Alfred Olufemi Bosede
Some common fixed point theorems in normed linear spaces


Persistent URL: http://dml.cz/dmlcz/140733

Terms of use:
© Palacký University Olomouc, Faculty of Science, 2010

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
Some Common Fixed Point Theorems in Normed Linear Spaces

Alfred Olufemi BOSEDE

Department of Mathematics, Lagos State University
Ojo, Nigeria
e-mail: aolubosede@yahoo.co.uk

(Received August 26, 2009)

Abstract

In this paper, we establish some generalizations to approximate common fixed points for selfmappings in a normed linear space using the modified Ishikawa iteration process with errors in the sense of Liu [10] and Rafiq [14]. We use a more general contractive condition than those of Rafiq [14] to establish our results. Our results, therefore, not only improve a multitude of common fixed point results in literature but also generalize some of the results of Berinde [3], Rhoades [15] and recent results of Rafiq [14].

Key words: Common fixed point, contractive condition, Mann and Ishikawa iterations.

2000 Mathematics Subject Classification: 47H10, 54H25

1 Introduction

Let $K$ be a nonempty closed convex subset of a normed linear space $E$ and $T: K \to K$ a selfmap. For arbitrary $x_0$ in $K$, we define Mann [11] iteration process $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = (1 - b_n)x_n + b_nTx_n, \quad n = 0, 1, 2, \ldots$$

Ishikawa [6] iteration process $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = (1 - b_n)x_n + b_nTy_n$$
$$y_n = (1 - b'_n)x_n + b'_nTx_n, \quad n = 0, 1, 2, \ldots$$

where $x_0 \in K$ is arbitrary, $\{b_n\}$ and $\{b'_n\}$ being sequences of real numbers in $[0,1]$. 
The concept of Ishikawa iteration process with errors was introduced by Liu [10] and is the sequence \( \{x_n\}_{n=0}^\infty \) defined by
\[
x_{n+1} = (1 - b_n)x_n + b_n Ty_n + u_n,
\]
\[
y_n = (1 - b_n)x_n + b_n Tx_n + v_n, \quad n = 0, 1, 2, \ldots
\]
where \( x_0 \in K \) is arbitrary, \( \{b_n\} \) and \( \{b'_n\} \) being sequences of real numbers in \([0,1]\) while \( \{u_n\} \) and \( \{v_n\} \) satisfy
\[
\sum_{n=0}^\infty \|u_n\| < \infty \quad \text{and} \quad \sum_{n=0}^\infty \|v_n\| < \infty
\]
respectively. We observe that (3) contains (1) and (2). We also observe that (3) contains the Mann iteration process with errors given by
\[
x_{n+1} = (1 - b_n)x_n + b_n Tx_n + u_n, \quad n = 0, 1, 2, \ldots
\]

Das and Debata [5] generalized the Ishikawa iteration processes from the case of one self mapping to the case of two self mappings \( S \) and \( T \) of \( K \) given by
\[
x_{n+1} = (1 - b_n)x_n + b_n Sy_n
\]
\[
y_n = (1 - b'_n)x_n + b'_n Tx_n, \quad n = 0, 1, 2, \ldots
\]

By using Iteration (5), Das and Debata [5] established the common fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space. Several other researchers such as Takahashi and Tamura [21] investigated iteration (5) in a strictly convex Banach space, for the case of two nonexpansive mappings under different assumptions and contractive conditions.

Later, Rafiq [14] studied the two-step iteration process with errors in the sense of Liu [10] by using the following sequence \( \{x_n\}_{n=0}^\infty \) defined by
\[
x_{n+1} = b_n Sy_n + (1 - b_n)x_n + u_n
\]
\[
y_n = b'_n Tx_n + (1 - b'_n)x_n + v_n, \quad n = 0, 1, 2, \ldots
\]
where \( x_0 \in K \) is arbitrary, \( \{u_n\} \) and \( \{v_n\} \) are two summable sequences in \( K \).

We observe that iteration (6) contains all the iteration processes (1)–(5) as special cases.

In 1972, Zamfirescu [23] proved the following result.

**Theorem 1** Let \((E, d)\) be a complete metric space and \(T: E \to E\) be a mapping for which there exist real numbers \(a, b\) and \(c\) satisfying \(0 \leq a < 1, 0 \leq b, c < 0.5\) such that, for each \(x, y \in E\), at least one of the following is true:
\[
(Z_1) \quad d(Tx, Ty) \leq ad(x, y);
\]
\[
(Z_2) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)];
\]
\[
(Z_3) \quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)].
\]
Then, \(T\) is a Picard mapping.
An operator $T$ satisfying the contractive conditions $(Z_1), (Z_2)$ and $(Z_3)$ in Theorem 1 above is called a Zamfirescu operator.

