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The Multicores in Metric Spaces and Their Application in Fixed Point Theory

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Abstract

This paper discusses the notion, the properties and the application of multicores, i.e. some compact sets contained in metric spaces.

Key words: Lefschetz number, fixed point, topological vector spaces, Klee admissible spaces, absolute neighborhood multi-retracts, approximative absolute neighborhood multi-retracts, multicore.

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1 Introduction

The compactness of a map is a fundamental and important assumption of the fixed point theory. Thus, we should be interested in the properties of compact sets in metric spaces and not only in the properties of the spaces themselves. In the paper [7] G. Fournier and A. Granas consider a topological space of $NES(compact)$ type and prove that this type of a space is a Lefschetz space, i.e. every compact map $f\colon X \to X$ is a Lefschetz map. From this proof it results that every compact set in this space, especially a set $\overline{f(X)}$, has a property thanks to which a map f is a Lefschetz map. Hence the idea of the introduction of the notion of multicores, i.e. certain compact sets in metric spaces. In the paper, we examine three types of multicores. It proves that every metric space that has at least one point of convergence contains all types of multicores. There is a metric space that is not of AANMR type but its every compact subset is one of the three types of multicores. Finally, it is worth to note that every admissible and compact multivalued map $\varphi\colon X \multimap X$ for which $\overline{\varphi(X)}$ is a respective multicore, is a Lefschetz map. Thanks to the Dugundji theorem about the extending of continuous maps, in the definition of AR, ANR and AANR_C (aproximative ANR in the sense of Clapp) we can use locally convex spaces instead of normed spaces (see [20]).

2 Preliminaries

Throughout this paper all topological spaces are assumed to be metric. We shall assume that all single-valued mappings considered in the paper are continuous. Let H_* be the Čech homology functor with compact carriers and coefficients in the field of rational numbers Q from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f\colon X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$ where $f_q\colon H_q(X) \to H_q(Y)$ (see [2] and [8]). A space X is acyclic if:

(i) X is non-empty,

(ii) $H_q(X) = 0$ for every $q \geq 1$ and

(iii) $H_0(X) \approx \mathbb{Q}$.

A continuous mapping $f\colon X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is non-empty and compact. A proper map $p\colon X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. Let X and Y be two spaces and assume that for every $x \in X$ a non-empty closed subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi\colon X \multimap Y$ is a multi-valued mapping. For a multi-valued mapping $\varphi\colon X \multimap Y$ and a subset $U \subset Y$, we let:

$$\varphi^{-1}(U) = \{x \in X;\ \varphi(x) \subset U\}.$$

If for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open, then φ is called an upper semi-continuous mapping; we shall write φ is u.s.c.

Proposition 2.1 (see [2, 8]) *Assume that $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap T$ are u.s.c. mappings with compact values and $p\colon Z \to X$ is a Vietoris mapping. Then:*

(2.1.1) *for any compact $A \subset X$, the image $\varphi(A) = \bigcup_{x\in A} \varphi(x)$ of the set A under φ is a compact set;*

(2.1.2) *the composition $\psi \circ \varphi\colon X \multimap T$, $(\psi \circ \varphi)(x) = \bigcup_{y\in\varphi(x)} \psi(y)$, is an u.s.c. mapping;*

(2.1.3) *the mapping $\varphi_p\colon X \multimap Z$, given by the formula $\varphi_p(x) = p^{-1}(x)$, is u.s.c.*

Let $\varphi\colon X \multimap Y$ be a multivalued map. A pair (p, q) of single-valued, continuous map of the form is called a selected pair of φ (written $(p, q) \subset \varphi$) if the following two conditions are satisfied:

(i) p is a Vietoris map,

(ii) $q(p^{-1}(x)) \subset \varphi(x)$ for any $x \in X$.

Definition 2.2 A multivalued mapping $\varphi\colon X \multimap Y$ is called admissible provided there exists a selected pair (p, q) of φ.

Remark 2.3 We can assume that an admissible multivalued mapping $\varphi\colon X \multimap Y$ is u.s.c. and for each $x \in X$ $\varphi(x)$ is compact, because in the fixed point theory it is sufficient to consider some multivalued admissible selector $\psi\colon X \multimap Y$, such that for every $x \in X$:

(i) $\psi(x) \subset \varphi(x)$,

(ii) $q(p^{-1}(x)) = \psi(x)$, where $(p, q) \subset \varphi$ the fixed pair of selectors of the mapping φ.

Theorem 2.4 (see [8]) *Let $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ be two admissible maps. Then the composition $\psi \circ \varphi\colon X \multimap Z$ is an admissible map.*

Lemma 2.5 (see [8]) *If $\varphi\colon X \multimap Y$ is an admissible map, $Y_0 \subset Y$ and $X_0 = \varphi^{-1}(Y_0)$, then the contraction $\varphi_0\colon X_0 \multimap Y_0$ of φ to the pair (X_0, Y_0) is an admissible map.*

Theorem 2.6 (see [2]) *If $p\colon X \to Y$ is a Vietoris map, then an induced mapping*

$$p_*\colon H_*(X) \to H_*(Y)$$

is a linear isomorphism.

Let $u\colon E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E\colon u^n(x) = 0 \text{ for some } n\}$, where u^n is the nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u}\colon \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u admissible provided $\dim \widetilde{E} < \infty$.

Let $u = \{u_q\}\colon E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a Leray endomorphism if

(i) all u_q are admissible,

(ii) almost all $\widetilde{E_q}$ are trivial.

