omega$-resolvable (see Theorem 3.5 in [26]). If $Y$ is not empty and is not crowded, we can find an ordinal number $\alpha > 0$ and, for each $\beta < \alpha$, an $\omega$-resolvable subspace $M_\beta$ of $X$ such that $X_0 = \text{cl}_X(Y \cup \text{cl}_X(\bigcup_{\beta < \alpha} M_\beta))$ is almost-$\omega$-resolvable. In fact, let $D_0$ be the set of isolated points in $Y_0 = Y$. For each $x \in D_0$, there is an open set $A_x$ in $X$ such that $A_x \cap Y_0 = \{x\}$. Observe that $A_x \setminus \{x\}$ is a dense subset of $A_x$ and it satisfies the conditions in Theorem 3.5, so it is $\omega$-resolvable. Thus, $M_0 = \text{cl}_X(\bigcup_{x \in D_0} A_x)$ is an $\omega$-resolvable space. Assume that we have already constructed $\omega$-resolvable subspace $M_\beta$ of $X$ with $\beta < \gamma$. Put $Y_\gamma = Y \setminus \text{cl}_X(\bigcup_{\beta < \gamma} M_\beta)$. If $Y_\gamma$ is empty or crowded, we take $\alpha = \gamma$, and in this case $\text{cl}_X(Y \cup \text{cl}_X(\bigcup_{\beta < \gamma} M_\beta))$ is almost-$\omega$-resolvable because $Y_\gamma$ is empty or crowded and $\sigma$-locally finite. If $Y_\gamma$ is not empty and is not crowded, let $D_\gamma$ be the set of isolated points in $Y_\gamma$. For each $x \in D_\gamma$ there is an open set $A_x$ in $X$ such that $A_x \cap Y_\gamma = \{x\}$ and $A_x \cap \text{cl}_X(\bigcup_{\beta < \gamma} M_\beta) = \emptyset$. Again $A_x \setminus \{x\}$ is a dense subset of $A_x$ and it is $\omega$-resolvable because of Theorem 3.5. Thus, $M_\gamma = \text{cl}_X(\bigcup_{x \in D_\gamma} A_x)$ is an $\omega$-resolvable space. Continuing with this process we have to find an ordinal number $\alpha$ for which $X_0 = \text{cl}_X(Y \setminus \text{cl}_X(\bigcup_{\beta < \alpha} M_\beta))$ is almost-$\omega$-resolvable.

Now, if $X_1 = X \setminus X_0$ is not empty, then it is a crowded space and $\mathcal{N}_1 = \{N \in \mathcal{N} : N \subseteq X_1\}$ is a $\pi$-network in $X_1$ with infinite elements and $|\mathcal{N}_1| < c$. Then, again by Theorem 3.5, $X_1$ is $\omega$-resolvable. Therefore, $X = X_0 \cup X_1$, and $X_0, X_1$ satisfy the conditions of our proposition. □

3.14 Questions. (1) Let $X$ be a crowded space with cardinality $< c$. Does $\text{MA} \Rightarrow \text{CH}$ imply that $X$ is almost-$\omega$-resolvable?

(2) Is there a combinatorial axiom on $\omega_1$ ensuring that every card-homogeneous topology in $\omega_1$ is almost-$\omega$-resolvable?

(3) Does $\diamondsuit$ imply that every card-homogeneous topology in $\omega_1$ is almost-$\omega$-resolvable?

4. Martin’s Axiom, cellularity and $\omega$-resolvable Baire spaces

It is well known that $\text{MA}(\omega_1)$ implies that a Souslin line does not exist. That is, $\text{MA}(\omega_1) \Rightarrow \text{SH}$. We show that it is enough to assume $\text{SH}$ in order to prove that every $T_2$ space with countable cellularity is almost-$\omega$-resolvable.

4.1 Theorem [SH]. Every crowded $T_2$ space with countable cellularity is almost-$\omega$-resolvable.

Proof: Let $a_0 \in X$ and $F_0 = \{a_0\}$. Let $\mathcal{C}_0$ be a maximal cellular family of open sets in $X \setminus F_0$ containing at least two elements. Let $X_0$ be equal to $\bigcup \mathcal{C}_0$. Assume that we have already constructed, by recursion, families $\{\mathcal{C}_\alpha : \alpha < \gamma\}$, $\{X_\alpha : \alpha < \gamma\}$ and $\{F_\alpha : \alpha < \gamma\}$, such that

(1) for all $\alpha < \gamma$, $\mathcal{C}_\alpha$ is a maximal cellular collection of open sets in $X$;
(2) if \( \alpha < \xi < \gamma \), then \( C_\xi \) properly refines \( C_\alpha \);

(3) if \( \alpha < \xi < \gamma \) and \( C \in C_\alpha \), then \( C_\xi \) contains a maximal cellular family of proper open sets of \( C \) having more than one element;

(4) \( X_\alpha = \bigcup C_\alpha \) for each \( \alpha < \gamma \);

(5) the family \( \{ X_\alpha : \alpha < \gamma \} \) is a strictly decreasing \( \gamma \)-sequence of open sets in \( X \);

(6) \( F_\alpha \neq \emptyset \) for every \( \alpha < \gamma \);

(7) \( F_\alpha \subseteq (\bigcap_{\xi<\alpha} X_\xi) \setminus X_\alpha \) for all \( \alpha < \gamma \);

(8) \( \text{int}(F_\alpha) = \emptyset \) for all \( \alpha < \gamma \).