**Remark 1** The proof of this Theorem is contained in Berinde [2]. Indeed, if

$$
\delta = \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\},
$$

(7)

in Theorem 1, we obtain

$$
0 \leq \delta < 1.
$$

(8)

Then, for all $x, y \in E$, and by using $Z_2$, it was proved in Berinde [2] that

$$
d(Tx, Ty) \leq 2\delta d(x, Tx) + \delta d(x, y),
$$

(9)

and by using $Z_3$, we obtain

$$
d(Tx, Ty) \leq 2\delta d(x, Ty) + \delta d(x, y),
$$

(10)

where $0 \leq \delta < 1$ is as defined by (7).

**Remark 2** If $(E, \|\cdot\|)$ is a normed linear space, then (9) becomes

$$
\|Tx - Ty\| \leq 2\delta \|x - Tx\| + \delta \|x - y\|,
$$

(11)

for all $x, y \in E$ and where $0 \leq \delta < 1$ is as defined by (7).

In 2008, Rafiq [14] proved a convergence theorem and some corollaries to approximate common fixed points of quasi-contractive operators on a normed space by using iteration (6) and under the assumption that the two self mappings $S$ and $T$ satisfy the conditions of a Zamfirescu operator.

Our aim in this paper is to establish some common fixed point theorems by using a more general contractive condition than those of Rafiq [14]. We shall use iteration (6) and employ the following contractive definition: Let $K$ be a nonempty closed convex subset of a normed linear space $E$ and $T: K \to K$ a selfmap of $K$. There exist a constant $L \geq 0$ such that $\forall x, y \in K$, we have

$$
\|Tx - Ty\| \leq e^L \|x - Tx\| (2\delta \|x - Tx\| + \delta \|x - y\|),
$$

(12)

where $0 \leq \delta < 1$ is as defined by (7) and $e^x$ denotes the exponential function of $x \in K$.

**Remark 3** The contractive condition (12) is more general than those of Rafiq [14] and others in the following sense:

If $L = 0$ in the contractive condition (12), then we obtain

$$
\|Tx - Ty\| \leq 2\delta \|x - Tx\| + \delta \|x - y\|
$$

which is the Zamfirescu contraction condition used by Rafiq [14], where

$$
\delta = \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\}, \quad 0 \leq \delta < 1,
$$

while constants $a, b$ and $c$ are as defined in Theorem 1 above.
The following lemma contained in Liu [10] will be required in the sequel.

**Lemma 1** Let \( \{\rho_n\} \), \( \{s_n\} \), \( \{t_n\} \) and \( \{k_n\} \) be sequences of nonnegative numbers satisfying
\[
\rho_{n+1} \leq (1 - s_n)\rho_n + s_n t_n + k_n,
\]
for all \( n \geq 1 \). If
\[
\sum_{n=0}^{\infty} s_n = \infty, \quad \lim_{n \to \infty} t_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} k_n < \infty
\]
hold, then
\[
\lim_{n \to \infty} \rho_n = 0.
\]

**2 Main results**

**Theorem 2** Let \( K \) be a nonempty closed convex subset of a normed linear space \( E \). Suppose that \( S, T : K \to K \) are two selfmappings of \( K \) satisfying the contractive condition (12). Suppose also that \( \{x_n\}_{n=0}^{\infty} \) is a sequence defined iteratively by (6).

Let \( F_S \cap F_T \neq \emptyset \), where \( F_S \) and \( F_T \) are the sets of fixed points of \( S \) and \( T \) respectively.

If in iteration (6) we have,
\[
\sum_{n=0}^{\infty} b_n = \infty, \quad \sum_{n=0}^{\infty} \|u_n\| < \infty \quad \text{and} \quad \lim_{n \to \infty} \|v_n\| = 0,
\]
then \( \{x_n\}_{n=0}^{\infty} \) converges strongly to a common fixed point of \( S \) and \( T \).

**Proof** Since \( S \) and \( T \) satisfy the contractive definition (12), then for \( x, y \in K \), we have
\[
\|Sx - Sy\| \leq e^L\|x - Sx\| (2\delta \|x - Sx\| + \delta \|x - y\|)
\]
and
\[
\|Tx - Ty\| \leq e^L\|x - Tx\| (2\delta \|x - Tx\| + \delta \|x - y\|)
\]
where \( L \geq 0 \) and \( 0 \leq \delta < 1 \) is as defined by (7).

By assumption, \( F_S \cap F_T \neq \emptyset \). Let \( p \in F_S \cap F_T \).