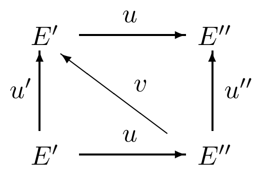
For such an u, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q tr(\widetilde{u_q}),$$

where $tr(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (comp. [2]). The following important property of a Leray endomorphism is a consequence of the well-known formula $tr(u \circ v) = tr(v \circ u)$ for the ordinary trace. An endomorphism $u\colon E \to E$ of a graded vector space E is called weakly nilpotent if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. Since, for a weakly nilpotent endomorphism $u\colon E \to E$, we have $N(u) = E$, we get:

Proposition 2.7 *If $u\colon E \to E$ is a weakly nilpotent endomorphism, then $\Lambda(u) = 0$.*

Proposition 2.8 *Assume that, in the category of graded vector spaces, the following diagram commutes*



If one of u', u'' is a Leray endomorphism, then so is the other; and $\Lambda(u') = \Lambda(u'')$.

Let $\varphi\colon X \multimap X$ be an admissible map. Let $(p,q) \subset \varphi$, where $p\colon Z \to X$ is a Vietoris mapping and $q\colon Z \to X$ a continuous map. Assume that $q_* \circ p_*^{-1}\colon H_*(X) \to H_*(X)$ is a Leray endomorphism for all pairs $(p,q) \subset \varphi$. For such a φ, we define the Lefschetz set $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) = \{\Lambda(q_* p_*^{-1});\ (p,q) \subset \varphi\}.$$

Let us observe that if X is an acyclic or, in particular, contractible space, then for every admissible map $\varphi\colon X \multimap X$ and for any pair $(p,q) \subset \varphi$ the endomorphism $q_* p_*^{-1}\colon H_*(X) \to H_*(X)$ is a Leray endomorphism and $\Lambda(q_* p_*^{-1}) = 1$.

Theorem 2.9 (see [8]) *If $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap T$ are admissible, then the composition $\psi \circ \varphi\colon X \multimap T$ is admissible and for every $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$ there exists a pair $(p,q) \subset \psi \circ \varphi$ such that $q_{2*} p_{2*}^{-1} \circ q_{1*} p_{1*}^{-1} = q_* p_*^{-1}$.*

Definition 2.10 An admissible map $\varphi\colon X \multimap X$ is called a Lefschetz map provided the generalized Lefschetz set $\Lambda(\varphi)$ of φ is well defined and $\Lambda(\varphi) \neq \{0\}$ implies that the set $\mathrm{Fix}(\varphi) = \{x \in X : x \in \varphi(x)\}$ is non-empty.

Theorem 2.11 (see [17]) *Let U be an open subset of a normed space E and let X be a compact subset U. Then for each sufficiently small $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a mapping $p_\varepsilon\colon X \to U$ such that:*

2.2.1 $\|x - p_\varepsilon(x)\| < \varepsilon$ *for all* $x \in X$,

2.2.2 $p_\varepsilon(X) \subset K_\varepsilon$,

2.2.3 p_ε *is homotopic to* i, *where* $i\colon X \to U$ *is an inclusion.*

Let Y be a metric space and let $\mathrm{Id}_Y\colon Y \to Y$ be a map given by formula $\mathrm{Id}_Y(y) = y$ for each $y \in Y$.

Definition 2.12 A map $r\colon X \to Y$ of a space X onto a space Y is said to be an r-map if there is a map $s\colon Y \to X$ such that $r \circ s = \mathrm{Id}_Y$.

Definition 2.13 A metric space X is called an absolute neighborhood retract (notation: $X \in \mathrm{ANR}$) provided there exists an open subset U of some normed space E and an r-map $r\colon U \to X$ from U onto X.

Definition 2.14 A metric space X is called an absolute retract (notation: $X \in$ AR) provided there exists a normed space E and an r-map $r\colon E \to X$ from E onto X.

Let $A \subset X$ be a nonempty set. We shall say that A is a retract of X if there exists a continuous map $r\colon X \to A$ such that for each $x \in A$ $r(x) = x$. A nonempty set $B \subset X$ is a neighborhood retract in X if there exists an open set $U \subset X$ such that $B \subset U$ and B is a retract of U.

Theorem 2.15 (see [8]) $X \in$ ANR *if and only if for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, the set $h(X)$ is a neighborhood retract in Y.*

Theorem 2.16 (see [8]) $X \in AR$ *if and only if for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, the set $h(X)$ is a retract in Y.*

Now we shall recall a generalization of the concept of absolute neighborhood retracts, which was introduced by Clapp.

Definition 2.17 We shall say that a compact metric space X is an approximative absolute neighborhood retract in the sense of Clapp (notation: $X \in \mathrm{AANR}_C$) provided for every $\varepsilon > 0$ there exists an open subset U of some normed linear space E and two maps $r_\varepsilon\colon U \to X$, $s_\varepsilon\colon X \to U$ such that $d(x, r_\varepsilon(s_\varepsilon(x))) < \varepsilon$ for any $x \in X$.

Theorem 2.18 (see [8]) $X \in \mathrm{AANR}_C$ *if and only if for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \supset h(X)$ of X and $r_\varepsilon\colon U_\varepsilon \to h(X)$ such that for each $y \in h(X)$ $d(r_\varepsilon(y), y) < \varepsilon$.*

Definition 2.19 Let E be a topological vector space. We shall say that E is a Klee admissible space provided for any compact subset $K \subset E$ and for any open neighborhood V of $0 \in E$ there exists a map $\pi_V\colon K \to E$ such that the following two conditions are satisfied:

(2.19.1) $\pi_V(x) \in (x + V)$, for any $x \in K$,

(2.19.2) there exists a natural number $n = n_K$ such that $\pi_V(K) \subset E^n$, where E^n is an n-dimensional subspace of E.

Definition 2.20 We shall say that a topological vector space E is locally convex provided that for each $x \in E$ and for each open set $U \subset E$ such that $x \in U$ there exists an open and convex set $V \subset E$ such that $x \in V \subset U$.

It is clear that if E is a normed space then E is locally convex.