If \( \gamma \) is a successor ordinal, say \( \gamma = \xi + 1 \), take for each \( C \in C_\xi \) a point \( a_\xi^C \in C \). Now, take a maximal cellular family of open proper subsets in \( C \setminus \{ a_\xi^C \} \) with more than one element, \( C_\gamma \). (This is possible because \( C \) is \( T_2 \) and infinite). Put \( C_\gamma = \bigcup_{C \in C_\gamma} C_\xi \), \( \gamma = \bigcup C_\gamma \) and \( F_\gamma = \{ a_\xi^C : C \in C_\xi \} \).

If \( \gamma \) is a limit ordinal, analyse the set \( \bigcap_{\xi<\gamma} X_\xi \): if \( \text{int}(\bigcap_{\xi<\gamma} X_\xi) = \emptyset \), declare our process finished; and if \( \text{int}(\bigcap_{\xi<\gamma} X_\xi) \) is not empty, take a point \( \alpha > 0 \) \( \in \text{int}(\bigcap_{\xi<\gamma} X_\xi) \) and take a maximal cellular family \( C_\gamma \) with cardinality bigger than one of open proper subsets in \( \text{int}(\bigcap_{\xi<\gamma} X_\xi) \setminus F_\gamma \), where \( F_\gamma = \{ a_\xi^C \} \). Put \( X_\gamma = \bigcup C_\gamma \).

In this way we can find an ordinal number \( \alpha_0 \) and families \( \mathcal{C} = \{ C_\alpha : \alpha < \alpha_0 \} \), \( \mathcal{Y} = \{ X_\alpha : \alpha < \alpha_0 \} \) and \( \mathcal{F} = \{ F_\alpha : \alpha < \alpha_0 \} \) satisfying properties from (1) to (8) above where \( \alpha_0 \) is an ordinal number such that \( \text{int}(\bigcap_{\xi<\alpha_0} X_\xi) = \emptyset \) and for each \( \alpha < \alpha_0 \), \( \text{int}(\bigcap_{\xi<\alpha_0} X_\xi) \neq \emptyset \).

First, observe that \( \alpha_0 \) must be a limit ordinal and every \( X_\alpha \) is an open set of \( X \). Now, consider the collection \( \mathcal{Y} = \{ Y_\alpha : \alpha < \alpha_0 \} \) of subspaces of \( X \) where \( Y_0 = X \setminus X_0 \) and \( Y_\alpha = (\bigcap_{\xi<\alpha} X_\xi) \setminus X_\alpha \) if \( \alpha > 0 \). We have that \( F_\alpha \subseteq Y_\alpha \) and \( \text{int}(Y_\alpha) = \emptyset \) for every \( \alpha < \alpha_0 \).

The set \( \bigcup_{\alpha<\alpha_0} C_\alpha \) with the order relation \( \subseteq \) is a tree \( T \) and each element in it has at least two immediate successors.

**Claim 1.** The height of \( T \), \( \alpha_0 \), is at most \( c(X)^+ = \omega_1 \).

In fact, if \( \alpha_0 > \omega_1 \), then \( C_{\omega_1} \neq \emptyset \). Take \( C_{\omega_1} \in C_{\omega_1} \). Let \( C = \{ C \in T : C \supseteq C_{\omega_1} \) and \( C \neq C_{\omega_1} \} \). Since \( T \) is a tree, \( C \) is a well ordered set with order type \( \omega_1 \). We can rename \( C \) as \( \{ C_\alpha : \alpha < \omega_1 \} \) where \( C_\alpha \) is the only element in \( C_\alpha \) which belongs to \( C \). For each \( \alpha < \omega_1 \), there is \( A_{\alpha+1} \subseteq C_{\alpha+1} \) such that \( A_{\alpha+1} \subseteq C_\alpha \) and \( A_{\alpha+1} \cap C_{\alpha+1} = \emptyset \). The set \( A = \{ A_{\alpha+1} : \alpha < \omega_1 \} \) is an antichain in \( T \). Indeed, let \( A_{\alpha+1} \) and \( A_{\xi+1} \) be two different elements of \( A \). Assume that \( \alpha < \xi \). Hence, \( A_{\xi+1} \subseteq C_\xi \) and \( C_\xi \subseteq C_{\alpha+1} \). But \( C_{\alpha+1} \cap A_{\alpha+1} = \emptyset \). Therefore, \( A_{\alpha+1} \cap A_{\xi+1} = \emptyset \). This means that \( c(X) > \aleph_0 \), which is a contradiction. We get that every chain and every antichain of \( T \) has cardinality \( \leq \aleph_0 \). Since we are assuming the Souslin’s Hypothesis, there are no Souslin trees. Therefore \( \alpha_0 < \omega_1 \).

It is not difficult to prove that the set \( Z = X \setminus X_{\alpha_0} \) is equal to \( \bigcup_{\alpha<\alpha_0} Y_\alpha \) and that the collection \( \{ Y_\alpha : \alpha < \alpha_0 \} \) is a partition of \( Z \).
Claim 2. The collection \( \{Y_\alpha : \alpha < \alpha_0\} \cup \{X_{\alpha_0}\} \) is an almost-\( \omega \)-resolution for \( X \); that is, \( X \) is almost-\( \omega \)-resolvable.