Therefore, for arbitrary \( x_0 \in K \) and by using iteration process (6), we get
\[
x_{n+1} - p = (1 - b_n)x_n + b_n Sy_n + u_n - p
\]
\[
= (1 - b_n)x_n + b_n Sy_n - b_n p - (1 - b_n)p + u_n
\]
\[
= (1 - b_n)(x_n - p) + b_n(Sy_n - p) + u_n
\]
and hence,
\[
\|x_{n+1} - p\| = \|(1 - b_n)(x_n - p) + b_n(Sy_n - p) + u_n\|
\]
\[
\leq (1 - b_n)\|x_n - p\| + b_n \|Sy_n - p\| + \|u_n\|
\]
\[
= (1 - b_n)\|x_n - p\| + b_n \|Sy_n - Sp\| + \|u_n\|
\]
\[
= (1 - b_n)\|x_n - p\| + b_n \|Sp - Sy_n\| + \|u_n\|
\]
By using (13), we obtain
\[ \|x_{n+1} - p\| \leq (1 - b_n) \|x_n - p\| + b_n[ e^{L\|p - Sp\|} (2\delta \|p - Sp\| + \delta \|p - y_n\|) ] + \|u_n\| \]
\[ = (1 - b_n) \|x_n - p\| + b_n[ e^{L\|p - p\|} (2\delta \|p - p\| + \delta \|y_n - p\|) ] + \|u_n\| \]
\[ = (1 - b_n) \|x_n - p\| + b_n[ e^{L(0)} (2\delta(0) + \delta \|y_n - p\|) ] + \|u_n\| \]
\[ = (1 - b_n) \|x_n - p\| + b_n\delta \|y_n - p\| + \|u_n\| \]

Therefore,
\[ \|x_{n+1} - p\| \leq (1 - b_n) \|x_n - p\| + b_n\delta \|y_n - p\| + \|u_n\|. \tag{15} \]

Similarly, by using iteration process (6), we obtain
\[ \|y_n - p\| = \|(1 - b'_n)(x_n - p) + b'_n (Tx_n - p) + v_n\| \]
\[ \leq (1 - b'_n) \|x_n - p\| + b'_n \|Tx_n - p\| + \|v_n\| \]
\[ = (1 - b'_n) \|x_n - p\| + b'_n \|Tx_n - Tp\| + \|v_n\| \]
\[ = (1 - b'_n) \|x_n - p\| + b'_n \|Tp - Tx_n\| + \|v_n\| \]

By using (14), we get
\[ \|y_n - p\| \leq (1 - b'_n) \|x_n - p\| + b'_n[ e^{L\|p - Tp\|} (2\delta \|p - Tp\| + \delta \|p - x_n\|) ] + \|v_n\| \]
\[ = (1 - b'_n) \|x_n - p\| + b'_n[ e^{L\|p - p\|} (2\delta \|p - p\| + \delta \|x_n - p\|) ] + \|v_n\| \]
\[ = (1 - b'_n) \|x_n - p\| + b'_n[ e^{L(0)} (2\delta(0) + \delta \|x_n - p\|) ] + \|v_n\| \]
\[ = (1 - b'_n) \|x_n - p\| + b'_n\delta \|x_n - p\| + \|v_n\| \]

which implies that
\[ \|y_n - p\| \leq (1 - b'_n + b'_n\delta) \|x_n - p\| + \|v_n\|. \tag{16} \]

By observing that \(0 \leq b'_n \leq 1\), \(0 \leq \delta < 1\) and since \(0 \leq (1 - b'_n + b'_n\delta) < 1\), we obtain
\[ \|y_n - p\| \leq \|x_n - p\| + \|v_n\|. \tag{17} \]

Substitute (17) into (15) yields
\[ \|x_{n+1} - p\| \leq (1 - b_n) \|x_n - p\| + b_n\delta \|x_n - p\| + b_n\delta \|v_n\| + \|u_n\|. \]

and hence,
\[ \|x_{n+1} - p\| \leq (1 - b_n + b_n\delta) \|x_n - p\| + b_n\delta \|v_n\| + \|u_n\|. \tag{18} \]
By applying Lemma 1 and using the fact that
\[ 0 \leq b_n \leq 1, \quad 0 \leq \delta < 1, \quad 0 \leq (1 - b_n + b_n \delta) < 1, \]
\[ \sum_{n=0}^{\infty} b_n = \infty, \quad \sum_{n=0}^{\infty} \|u_n\| < \infty \quad \text{and} \quad \lim_{n \to \infty} \|v_n\| = 0, \]
we obtain
\[ \lim_{n \to \infty} \|x_{n+1} - p\| = 0 \]
which implies that \( \{x_n\}_{n=0}^{\infty} \) converges strongly to a common fixed point of \( S \) and \( T \).