Theorem 2.21 (see [2, 7]) *Let E be locally convex. Then E is a Klee admissible space.*

Theorem 2.22 (see [9]) *Let E be a Klee admissible space. For each compact subset $K \subset E$ and for any open set $U \subset E$ such that $K \subset U$ there exists a continuous map $\pi_K : K \to U$ such that the following conditions are satisfied:*

2.22.1 $\pi_K(K) \subset E^n$, *where E^n is an n-dimensional subspace of E,*

2.22.2 $\pi_K : K \to U$ *and $i : K \to U$ are homotopic, where $i : K \to U$ is an inclusion.*

The following theorem is obvious.

Theorem 2.23 *Let E_s be a locally convex space for every $s \in S$. Then the space $E = \prod_{s \in S} E_s$ is a locally convex space.*

Theorem 2.24 (see [9]) *Let U be an open subset in a Klee admissible space E and $\varphi : U \multimap U$ be an admissible and compact map, then φ is a Lefschetz map.*

Definition 2.25 A metric space X is of finite type provided that for almost every $q \in N$ $H_q(X) = \{0\}$ and for any $q \in N$ $\dim H_q(X) < \infty$.

Theorem 2.26 ([8]) *Let X be a compact metric space of finite type. Then there exists $\varepsilon > 0$ such that for every two maps $f, g : Y \to X$, where Y is a Hausdorff space, the condition $d(f(y), g(y)) < \varepsilon$ for each $y \in Y$ implies $f_* = g_*$.*

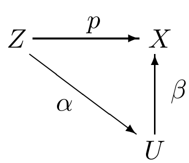
Definition 2.27 (see [19]) A map $r : X \to Y$ of a space X onto a space Y is said to be an mr-map if there is an admissible map $\varphi : Y \multimap X$ such that $r \circ \varphi = \mathrm{Id}_Y$.

In the definitions below instead of normed spaces (see [19]), we will use locally convex spaces (see [20]).

Definition 2.28 (see [19, 20]) A metric space X is called an absolute multi-retract (notation: $X \in$ AMR) provided there exists a locally convex space E and an mr-map $r : E \to X$ from E onto X.

Definition 2.29 (see [19, 20]) A metric space X is called an absolute neighborhood multi-retract (notation: $X \in$ ANMR) provided there exists an open subset U of some locally convex space E and an mr-map $r : U \to X$ from U onto X.

Theorem 2.30 (see [19, 20]) *A space X is an* ANMR *if and only if there exists a metric space Z and a Vietoris map $p : Z \to X$ which factors through an open subset U of some locally convex space E, i.e. there are two continuous maps α and β such that the following diagram*



is commutative.

Definition 2.31 (see [20]) Let X be a compact space. We shall say that X is an approximative ANMR (we write AANMR) provided that for any $\varepsilon > 0$ there exists a locally convex space E_ε and an open set $U_\varepsilon \subset E_\varepsilon$, a map $r_\varepsilon\colon U_\varepsilon \to X$ and an admissible map $\varphi_\varepsilon\colon X \multimap U_\varepsilon$ such that for any $x \in X$

$$r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon),$$

where $B(x, \varepsilon)$ is an open ball in X of a center in x and of a radius $\varepsilon > 0$.

Theorem 2.32 (see [20]) *A space X is an* AANMR *if and only if for any $\varepsilon > 0$ there exists a space Z_ε, a Vietoris map $p_\varepsilon\colon Z_\varepsilon \to X$, a locally convex space E_ε, an open set $U_\varepsilon \subset E_\varepsilon$, and maps $r_\varepsilon\colon U_\varepsilon \to X$, $q_\varepsilon\colon Z_\varepsilon \to U_\varepsilon$ such that for any $z \in Z_\varepsilon$*

$$d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon.$$

Theorem 2.33 (see [8]) *Let X and Y be acyclic and compact spaces. Then $X \times Y$ is a compact and acyclic space.*

3 The multicores in metric spaces

We shall present the definition of a multicore in a metric space.

Definition 3.1 We shall say that a compact set $K \subset X$ is an absolute multicore (we write $K \in AMC(X)$) provided that there exists a metric space Z, a locally convex space E and maps $r\colon E \to X$, $q\colon Z \to E$ such that the following conditions are satisfied:

(3.1.1) for any $z \in Z$, $r(q(z)) \in K$,

(3.1.2) a map $p\colon Z \to K$ given by $p(z) = r(q(z))$ for any $z \in Z$ is Vietoris.

Definition 3.2 We shall say that a compact set $K \subset X$ is an absolute neighborhood multicore (we write $K \in \text{ANMC}(X)$) provided that there exists a metric space Z, an open subset U of some locally convex space E and maps $r\colon U \to X$, $q\colon Z \to U$ such that the following conditions are satisfied:

(3.2.1) for any $z \in Z$, $r(q(z)) \in K$,

(3.2.2) a map $p\colon Z \to K$ given by $p(z) = r(q(z))$ for any $z \in Z$ is Vietoris.

Definition 3.3 We shall say that a compact set $K \subset X$ is an approximative absolute neighborhood multicore (we write $K \in \text{AANMC}(X)$) provided that for any $\varepsilon > 0$ there exists a metric space Z_ε, a locally convex space E_ε, an open set $U_\varepsilon \subset E_\varepsilon$, a Vietoris map $p_\varepsilon\colon Z_\varepsilon \to K$ and maps $r_\varepsilon\colon U_\varepsilon \to X$, $q_\varepsilon\colon Z_\varepsilon \to U_\varepsilon$ such that the following conditions are satisfied:

(3.3.1) for any $z \in Z_\varepsilon$, $r_\varepsilon(q_\varepsilon(z)) \in K$,

(3.3.2) for any $z \in Z_\varepsilon$, $d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon$.