The collection \( Y = \{Y_\alpha : \alpha < \alpha_0\} \cup \{X_{\alpha_0}\} \) is a countable partition of \( X \). Assume that \( A \) is a non-empty open set of \( X \) and \( |\{\alpha < \alpha_0 : A \cap Y_\alpha \neq \emptyset\}| < \aleph_0 \). Assume that \( H = \{\alpha < \alpha_0 : A \cap Y_\alpha \neq \emptyset\} \) is equal to \( \{\xi_1, \ldots, \xi_n\} \) with \( \xi_1 < \xi_2 < \cdots < \xi_n \).

If \( B = A \cap X_{\alpha_0} \neq \emptyset \), then \( A \cap X_{\xi_n} = B \). But \( A \) and \( X_{\xi_n} \) are open sets in \( X \), so \( B \) is a non-empty open set in \( X \), contradicting the fact that \( \text{int}(X_{\alpha_0}) = \emptyset \). This means that \( A \cap X_{\alpha_0} \) must be empty.

Now, let \( B = A \cap Y_{\xi_n} \). \( B \) is not empty and \( A \cap X_{\xi_n-1} = B \). Thus, \( B \) is a non-empty open set in \( X \) which does not intersect any member of \( \mathcal{C}_{\xi_n} \). If \( \xi_n = \alpha + 1 \), \( \mathcal{C}_{\xi_n} \) is a maximal cellular collection of open sets contained in \( (\bigcup \mathcal{C}_\alpha) \setminus \{a_\alpha^C : C \in \mathcal{C}_\alpha\} \). Hence, \( B \cap \{a_\alpha^C : C \in \mathcal{C}_\alpha\} \neq \emptyset \). Let \( a_\alpha^C \in B \). We have that \( M = (C \cap B) \setminus \{a_\alpha^C\} \) is an open set contained in \( X_\alpha \setminus \{a_\alpha^C : C \in \mathcal{C}_\alpha\} \) and no element in \( \mathcal{C}_\xi \) intersects \( M \). By maximality of \( \mathcal{C}_\xi \), we must have that \( M \) is empty; that is, \( C \cap B = \{a_\alpha^C\} \), and this is not possible because \( X \) does not have isolated points.

Now assume that \( \xi_n \) is a limit ordinal. Since \( B \) is open and \( B \subseteq \bigcap_{\xi < \xi_n} X_\xi \), \( B \) must be contained in \( \text{int}(\bigcap_{\xi < \xi_n} X_\xi) \). Since \( \{a_{\xi_n}\} \) is closed and \( B \) does not intersects any element of \( \mathcal{C}_{\xi_n} \) which is a maximal cellular family of open sets contained in the set \( \text{int}(\bigcap_{\xi < \xi_n} X_\xi) \setminus \{a_{\xi_n}\} \), \( B \) must be equal to \( \{a_{\xi_n}\} \), which is again a contradiction.

Therefore, \(|\{\xi < \alpha_0 : A \cap Y_\xi \neq \emptyset\}| \) must be equal to \( \aleph_0 \). \( \square \)

Since the cellularity of a space is a monotone function when it is applied on dense subspaces, and using Theorem 1.3, we conclude:

4.2 Corollary [SH]. Every \( T_2 \) Baire space with \( c(X) \leq \aleph_0 \) is \( \omega \)-resolvable.

Example 4.3 in [26] (see Example 2.3 above) gives us a space which is Baire, \( T_1 \) with countable cellularity but it is not almost-\( \omega \)-resolvable. This example is constructed assuming the existence of measurable cardinals. Moreover, there is a model \( M \) in which SH holds and there are measurable cardinals. So we cannot get anything stronger than our results of this section by assuming only \( T_1 \). Furthermore, we cannot erase the Baire condition in Corollary 4.2 because there is in ZFC a Tychonoff, countable irresolvable space (see Examples 2.5). Finally, in 2.2 we list an example of a space with cellularity \( \leq \aleph_1 \) which is Baire and is not almost-\( \omega \)-resolvable. This last example is given by assuming the existence of an \( \omega_1 \)-complete ideal over \( \omega_1 \) which has a dense set of cardinality \( \omega_1 \). Hence, it is natural to ask:

4.3 Question. Does MA imply that every crowded \( T_2 \) space of cellularity \( < c \) is almost-\( \omega \)-resolvable?
In this question, we cannot change “almost-$\omega$-resolvable” for “resolvable” since there is in ZFC an irresolvable countable space.

5. Almost-$\omega$-irresolvable spaces

A space is *almost-$\omega$-irresolvable* if it is not almost-$\omega$-resolvable. In a similar way we define almost irresolvable spaces.

5.1 Proposition. If $X$ is almost-$\omega$-irresolvable, then there is a non-empty open subset $U$ of $X$ which is hereditarily almost-$\omega$-irresolvable.

**Proof:** Let $U$ be the collection of all almost-$\omega$-resolvable subspaces $Y$ of $X$. The set $Z = \text{cl}_X(\bigcup U)$ is almost-$\omega$-resolvable and $U = X \setminus Z$ is not empty and satisfies the requirements.