This completes the proof. \( \square \)

**Remark 4** Our result in Theorem 2 is a generalization of Theorem 2.1 of Rafiq [14].

**Theorem 3** Let \( K \) be a nonempty closed convex subset of a normed linear space \( E \). Suppose that \( S: K \to K \) is a selfmap of \( K \) satisfying the contractive condition (12). Suppose also that \( \{x_n\}_{n=0}^{\infty} \) is a sequence defined iteratively by (4).

Let \( F_S \) be the set of fixed points of \( S \) such that \( F_S \neq \emptyset \). If in iteration (4) we have, \( \sum_{n=0}^{\infty} b_n = \infty \) and \( \sum_{n=0}^{\infty} \|u_n\| < \infty \), then \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the unique fixed point of \( S \).

**Proof** By assumption, \( F_S \neq \emptyset \). Let \( p \in F_S \). Therefore, for arbitrary \( x_0 \in K \) and by using iteration process (4), we get
\[
\begin{align*}
x_{n+1} - p &= (1 - b_n)x_n + b_n Sx_n + u_n - p \\
&= (1 - b_n)x_n + b_n Sx_n - b_n p - (1 - b_n)p + u_n \\
&= (1 - b_n)(x_n - p) + b_n (Sx_n - p) + u_n
\end{align*}
\]

and hence,
\[
\begin{align*}
\|x_{n+1} - p\| &= \|(1 - b_n)(x_n - p) + b_n (Sx_n - p) + u_n\| \\
&\leq (1 - b_n) \|x_n - p\| + b_n \|Sx_n - p\| + \|u_n\| \\
&= (1 - b_n) \|x_n - p\| + b_n \|Sx_n - Sp\| + \|u_n\| \\
&= (1 - b_n) \|x_n - p\| + b_n \|Sp - Sx_n\| + \|u_n\|
\end{align*}
\]

Since \( S \) satisfies the contractive condition (12), we get
\[
\begin{align*}
\|x_{n+1} - p\| &\leq (1 - b_n) \|x_n - p\| + b_n [e^{L \|p - Sp\|} (2\delta \|p - Sp\| + \delta \|p - x_n\|)] + \|u_n\| \\
&= (1 - b_n) \|x_n - p\| + b_n [e^{L \|p - p\|} (2\delta \|p - p\| + \delta \|x_n - p\|)] + \|u_n\| \\
&= (1 - b_n) \|x_n - p\| + b_n [e^{L(0)} (2\delta(0) + \delta \|x_n - p\|)] + \|u_n\| \\
&= (1 - b_n) \|x_n - p\| + b_n [e^{0} (0 + \delta \|x_n - p\|)] + \|u_n\| \\
&= (1 - b_n) \|x_n - p\| + b_n \delta \|x_n - p\| + \|u_n\|
\end{align*}
\]
and hence,
\[ \|x_{n+1} - p\| \leq (1 - b_n + b_n \delta) \|x_n - p\| + \|u_n\|. \]

By using Lemma 1 and the fact that
\[ 0 \leq b_n \leq 1, \quad 0 \leq \delta < 1, \quad 0 \leq (1 - b_n + b_n \delta) < 1, \]
\[ \sum_{n=0}^{\infty} b_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|u_n\| < \infty, \]
we obtain
\[ \lim_{n \to \infty} \|x_{n+1} - p\| = 0 \]
which implies that \( \{x_n\}_{n=0}^{\infty} \) converges strongly to a fixed point of \( S \).

To prove the uniqueness, we take \( p_1, p_2 \in F_S \) and assume that \( p_1 \neq p_2 \).
By using the contractive condition (12) and \( 0 \leq \delta < 1 \), we get
\[ \|p_1 - p_2\| = \|Sp_1 - Sp_2\| \]
\[ \leq e^{L\|p_1 - Sp_1\|}(2\delta \|p_1 - Sp_1\| + \delta \|p_1 - p_2\|) \]
\[ = e^{L\|p_1 - p_1\|}(2\delta \|p_1 - p_1\| + \delta \|p_1 - p_2\|) \]
\[ = e^{L(0)}(2\delta(0) + \delta \|p_1 - p_2\|) \]
\[ = e^{0}(0 + \delta \|p_1 - p_2\|) \]
\[ = \delta \|p_1 - p_2\| \]
\[ < \|p_1 - p_2\| \]
which is a contradiction. Hence, \( p_1 = p_2 \).

This completes the proof. \( \square \)

**Remark 5** The uniqueness result in Theorem 3 is a generalization of Corollary 2.2 of Rafiq [14].

**References**