We observe that for any metric space X we have

$$\emptyset \neq \mathrm{AMC}(X) \subset \mathrm{ANMC}(X) \subset \mathrm{AANMC}(X),$$

since $\{x\} \in \mathrm{AMC}(X)$ for each $x \in X$ (see 3.5). The following theorem consists of some properties of multicores. Let $C_1(X) \equiv AMC(X)$, $C_2(X) \equiv \mathrm{ANMC}(X)$ and $C_3(X) \equiv \mathrm{AANMC}(X)$. We will denote the set

$$\{K_1 \times K_2 \colon K_1 \in C_i(X_1) \text{ and } K_2 \in C_i(X_2)\},$$

with $C_i(X_1, X_2)$ whereas the set

$$\{K_1 \times K_2 \times \ldots \times K_n \times \ldots \colon K_j \in C_i(X_j),\ j = 1, 2, \ldots, n, \ldots\}$$

will be denoted by $C_i(X_1, X_2, \ldots, X_n, \ldots)$, $i = 1, 2, 3$.

Let $K(X) = \{K \subset X \colon K \text{ is a nonempty and compact set}\}$.

Theorem 3.4

3.4.1 *$C_1(X) \subset C_2(X) \subset C_3(X)$.*

3.4.2 *Let $X \subset Y$. Then $C_i(X) \subset C_i(Y)$, $i = 1, 2, 3$.*

3.4.3 *$C_i(X_1, X_2) \subset C_i(X_1 \times X_2)$, $i = 1, 2, 3$.*

3.4.4 *$C_3(X_1, X_2, \ldots, X_n, \ldots) \subset C_3(\prod_{n=1}^{\infty} X_n)$.*

3.4.5 *Let $K \in C_i(X)$. Then for each compact set $A \subset K$, $A \in C_i(X)$, $i = 1, 2$.*

3.4.6 *Let $K_1, K_2 \in C_i(X)$ and $K_1 \cap K_2 = \emptyset$. Then $(K_1 \cup K_2) \in C_i(X)$, $i = 2, 3$.*

3.4.7 *Let $V \subset X$ be an open set and let $K \subset V$ be a compact set. Then*

$$(K \in C_i(V)) \Leftrightarrow (K \in C_i(X)), \quad i = 2, 3.$$

3.4.8 *Let $X = \bigcup_{n=1}^{\infty} X_n$, where X_n is open of X and $X_n \subset X_{n+1}$ for any n. Then*

$$C_i(X) = \bigcup_{n=1}^{\infty} C_i(X_n), \quad i = 2, 3.$$

3.4.9 *Let $p \colon X \to Y$ be a Vietoris map. Then for each compact set $K \subset Y$*

$$(p^{-1}(K) \in C_i(X)) \Rightarrow (K \in C_i(Y)), \quad i = 1, 2, 3.$$

3.4.10 $(X \in \mathrm{AMR}) \Rightarrow (C_1(X) = K(X))$,
$(X \in \mathrm{ANMR}) \Rightarrow (C_2(X) = K(X))$,
$(X \in \mathrm{AANMR}) \Leftrightarrow (X \in C_3(X))$.

If the metric space X is compact, we can substitute the above implications with equivalence.

3.4.11 *Let $K \subset X$ be a compact set. Then*
$(K \in \mathrm{AMR}) \Rightarrow (K \in C_1(X))$,
$(K \in \mathrm{ANMR}) \Rightarrow (K \in C_2(X))$,
$(K \in \mathrm{AANMR}) \Rightarrow (K \in C_3(X))$.

Proof The properties 3.4.1 and 3.4.2 are obvious.

We will show the property 3.4.3 for $i = 3$. Let $K_1 \in C_3(X_1)$ and $K_2 \in C_3(X_2)$. Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{2}$. Then there exist locally convex spaces E^1_δ, E^2_δ, metric spaces Z^1_δ, Z^2_δ open sets $U^1_\delta \subset E^1_\delta$, $U^2_\delta \subset E^2_\delta$, maps $r^1_\delta \colon U^1_\delta \to X_1$, $r^2_\delta \colon U^2_\delta \to X_2$, $q^1_\delta \colon Z^1_\delta \to U^1_\delta$, $q^2_\delta \colon Z^2_\delta \to U^2_\delta$ and Vietoris maps $p^1_\delta \colon Z^1_\delta \to K_1$, $p^2_\delta \colon Z^2_\delta \to K_2$ such that $r^1_\delta(q^1_\delta(z)) \in K_1$, $r^2_\delta(q^2_\delta(z)) \in K_2$ and

$$d(r^1_\delta(q^1_\delta(z)), p^1_\delta(z)) < \delta \quad \text{for each } z \in Z^1_\delta$$

and

$$d(r^2_\delta(q^2_\delta(z)), p^2_\delta(z)) < \delta \quad \text{for each } z \in Z^2_\delta.$$

Let $X = X_1 \times X_2$ and let $K = K_1 \times K_2$. Then the metric d in X given by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

where d_1 and d_2 are metrics in X_1 and X_2 respectively. We define

$$E_\varepsilon = E^1_\delta \times E^2_\delta, \quad Z_\varepsilon = Z^1_\delta \times Z^2_\delta, \quad U_\varepsilon = U^1_\delta \times U^2_\delta,$$

$r_\varepsilon \colon U_\varepsilon \to X$ given by $r_\varepsilon(u_1, u_2) = (r^1_\delta(u_1), r^2_\delta(u_2))$ for each $(u_1, u_2) \in U^1_\delta \times U^2_\delta$,

$q_\varepsilon \colon Z_\varepsilon \to U_\varepsilon$ given by $q_\varepsilon(z_1, z_2) = (q^1_\delta(z_1), q^2_\delta(z_2))$ for each $(z_1, z_2) \in Z^1_\delta \times Z^2_\delta$,

$p_\varepsilon \colon Z_\varepsilon \to K$ given by $p_\varepsilon(z_1, z_2) = (p^1_\delta(z_1), p^2_\delta(z_2))$ for each $(z_1, z_2) \in Z^1_\delta \times Z^2_\delta$.

From 2.33 the map p_ε is Vietoris. It is clear that maps r_ε, q_ε and p_ε satisfy the definition 3.3. Hence $K \in C_3(X_1 \times X_2)$. For $i = 1, 2$ the proof is analogous.