5.2 Proposition. If $X$ is open hereditarily almost-$\omega$-irresolvable, then $X$ is a Baire space.

**Proof:** Let $\{U_n : n < \omega\}$ be a sequence of open and dense subsets of $X$. We can choose this sequence to be $\subseteq$-decreasing. Denote by $F$ the set $\bigcap_{n<\omega} U_n$. We claim that $F$ is dense in $X$. In fact, if for a $k < \omega$, $\text{cl}_X F \supseteq U_k$, then $\text{cl}_X F \supseteq \text{cl}_X U_k = X$ and $F$ is dense. Now, assume that for each $n < \omega$, $U_n \setminus \text{cl}_X F$ is not empty. In this case, the collection $T = \{i < \omega : (U_i \setminus U_{i+1}) \cap (X \setminus \text{cl}_x F) \neq \emptyset\}$ is infinite. For each $i \in T$, we put $T_i = (U_i \setminus U_{i+1}) \cap (X \setminus \text{cl}_x F)$. The collection $\{T_i : i < \omega\}$ forms an almost-$\omega$-resolution of $X \setminus \text{cl}_X F$. But this is not possible.

5.3 Corollary. If there is an almost resolvable space $X$ which is almost-$\omega$-irresolvable, then there is a resolvable Baire open subspace $U$ of $X$ which is hereditarily almost-$\omega$-irresolvable.

**Proof:** Let $X$ be an almost-resolvable almost-$\omega$-irresolvable space. The space $X$ contains a non-empty open subspace $U$ which is hereditarily almost-$\omega$-irresolvable. By Proposition 5.2, $U$ is a Baire space; so, it is resolvable being almost resolvable.

5.4 Corollary. There is an almost resolvable space $X$ which is almost-$\omega$-irresolvable if and only if there is an almost resolvable Baire space which is hereditarily almost-$\omega$-irresolvable.

As a consequence of the previous result, we have that almost resolvability and almost-$\omega$-resolvability coincide in the class of spaces $X$ in which every open subset is not a Baire space. Even more was obtained in [2, Corollary 5.21]: every space which does not contain a Baire open subspace is almost-$\omega$-resolvable.

5.5 Proposition. Let $X$ be a $T_1$ space. Then $X$ is hereditarily resolvable if and only if $X$ is hereditarily $\omega$-resolvable.
ω-resolvability of Baire spaces

PROOF: Let $Y$ be a crowded subspace of $X$ and assume that $Y$ is not ω-resolvable. Then, there is $k \in \omega$ with $k > 1$ such that $X$ is $k$-resolvable but $X$ is not $(k + 1)$-resolvable [15]. So there are $D_0, \ldots, D_{k-1}$ dense and pairwise disjoint subspaces of $Y$. But, then, each $D_i$ is crowded and irresolvable, a contradiction. □

5.6 Proposition. Let $X$ have the property that every of its crowded subspaces is Baire. Then $X$ is hereditarily ω-resolvable iff $X$ is hereditarily resolvable iff $X$ is hereditarily almost-ω-resolvable iff $X$ is hereditarily almost resolvable.

Several results established in [2, Section 5] and [26, Section 4] relate Baire irresolvable spaces with the property of almost-ω-resolvability (see also [1, Section 3]). In the following theorem we obtain the most general possible result in the mood of these propositions.

5.7 Theorem. For crowded $T_1$ spaces and for a crowded-hereditarily topological property $P$, the following assertions are equivalent:

1. every Baire space with $P$ is ω-resolvable,
2. every Baire space with $P$ is resolvable,
3. every space with $P$ is almost-ω-resolvable,
4. every space with $P$ is almost resolvable.

PROOF: The implications (1) ⇒ (2) and (3) ⇒ (4) are evident.

(2) ⇒ (3): Assume that $X$ is not almost-ω-resolvable and satisfies $P$. The space $X$ contains an open and non-empty subset $U$ which is hereditarily almost-ω-irresolvable. By Proposition 5.5, $U$ is not hereditarily resolvable, so there is a crowded subspace $Y$ which is not resolvable. Observe that $Y$ is hereditarily almost-ω-irresolvable, then $Y$ is an irresolvable Baire space because of Proposition 5.2. Since $P$ is a crowded-hereditarily topological property, $Y$ satisfies $P$ too.

(4) ⇒ (2): Assume that $X$ is a Baire space with $P$. By hypothesis, $X$ is almost resolvable and every Baire almost resolvable space is resolvable (see [2, Corollary 5.4]).

(3) ⇒ (1): Assume that $X$ is a Baire space with $P$. By hypothesis, every crowded subspace $Y$ of $X$ has $P$ and so it is almost-ω-resolvable; hence $X$ is ω-resolvable because of Theorem 1.3. □

Taking $P$ equal to “$X$ is a crowded topological space”, we have:

5.8 Corollary. For crowded $T_1$ spaces, the following assertions are equivalent:

1. every Baire space is ω-resolvable,
2. every Baire space is resolvable,
3. every space is almost-ω-resolvable,
4. every space is almost resolvable.
A space is \textit{locally homogeneous} if each of its points has a homogeneous neighborhood. For a cardinal number \( \kappa \geq 1 \), we will say that \( X \) is \textit{exactly} \( \kappa \)-resolvable, in symbols \( \text{E}_\kappa \text{R} \), if \( X \) is \( \kappa \)-resolvable but is not \( \kappa^+ \)-resolvable. The space \( X \) is said to be \( \text{OE}_\kappa \text{R} \) if every non-empty open set in \( X \) is \( \text{E}_\kappa \text{R} \). The concept and examples of \( \text{E}_n \text{R} \) spaces for \( n \in \omega \) have existed in the literature for some time (see, for example, \cite{10} and \cite{8}). It is clear that the \( \text{OE}_\kappa \text{R} \) spaces are \( \text{E}_\kappa \text{R} \). The above definitions can be viewed as natural generalizations of the concepts of irresolvable and open-hereditarily irresolvable spaces since \( \text{E}_1 \text{R} \) and irresolvability are the same concept and \( \text{OE}_1 \text{R} \) and open-hereditarily irresolvability coincide.