3.4.4 Let (X_n, d_n) be a metric space for each $n \in \mathbb{N}$ and let $K = \prod_{n=1}^\infty K_n$, where $K_n \in C_3(X_n)$ for all n. Assume that for any n and for all $x_n, y_n \in X_n$ $d_n(x_n, y_n) \leq 1$. We define the metric in a space X given by:

$$d(x, y) = \sum_{n=1}^\infty \frac{d_n(x_n, y_n)}{2^n},$$

where $x = (x_1, x_2, \ldots, x_n, \ldots)$, $y = (y_1, y_2, \ldots, y_n, \ldots)$. Let $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2}$. From the definition 3.3 for any n we get $r^n_\delta \colon U^n_\delta \to X_n$, $q^n_\delta \colon Z^n_\delta \to U^n_\delta$ and a Vietoris map $p^n_\delta \colon Z^n_\delta \to K_n$ such that for all $z_n \in Z^n_\delta$

$$d_n(r^n_\delta(q^n_\delta(z_n)), p^n_\delta(z_n)) < \delta,$$

where $U^n_\delta \subset E^n_\delta$ is an open subset in some locally convex space.

Let $E_\varepsilon = \prod_{n=1}^\infty E^n_\delta$ (from 2.23 E_ε is a locally convex space) and let $Z_\varepsilon = \prod_{n=1}^\infty Z^n_\delta$. We observe that the space Z_ε is compact. There exists a natural number n_0 such that for any $n \geq n_0$

$$\sum_{n=n_0+1}^\infty \frac{d_n(x_n, y_n)}{2^n} < \delta = \frac{\varepsilon}{2}.$$

We define an open set in the space E_ε given by:

$$U_\varepsilon = \prod_{i=1}^{n_0} U_\delta^i \times \prod_{n=n_0+1}^{\infty} E_\delta^n.$$

Let $r_\varepsilon \colon U_\varepsilon \to X$ be given by:

$$r_\varepsilon(x_1, x_2, \dots, x_n, \dots) = (r_\delta^1(x_1), r_\delta^2(x_2), \dots, r_\delta^{n_0}(x_{n_0}), y_{n_0+1}, \dots, y_m, \dots)$$

for each $(z_1, z_2, \dots, z_n, \dots) \in Z_\varepsilon$, where $y_m \in X_m$ for all $m > n_0$ are stationary points and let $q_\varepsilon \colon Z_\varepsilon \to U_\varepsilon$ be given by:

$$q_\varepsilon(z_1, z_2, \dots, z_n, \dots) = (q_\delta^1(z_1), q_\delta^2(z_2), \dots, q_\delta^n(z_n), \dots)$$

for each $(z_1, z_2, \dots, z_n, \dots) \in Z_\varepsilon$. A Čech homology theory is continuous, therefore a map $p_\varepsilon \colon Z_\varepsilon \to X$ given by

$$p_\varepsilon(z_1, z_2, \dots, z_n, \dots) = (p_\delta^1(z_1), p_\delta^2(z_2), \dots, p_\delta^n(z_n), \dots)$$

for each $(z_1, z_2, \dots, z_n, \dots) \in Z_\varepsilon$ is a Vietoris map (see [20]) since

$$p_\varepsilon^{-1}(x_1, x_2, \dots, x_n, \dots) = \prod_{n=1}^{\infty} (p_\delta^n)^{-1}(x_n)$$

for any $x = (x_1, x_2, \dots, x_n, \dots) \in X$. It is clear that maps r_ε, q_ε and p_ε satisfy the definition 3.3.

3.4.5 Let $K \in C_2(X)$ and let $A \subset K$ be a compact set. From definition 3.2 there exists a locally convex space E', a metric space Z', an open set $U' \subset E'$, maps $r' \colon U' \to X$, $q' \colon Z' \to U'$ such that for all $z \in Z'$ $r'(q'(z)) \in K$ and the map $p' \colon Z' \to K$ given by $p'(z) = r'(q'(z))$ for each $z \in Z'$ is Vietoris. Let $E = E'$, $Z = p'^{-1}(A)$, $U = U'$, $r = r'$ and $q = q'_{/Z}$. It is clear that maps r and q satisfy the definition 3.2. For $i = 1$ the proof is analogous.

3.4.6 We prove the property for $i = 3$. The proof for $i = 2$ is analogous. From the assumption, for each $\varepsilon > 0$ there exists locally convex spaces $E_\varepsilon^1, E_\varepsilon^2$, metric spaces $Z_\varepsilon^1, Z_\varepsilon^2$, open sets $U_\varepsilon^1 \subset E_\varepsilon^1$, $U_\varepsilon^2 \subset E_\varepsilon^2$, maps $r_\varepsilon^1 \colon U_\varepsilon^1 \to X$, $r_\varepsilon^2 \colon U_\varepsilon^2 \to X$, $q_\varepsilon^1 \colon Z_\varepsilon^1 \to U_\varepsilon^1$, $q_\varepsilon^2 \colon Z_\varepsilon^2 \to U_\varepsilon^2$ and Vietoris maps $p_\varepsilon^1 \colon Z_\varepsilon^1 \to K_1$, $p_\varepsilon^2 \colon Z_\varepsilon^2 \to K_2$ such that for each $z \in Z_\varepsilon^1$ $r_\varepsilon^1(q_\varepsilon^1(z)) \in K_1$ and $d(r_\varepsilon^1(q_\varepsilon^1(z)), p_\varepsilon^1(z)) < \varepsilon$ and for each $z \in Z_\varepsilon^2$ $r_\varepsilon^2(q_\varepsilon^2(z)) \in K_2$ and $d(r_\varepsilon^2(q_\varepsilon^2(z)), p_\varepsilon^2(z)) < \varepsilon$.

Let $E_\varepsilon = E_\varepsilon^1 \times E_\varepsilon^2$, $U_\varepsilon = (U_\varepsilon^1 \times V_2) \cup (V_1 \times U_\varepsilon^2) \subset E_\varepsilon$, where $V_1 \subset E_\varepsilon^1$, $V_2 \subset E_\varepsilon^2$ are open sets such that $V_1 \cap U_\varepsilon^1 = \emptyset$ and $V_2 \cap U_\varepsilon^2 = \emptyset$ and let

$$Z_\varepsilon = (Z_\varepsilon^1 \times \{s_2\}) \cup (\{s_1\} \times Z_\varepsilon^2) \subset Z_\varepsilon^1 \times Z_\varepsilon^2,$$

where $(s_1, s_2) \in Z_\varepsilon^1 \times Z_\varepsilon^2$ such that $s_1 \neq s_2$. We observe that

$$(U_\varepsilon^1 \times V_2) \cap (V_1 \times U_\varepsilon^2) = \emptyset \quad \text{and} \quad (Z_\varepsilon^1 \times \{s_2\}) \cap (\{s_1\} \times Z_\varepsilon^2) = \emptyset.$$

We define:

$$r_\varepsilon\colon U_\varepsilon \to X,\ \text{given by } r_\varepsilon(x,y) = \begin{cases} r^1_\varepsilon(x), & \text{for } (x,y) \in U^1_\varepsilon \times V_2 \\ r^2_\varepsilon(y), & \text{for } (x,y) \in V_1 \times U^2_\varepsilon, \end{cases}$$

$$q_\varepsilon\colon Z_\varepsilon \to U_\varepsilon \text{ given by } q_\varepsilon(z,t) = \begin{cases} (q^1_\varepsilon(z), v_2) & \text{for } (z,t) \in Z^1_\varepsilon \times \{s_2\} \\ (v_1, q^2_\varepsilon(t)), & \text{for } (z,t) \in \{s_1\} \times Z^2_\varepsilon, \end{cases}$$

$$p_\varepsilon\colon Z_\varepsilon \to K_1 \cup K_2 \text{ given by } p_\varepsilon(z,t) = \begin{cases} p^1_\varepsilon(z), & \text{for } (z,t) \in Z^1_\varepsilon \times \{s_2\} \\ p^2_\varepsilon(t), & \text{for } (z,t) \in \{s_1\} \times Z^2_\varepsilon, \end{cases}$$

where $(v_1, v_2) \in V_1 \times V_2$ is a stationary point. It is clear that maps $r_\varepsilon, q_\varepsilon, p_\varepsilon$ satisfy the definition 3.3.

3.4.7 Let $V \subset X$ be an open set and let $K \subset V$ be a compact set. It is clear that if $(K \in C_i(V)) \Rightarrow (K \in C_i(X))$, $i = 2, 3$. Assume that $K \in C_3(X)$. Then for each $\varepsilon > 0$ there exists a locally convex space E'_ε, an open set $U'_\varepsilon \subset E'_\varepsilon$, a metric space Z'_ε, maps $r'_\varepsilon\colon U'_\varepsilon \to X$, $q'_\varepsilon\colon Z'_\varepsilon \to U'_\varepsilon$ and a Vietoris map $p'_\varepsilon\colon Z'_\varepsilon \to K$ such that $r'_\varepsilon(q'_\varepsilon(z)) \in K$ for all $z \in Z'_\varepsilon$ and

$$d(r'_\varepsilon(q'_\varepsilon(z)), p'_\varepsilon(z)) < \varepsilon$$

for each $z \in Z'_\varepsilon$. Let $E_\varepsilon = E'_\varepsilon$, $U_\varepsilon = r'^{-1}_\varepsilon(V)$, $Z_\varepsilon = Z'_\varepsilon$, $r_\varepsilon = (r'_\varepsilon)_{/U_\varepsilon}$, $q_\varepsilon = q'_\varepsilon$ and $p_\varepsilon = p'_\varepsilon$. We observe that maps r_ε, q_ε and p_ε satisfy the definition 3.3 for $X = V$. Hence $K \in C_3(V)$ and the proof is complete. For $i = 2$ the proof is analogous.

3.4.8 Let $K \in C_i(X)$, $i = 2, 3$. Then there exists n such that $K \subset X_n$. From 3.4.7 we get that $K \in C_i(X_n)$, $i = 2, 3$. Hence $K \in \bigcup_{n=1}^\infty C_i(X_n)$ and $C_i(X) \subset \bigcup_{n=1}^\infty C_i(X_n)$, $i = 2, 3$. We observe from 3.4.2 that for any n $C_i(X_n) \subset C_i(X)$, $i = 2, 3$, so $\bigcup_{n=1}^\infty C_i(X_n) \subset C_i(X)$, $i = 2, 3$.

3.4.9 Let $p\colon X \to Y$ be a Vietoris map and let $K \subset Y$ be a compact set. Assume that $p^{-1}(K) \in C_3(X)$. Let $\varepsilon > 0$. The map $p\colon p^{-1}(K) \to K$ is uniformly continuous, so there exists $\delta > 0$ such that

$$(d(z_1, z_2) < \delta) \Rightarrow (d(p(z_1), p(z_2)) < \varepsilon), \quad \text{for each } z_1, z_2 \in p^{-1}(K). \tag{3.1}$$

From assumption there exists a locally convex space E'_δ, an open set $U'_\delta \subset E'_\delta$, a metric space Z'_δ, maps $r'_\delta\colon U'_\delta \to X$, $q'_\delta\colon Z'_\delta \to U'_\delta$, a Vietoris map $p'_\delta\colon Z'_\delta \to p^{-1}(K)$ such that $r'_\delta(q'_\delta(z)) \in p^{-1}(K)$ for each $z \in Z'_\delta$ and $d(r'_\delta(q'_\delta(z)), p'_\delta(z)) < \delta$ for each $z \in Z'_\delta$. We define

$$E_\varepsilon = E'_\delta, \quad U_\varepsilon = U'_\delta, \quad Z_\varepsilon = Z'_\delta, \quad r_\varepsilon = p \circ r'_\delta, \ q_\varepsilon = q'_\delta, \quad p_\varepsilon = p \circ p'_\delta.$$

It is clear that p_ε is a Vietoris map (see [8]) and $r_\varepsilon(q_\varepsilon(z)) \in K$ for all $z \in Z_\varepsilon$. From (3.1) we get

$$d(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) = d(p(r'_\delta(q'_\delta(z))), p(p'_\delta(z))) < \varepsilon.$$

For $i = 1, 2$ the proof is analogous.