It was proved in \cite[Theorem 3.13]{1} that every locally homogeneous irresolvable space such that its cardinality is not a measurable cardinal is of the first category. Also, Li Feng and O. Masaveu \cite{13} proved that every crowded topological space \( X \) can be written as

\[
X = \Omega \cup \text{cl}_X \left( \bigcup_{n=1}^{\infty} O_n \right),
\]

where

1. for each \( n \), \( O_n \) is an open, possibly empty, subset of \( X \);
2. for each \( n \), if \( O_n \neq \emptyset \), then it is \( \text{OE}_n \text{R} \);
3. for \( n \neq m \), \( O_n \cap O_m = \emptyset \); and
4. \( \Omega \) is an open, possibly empty, \( \omega \)-resolvable subset of \( X \).

Thus we obtain the following:

\textbf{5.9 Proposition.} Every locally homogeneous Baire space of cardinality strictly less than the first measurable cardinal is resolvable.

\textbf{Proof:} Let \( X \) be a locally homogeneous Baire space. Write \( X \) as Feng and Masaveu say: \( X = \Omega \cup \text{cl}_X (\bigcup_{n=1}^{\infty} O_n) \). Assume that \( O_1 \) is not empty and take \( x \in O_1 \). There is a homogeneous neighborhood \( W \) of \( x \). (Observe that \( W \) has to be contained in \( X \setminus \text{cl}_X (\Omega \cup \bigcup_{n>1} O_n) \subseteq \text{int}_X \text{cl}_X O_1 \). On the other hand, \( O_1 \) is open hereditarily irresolvable, so \( \text{int}_X W \cap O_1 \) is irresolvable. Since \( \text{int}_X W \cap O_1 \) is a non-empty open subset of \( W \), \( W \) is irresolvable. By Theorem 3.13 in \cite{1}, \( W \) is of first category. In particular the open and non-empty subset \( O_1 \cap \text{int}_X W \) of \( X \) is of first category in itself, but this is not possible because \( X \) is a Baire space. Hence, \( O_1 = \emptyset \) and \( X \) is resolvable. \( \square \)

\textbf{5.10 Questions.} (1) Is every pseudocompact (resp., \( \check{\text{C}} \)ech-complete) Tychonoff space almost-\( \omega \)-resolvable in ZFC?

(2) Is every Baire locally homogeneous space (resp., homogeneous space, topological group) \( \omega \)-resolvable?

(3) For each \( n > 1 \), is there a Baire \( \text{OE}_n \text{R} \) space?
6. The infinite $\pi$-netweight and $\text{Seq}(u)$ spaces

We define the infinite $\pi$-networkweight of a crowded space $X$, $\pi\text{nw}^*(X)$, as the minimum infinite cardinal of a $\pi$-network with infinite elements. And $\pi\text{nw}(X)$ is the minimum infinite cardinal of a $\pi$-network in $X$. It is easy to prove that $\pi\text{nw}(X) = d(X)$ for every topological space $X$. Moreover, for a crowded space $X$, we have $d(X) \leq \pi\text{nw}^*(X) \leq \min\{d(X) \cdot \sup\{\pi\text{nw}^*(x, X) : x \in X\}, d(X) \cdot \pi(1)\}$, where $\text{nw}^*(x, X)$ and $\pi(1)$ were defined before Corollaries 3.9 and 3.10. Besides, for every metrizable space $X$ we have $d(X) = w(X)$. So, for a crowded metrizable space $X$, the equality $\pi\text{nw}^*(X) = \pi\text{nw}(X)$ always holds.

We have the same phenomenon for spaces of the form $C_p(X)$, the space of real continuous function defined on $X$ with the pointwise convergent topology (here, $X$ is not necessarily crowded). Indeed, for $f \in C_p(X)$, the sequence $(f_n)_{n<\omega}$ where $f_n = f + 1/n$, converges to $f$. So, if $D$ is a dense subset of $C_p(X)$ with cardinality equal to $d(C_p(X))$, the collection $\{\{f\} \cup \{f_i : i \geq n\} : f \in D, n < \omega\}$ is a $\pi$-network of cardinality $d(C_p(X))$ constituted by infinite elements. So, $\pi\text{nw}^*(C_p(X)) = \pi\text{nw}(C_p(X))$. In particular, for every cardinal number $\kappa$, $\pi\text{nw}^*(\mathbb{R}^\kappa) = d(\mathbb{R}^\kappa)$. The same can be said for spaces of the form $C_p(X, 2)$ where $X$ is an infinite zero-dimensional $T_2$ space. In fact, we can take an infinite discrete subspace $Y = \{x_n : n < \omega\}$ of $X$, and clopen subsets $\{V_n : n < \omega\}$ such that, for each $n < \omega$, $Y \cap V_n = \{x_n\}$. The characteristic functions $\chi_n$ constitute a sequence which converge to the constant function 0. So, in this case too, $\pi\text{nw}^*(C_p(X, 2)) = d(C_p(X, 2))$.