3.4.10 The third implication is obvious. We shall present the middle implication. It is obvious that $C_2(X) \subset K(X)$. From assumption and 2.30 there exists a locally convex space E', an open set $U' \subset E'$, a metric space Z' and maps $r' \colon U' \to X$, $q' \colon Z' \to U'$ such that $r' \circ q' \colon Z' \to X$ is a Vietoris map. Let $p' = r' \circ q'$ and let $K \in K(X)$. We define

$$E = E', \quad U = U', \quad r = r' \colon U \to X, \quad Z = p'^{-1}(K), \quad q = q'_{/Z} \colon Z \to U.$$

We observe that $r(q(z)) \in K$ for all $z \in Z$ and the map $s \colon Z \to K$ given by $s(z) = r(q(z))$ for each $z \in Z$ is Vietoris. For $i = 1$ the proof is analogous.

3.4.11 It is obvious. □

Now we shall present a few examples of multicores. Let X be a metric space. By $B(x_0, r)$ we denote an open ball, whereas by $K(x_0, r)$ a closed ball in X of a center in $x_0 \in X$ and of a radius $r > 0$. Let $\mathbb{R}$ denote the set of real numbers.

Example 3.5 Let $K = \{x\}$, where $x \in X$. Then $K \in AMC(X)$.

Justification: We define:

$E = \mathbb{R},\ r \colon E \to X$ given by $r(t) = x$ for each $t \in \mathbb{R}$,

$Z = K(0,1) \subset \mathbb{R},$ where $K(0,1)$ is a closed ball in $\mathbb{R}$,

$q \colon Z \to E$ given by $q(z) = z$ for each $z \in Z$,

$p \colon Z \to K$ given by $p(z) = x$ for each $z \in Z$.

Example 3.6 Let $K = \{x_1, x_2, \ldots, x_n\} \subset X$ be a finite set. Then $K \in$ ANMC(X).

Justification: We define:

$E = \mathbb{R}$, $U \subset E$,

$U = \bigcup_{i=1}^{n} B\left(i, \frac{1}{3}\right),$ where $B\left(i, \frac{1}{3}\right)$ is an open ball in $\mathbb{R},\ i = 1, 2, \ldots, n,$

$r \colon U \to X,$ given by $r(u) = x_i,$ for all $u \in B\left(i, \frac{1}{3}\right),\ i = 1, 2, \ldots, n,$

$Z = \bigcup_{i=1}^{n} K\left(i, \frac{1}{4}\right),$ where $K\left(i, \frac{1}{4}\right)$ is a closed ball in $\mathbb{R}$,

$q \colon Z \to U$ given by $q(z) = z$ for each $z \in Z$,

$p \colon Z \to K,$ given by $p(z) = x_i,$ for each $z \in K\left(i, \frac{1}{4}\right),\ i = 1, 2, \ldots, n.$

Example 3.7 Let $K = (\{x_n\}_{n=1}^{\infty} \cup \{x_0\}) \subset X$ where a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = x_0$. Then $K \in$ AANMC(X).

Justification: Let $\varepsilon > 0$. Then there exists n_0 such that for any $n > n_0$ $d(x_n, x_0) < \varepsilon$.

We define:

$$U_\varepsilon = \bigcup_{i=1}^{n_0} B\left(\frac{1}{i}, \frac{1}{2i(i+1)}\right) \cup \left(-1, \frac{2n_0+1}{2n_0(n_0+1)}\right), \text{ where } B\left(\frac{1}{i}, \frac{1}{2i(i+1)}\right)$$

is an open ball in $\mathbb{R}$ and $\left(-1, \frac{2n_0+1}{2n_0(n_0+1)}\right)$ is an open interval in $\mathbb{R}$,

$r_\varepsilon \colon U_\varepsilon \to X$ given by $r_\varepsilon(u) = x_i$ for $u \in B\left(\frac{1}{i}, \frac{1}{2i(i+1)}\right)$, $i = 1, 2, \ldots, n_0$

$r_\varepsilon(u) = x_0$ for $u \in \left(-1, \frac{2n_0+1}{2n_0(n_0+1)}\right)$,

$$Z_\varepsilon = \bigcup_{n=1}^{\infty} K\left(\frac{1}{n}, \frac{1}{3n(n+1)}\right) \cup \{0\}, \text{ where } K\left(\frac{1}{n}, \frac{1}{3n(n+1)}\right)$$

is a closed ball in $\mathbb{R}$,

$q_\varepsilon \colon Z_\varepsilon \to U_\varepsilon$ given by $q(z) = z$ for each $z \in Z_\varepsilon$,

$p_\varepsilon \colon Z_\varepsilon \to K$ given by $p_\varepsilon(z) = x_n$ for each $z \in K\left(\frac{1}{n}, \frac{1}{3n(n+1)}\right)$, $n = 1, 2, \ldots$

and $p_\varepsilon(0) = x_0$.