We have already mentioned that in [1] a dense countable subset $Y$ of $2^\omega$ which is irresolvable was constructed in ZFC. This space has $\pi\text{nw}(Y) = \aleph_0$, but every of its countable $\pi$-networks has to have finite elements, because otherwise $Y$ would be maximally resolvable (see Theorem 2.8(1)). The $\text{Seq}(u)$ spaces considered below are also examples of spaces of this kind.

We recall that for a $p \in \omega^*$, $\chi(p) = \min\{|b| : b$ is a base for $p\}$. Of course we can also define: $\pi\chi(p) = \min\{|b| : b$ is a $\pi$-base for $p\}$ where a family of infinite sets $\mathcal{G}$ in $\omega$ is a $\pi$-base for $p$ if every member of $p$ contains an element of $\mathcal{G}$. It is not difficult to prove that for every $p \in \omega^*$, $\pi\chi(p) \leq \chi(p)$ and $\pi\chi(p) > \aleph_0$. In fact, assume that $N_0, \ldots, N_k, \ldots$ are infinite subsets of $\omega$. By recursion, we can construct two sequences $A = \{a_0, \ldots, a_n, \ldots\}$ and $B = \{b_0, \ldots, b_n, \ldots\}$ such that the elements in $A \cup B$ are pairwise different, and for each $n < \omega$, $a_n, b_n \in N_n$. If $A \nsubseteq p$ then $A$ is an element of $p$ which does not contain any $N_k$. If $A \nsubseteq p$, then $\omega \setminus A$ belongs to $p$ and does not contain any $N_k$.

By $\text{Seq}$ we mean the set of all finite sequences of natural numbers. More precisely, for each natural number $n \in \omega$, let $^n\omega = \{t : t$ is a function and $t : n \to \omega\}$. Then $\text{Seq} = \bigcup_{n \in \omega} ^n\omega$. If $t \in \text{Seq}$, with domain $k = \{0, 1, \ldots, (k-1)\}$, and $n \in \omega$, let $t \sim n$ denote the function $t \cup \{(k, n)\}$. For every $t \in \text{Seq}$ let $u_t$ be a non-principal ultrafilter on $\omega$. By $\text{Seq}(\{u_t : t \in \text{Seq}\})$ we denote the space with underlying set $\text{Seq}$ and topology defined by declaring a set $U \subseteq \text{Seq}$ to be open if
and only if
\[(\forall t \in U) \{ n \in \omega : t \smallfrown n \in U \} \in u_t.\]

For short, we write \(\text{Seq}(u_t)\) instead of \(\text{Seq}\{u_t : t \in \text{Seq}\}\). We also consider the case where there is a single non-principal ultrafilter \(p\) in \(\omega\) such that \(u_t = p\) for all \(t \in \text{Seq}\), and in this case we write \(\text{Seq}(p)\) instead of \(\text{Seq}(u_t)\).

We use the following notation of W. Lindgren and A. Szymanski [20]; put \(L_n = \{ s \in \text{Seq} : \text{dom}(s) = n \}\), and for any \(s \in \text{Seq} \) the cone over \(s\) is defined by \(C(s) = \{ t \in \text{Seq} : s \subseteq t \}\). In particular, \(L_0 = \{ \emptyset \}\). We add some other notations: For each \(s \in L_n\), \(T(s) = \{ t \in L_{n+1} : s \subseteq t \}\). Observe that for every \(s \in \text{Seq}\), \(C(s)\) is a clopen subset of \(\text{Seq}(u_t)\).

It is well-known that for any choice of \(\{ u_t : t \in \text{Seq}\} \subseteq \omega^*\), the space \(\text{Seq}(u_t)\) is a zero-dimensional, extremally disconnected, Hausdorff space with no isolated points. By the way, \(\text{Seq}(p)\) is homogeneous and if \(p\) is Ramsey, there is a binary group operation \(+\) such that \((\text{Seq}(p), +)\) is a topological group (see [27]).

6.1 Proposition. Every \(\text{Seq}(u_t)\) space is \(\omega\)-resolvable.

Proof: In fact, let \(\{ E_n : n < \omega \}\) be a partition of \(\omega\) where each \(E_n\) is infinite. Set \(D_n = \bigcup_{i \in E_n} L_i\). Each \(D_n\) is dense in \(\text{Seq}(u_t)\) and \(D_n \cap D_m = \emptyset\) if \(n \neq m\). \(\square\)

6.2 Proposition. Let \(\{ u_t : t \in \text{Seq} \} \subseteq \omega^*\). Then, the infinite \(\pi\)-netweight of \(\text{Seq}(u_t)\) is not countable.

Proof: For each \(n < \omega\), each \(s \in L_n\), and each sequence \(S\) of subcollections of the form

\[
\{ B(s) \}, \{ B(s, i_{n+1}) : i_{n+1} \in B(s) \}, \{ B(s, i_{n+1}, i_{n+2}) : i_{n+1} \in B(s),
\]

\[i_{n+2} \in B(s, i_{n+1}) \}, \ldots, \{ B(s, i_{n+1}, \ldots, i_{n+k+1}) : i_{n+1} \in B(s),
\]

\[i_{n+1} \in B(s, i_{n+1}), \ldots, i_{n+k+1} \in B(s, i_{n+1}, \ldots, i_{n+k}) \}, \ldots
\]

where \(B(s) \in u_s\) and, if \(i_{n+1} \in B(s), i_{n+2} \in B(s, i_{n+1}), \ldots, i_{n+k} \in B(s, i_{n+1}, \ldots, i_{n+k-1}), B(s, i_{n+1}, \ldots, i_{n+k}) \in u_t\) with \(t = s \smallfrown i_{n+1} \ldots \smallfrown i_{n+k}\), we define a set \(V(s, S)\) as follows:

\[V(s, S) = \{ s \} \cup \{ t \in \text{Seq}(p) : m \in \omega, t \in L_{n+m+1}, s \subseteq t, t(n+1) \in B(s),
\]

\[t(n+2) \in B(s, t(n+1)), \ldots,
\]

\[t(n+m+1) \in B(s, t(n+1), t(n+2), \ldots, t(n+m)) \}.
\]

We call this set \(V(s, S)\) cascade of \(\text{Seq}(p)\) defined by \((s, S)\). Moreover, we will called each sequence \(S\), described as above, fan on \((s, (u_t))\).

Of course, the collection of cascades forms a base of clopen sets for \(\text{Seq}(u_t)\).
Claim 1. If \( \mathcal{N} = \{N_0, \ldots, N_k, \ldots\} \) is a countable set of infinite subsets of Seq\((u_t)\), then \( \mathcal{N} \) is not a \( \pi \)-network of Seq\((u_t)\).

We are going to prove Claim 1 in several lemmas.

Claim 1.1. If \( \mathcal{M} \) is a finite collection of subsets of Seq, then there is a non-empty open set \( A \) of Seq\((u_t)\) such that \( M \setminus A \neq \emptyset \) for all \( M \in \mathcal{M} \).

**Proof:** Take \( s_0, \ldots, s_n \) elements in Seq such that each \( M \) in \( \mathcal{M} \) contains one of these points. There is \( k < \omega \) such that \( s_i \in L_m \) implies \( m < k \) for all \( i \in \{0, \ldots, n\} \).

Take \( s \in L_k \). The cone \( C(s) \) is open and contains no element in \( \mathcal{M} \). \( \square \)

Claim 1.2. Assume that \( F \subseteq \text{Seq}(u_t) \) is such that \( |F \cap T(s)| \leq 1 \) for every \( s \in \text{Seq} \). Then, \( F \) is a proper closed subset of Seq\((u_t)\).

**Proof:** Let \( P \) be the set \( \{s < \text{Seq} : F \cap T(s) \neq \emptyset\} \). Let \( z_s \) be the only point belonging to \( F \cap T(s) \) for each \( s \in P \). Let \( x \in \text{Seq}(u_t) \setminus F \). Assume that \( x = (n_0, \ldots, n_k) \) (the argument is similar if \( x = \emptyset \)). Let

\[
S = \{\{B(x)\}, \{B(x, i_0) : i_0 \in B(x)\}, \{B(x, i_0, i_1) : i_0 \in B(x), i_1 \in B(x, i_0)\}, \ldots,
\{B(x, i_0, \ldots, i_{k+1}) : i_0 \in B(x), i_1 \in B(x, i_0), \ldots, i_{k+1} \in B(x, i_0, \ldots, i_k)\}, \ldots\}
\]

be a fan on \((x, (u_t))\). We claim that the set \( V(x, S) \setminus F \) is an open set. Indeed, if \( y \in V(x, S) \setminus F \), \( y \) is of the form \((n_0, \ldots, n_k, i_0, \ldots, i_{m+1})\) where \( m < \omega \), \( i_0 \in B(x), i_1 \in B(x, i_0), \ldots, i_{m+1} \in B(x, i_0, \ldots, i_m) \).

The set \( \{l < \omega : (n_0, \ldots, n_k, i_0, \ldots, i_{m+1}, l) \in V(x, s) \setminus F\} \) is equal to

\[
B(x, i_0, i_1, \ldots, i_{m+1}) \setminus F.
\]

Moreover, the set \( B(x, i_0, i_1, \ldots, i_{m+1}) \setminus F = G \) is either empty if \( F \cap T(x, i_0, i_1, \ldots, i_{m+1}) = \emptyset \), or \( G = \{z(x, i_0, i_1, \ldots, i_{m+1})\} \) if \( F \cap T(x, i_0, i_1, \ldots, i_{m+1}) \neq \emptyset \). Of course, in both cases, \( B(x, i_0, i_1, \ldots, i_{m+1}) \setminus F \) belongs to \( u_t \) where \( t = x \sim i_0 \sim \cdots \sim i_{m+1} \). This means that \( V(x, s) \setminus F \) is open. \( \square \)

Claim 1.3. Let \( \mathcal{M} = \{N \in \mathcal{N} : \forall s \in \text{Seq}(|N \cap T(s)| < \aleph_0)\} \). Then, there is a non-empty open set \( A \) of Seq\((u_t)\) such that \( N \setminus A \neq \emptyset \) for all \( N \in \mathcal{M} \).

**Proof:** First, we define in Seq a well order \( \subseteq \) as follows: \( \emptyset \) is the \( \subseteq \)-first element, and for two elements \( s \) and \( t \) different to \( \emptyset \), we define \( s \sqsubset t \) if either \( s \in L_{n+1} \), \( t \in L_{m+1} \) and \( n < m \), or \( n = m \) and \( s(n) < t(n) \).

Because of Claim 1.1, we can assume that \( \mathcal{M} \) is infinite. We faithfully enumerate \( \mathcal{M} \) as \( \{M_0, M_1, \ldots, M_k, \ldots\} \). Consider the set \( J = \{s \in \text{Seq} : \exists M \in \mathcal{M} \text{ such that } T(s) \cap M \neq \emptyset\} \). Because of the definition of \( \mathcal{M} \), we must have \( |J| = \aleph_0 \). Hence, we can enumerate \( J \) as \( \{s_m : m < \omega\} \) in such a way that \( s_0 \sqsubset s_1 \sqsubset \cdots \sqsubset s_n \sqsubset s_{n+1} \sqsubset \cdots \).
Let \( k_0 \) be the first natural number \( m \) such that \( M_m \cap (s_0) \neq \emptyset \). We take \( z_0 \in M_{k_0} \cap (s_0) \). Assume that we have already defined two finite sequences \( k_0, \ldots, k_l \) and \( z_0, \ldots, z_l \) such that

1. for each \( i \in \{0, \ldots, l - 1\} \), \( k_{i+1} \) is the first natural number \( m \in \omega \setminus \{k_0, \ldots, k_i\} \) such that \( M_m \cap (s_{i+1}) \neq \emptyset \), and
2. \( z_{i+1} \in M_{k_{i+1}} \cap (s_{i+1}) \) for each \( i \in \{0, \ldots, l - 1\} \).

We define now \( k_{l+1} \) as the first natural number \( m \in \omega \setminus \{k_0, \ldots, k_l\} \) such that \( M_m \cap (s_{l+1}) \neq \emptyset \). Take \( z_{l+1} \in M_{k_{l+1}} \cap (s_{l+1}) \).

Observe that \( \{k_i : i < \omega\} = \omega \). Indeed, assume that \( \{0, \ldots, m\} \subseteq \{k_i : i < \omega\} \) and \( \{k_{i_0}, \ldots, k_{i_m}\} = \{0, \ldots, m\} \). Let \( j \) be a natural number greater than \( k_{i_l} \) for all \( l \in \{0, \ldots, m\} \) and such that \( M_{m+1} \cap (s_j) \neq \emptyset \). Then we must have \( m+1 \in \{k_0, \ldots, k_j\} \).

We put \( F = \{z_i : i < \omega\} \). The set \( F \) satisfies the conditions required in Claim 1.2; so, \( F \) is a proper closed subset of \( \text{Seq}(u_1) \). Therefore, \( A = \text{Seq}(u_1) \setminus F \) is a non-empty open set which does not contain any of the sets \( M \in \mathcal{M} \).

**Claim 1.4.** Let \( \mathcal{O} = \mathcal{N} \setminus \mathcal{M} = \{N \in \mathcal{N} : \exists s \in \text{Seq}(|N \cap (s)| \geq \aleph_0)\} \). Then, there is an open set \( B \) of \( \text{Seq}(u_1) \) such that \( N \setminus B \neq \emptyset \) for all \( N \in \mathcal{O} \).

**Proof:** Let \( T = \{n < \omega : N_n \in \mathcal{O}\} \). The open set \( B \) will be an open cascade \( V(s, S) \) defined by \( (s, S) \) where \( s = \emptyset \) and the fan

\[
S = \{\{B(s)\}, \{B(s, i_1) : i_1 \in B(s)\}, \{B(s, i_1, i_2) : i_1 \in B(s), i_2 \in B(s, i_1)\}, \ldots, \{B(s, i_1, \ldots, i_{k+1}) : i_1 \in B(s), i_1 \in B(s, i_1), \ldots, i_{k+1} \in B(s, i_1, \ldots, i_k)\}, \ldots \}
\]

will be constructed by recursion.

Assume that we have already selected

\[
\{\{B(s)\}, \{B(s, i_1) : i_1 \in B(s)\}, \{B(s, i_1, i_2) : i_1 \in B(s), i_2 \in B(s, i_1)\}, \ldots, \{B(s, i_1, \ldots, i_k) : i_1 \in B(s), i_2 \in B(s, i_1), \ldots, i_k \in B(s, i_1, \ldots, i_{k-1})\}\}.
\]

For each sequence \( i_1 \in B(s), i_2 \in B(s, i_1), \ldots, i_{k+1} \in B(s, i_1, i_2, \ldots, i_k) \), consider the ultrafilter \( u_t \) where \( t = s \setminus i_1 \setminus \cdots \setminus i_k \), and consider the set \( P(s, i_1, \ldots, i_{k+1}) = \{n \in T : |N_n \cap (s, i_1, \ldots, i_k)| \geq \aleph_0\} \). If \( P(s, i_1, \ldots, i_{k+1}) \) is empty, we choose \( B(s, i_1, \ldots, i_{k+1}) \) to be an arbitrary element of \( u_t \). If \( P(s, i_1, \ldots, i_{k+1}) \) is not empty, there is \( B(s, i_1, \ldots, i_{k+1}) \in u_t \) such that \( N \setminus B(s, i_1, \ldots, i_{k+1}) \neq \emptyset \) for every \( n \in P(s, i_1, \ldots, i_{k+1}) \) because \( \pi_X(u_t) > \aleph_0 \).

We have already finished the description of the recursive process that define the fan \( S \). The set \( B = V(s, S) \) is the required open set.
We finished the proof of Claim 1 by saying that the open set $A \cap B$, where $A$ was defined in the proof of Claim 1.3 and $B$ in that of Claim 1.4, is not empty and does not contain any of the elements in $N$. □

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