We observe that if $X \in \mathrm{AMR}$ then

$$\mathrm{AMC}(X) = \mathrm{ANMC}(X) = \mathrm{AANMC}(X) \quad \text{(see 3.4.1, 3.4.10).}$$

however, if X is compact, $X \in \mathrm{ANMR}$ and $X \notin \mathrm{AMR}$ (see [19]) then

$$\mathrm{AMC}(X) \subset \mathrm{ANMC}(X) = \mathrm{AANMC}(X) \quad \text{(see 3.4.1, 3.4.10)}$$

and if $X = ((Y \times \{z_0\}) \cup (\{y_0\} \times Z)) \subset Y \times Z$, where $(y_0, z_0) \in Y \times Z$, $y_0 \neq z_0$, $Y \in \mathrm{AANMR}$, $Y \notin \mathrm{ANMR}$ (see [20]) and Z is compact, $Z \in \mathrm{ANMR}$, $Z \notin \mathrm{AMR}$ (see [19]) then

$$\mathrm{AMC}(X) \subset \mathrm{ANMC}(X) \subset \mathrm{AANMC}(X) \quad \text{(see 3.4.1, 3.4.10, 3.9).}$$

The above inclusions cannot be substituted with an equality. Now an important theorem shall be presented. First, however, we shall prove the following lemma.

Lemma 3.8 *Let E be a locally convex space and let U be an open set in E. Assume that a map $q \colon Z \to U$ induced a monomorphism $q_* \colon H_*(Z) \to H_*(U)$ where Z is a compact metric space. Then Z is of finite type.*

Proof Theorem 2.22 implies that for a compact set $K = q(Z) \subset U \subset E$ there exists a map $\pi_K \colon K \to U$ such that $\pi_K(K) \subset E^n$ and maps $\pi_K, i \colon K \to U$ are homotopic, where $E^n \subset E$ is an n-dimensional subspace of E and $i \colon K \to U$ is an inclusion. Let $i_1 \colon U \cap E^n \to U$ be an inclusion, $\widehat{q} \colon Z \to K$ given by $\widehat{q}(z) = q(z)$ for each $z \in Z$ and

$$t \colon Z \to U \cap E^n, \text{ given by } t(z) = \pi_K(\widehat{q}(z)) \text{ for each } z \in Z$$

then we have the following commutative diagram:

$$\begin{array}{ccc} H_*(Z) & \xrightarrow{q_*} & H_*(U) \\ & \searrow^{t_*} & \uparrow{\scriptstyle i_{1*}} \\ & & H_*(U \cap E^n). \end{array}$$

In the above diagram we get that $i_{1*} \circ t_* = q_*$ and, hence, t_* is a monomorphism. From the Schauder theorem, for a compact set $K_1 = \pi_K(K) \subset U \cap E^n = V$ and for sufficiently small $\varepsilon > 0$ there exists a projection $p_\varepsilon \colon K_1 \to V$ such that $p_\varepsilon(K_1) \subset K_\varepsilon$ and maps $p_\varepsilon, i_2 \colon K_1 \to V$ are homotopic, where $i_2 \colon K_1 \to V$ is an inclusion and K_ε is a polyhedron of finite type such that $K_\varepsilon \subset V$. We have the following commutative diagram:

$$\begin{array}{ccc} H_*(Z) & \xrightarrow{t_*} & H_*(V) \\ & \searrow^{r_*} & \uparrow{\scriptstyle i_{3*}} \\ & & H_*(K_\varepsilon), \end{array}$$

where $i_3 \colon K_\varepsilon \to V$ is an inclusion and $r \colon Z \to K_1$ given by $r(z) = p_\varepsilon(t(z))$ for each $z \in Z$. It is clear, that r_* is a monomorphism. Hence Z is a space of finite type. □

Theorem 3.9 *Let $K \in K(X)$ and let $i \colon K \to X$ be an inclusion such that $i_* \colon H_*(K) \to H_*(X)$ is a monomorphism. Then*

3.9.1 $(K \in \mathrm{AMC}(X)) \Rightarrow$*($K$ is acyclic),*

3.9.2 $(K \in \mathrm{ANMC}(X)) \Rightarrow$ *(K is of finite type).*

Proof 3.9.1 We have the following commutative diagram:

$$\begin{array}{ccc} H_*(Z) & \xrightarrow{r_* \circ q_*} & H_*(X) \\ & \searrow^{p_*} & \uparrow{\scriptstyle i_*} \\ & & H_*(K), \end{array}$$

where $p_* = (r \circ q)_*$. Hence a map $q \colon Z \to E$ induced a monomorphism $q_* \colon H_*(Z) \to H_*(E)$. Since $H_*(Z) \approx H_*(K)$, therefore the set K is acyclic.

3.9.2 Acting analogously as in 3.9.1 we get a monomorphism $q_* \colon H_*(Z) \to H_*(U)$. From 3.8 we get that Z is of finite type. Since $H_*(Z) \approx H_*(K)$, therefore the set K is of finite type. □

We recall that a metric space $X \in \mathrm{NES}$(compact metric) if for any compact metric space Y, for any closed subset $A \subset Y$ and for each continuous map $f \colon A \to X$ there exists an open set $U \subset Y$ and a continuous map $F \colon U \to X$ such that $A \subset U$ and for each $y \in A$ $F(y) = f(y)$.

Theorem 3.10 *Let X be a metric space and $X \in NES(compact\ metric)$. Then* $\mathrm{ANMC}(X) = K(X)$.

Proof Let $K \in K(X)$. We embed K into a Hilbert cube Q in a normed space E (in particular a locally convex space). Let us denote by $s\colon K \to \widetilde{K}$ the homeomorphism of K onto $\widetilde{K} \subset Q$. Consider the map $i' \circ s^{-1}\colon \widetilde{K} \to X$ where $s^{-1}\colon \widetilde{K} \to K$ is an inverse homeomorphism and $i'\colon K \to X$ is an inclusion. Since $X \in NES(compact\ metric)$, there is an open set $U \subset Q$ containing $\widetilde{K}$ and the extension $h\colon U \to X$ of $i' \circ s^{-1}$ over U. Let $j\colon \widetilde{K} \to U$ be inclusions. It is clear that $h \circ j = i' \circ s^{-1}$. Let $r'\colon E \to Q$ be a retraction and let $V = r'^{-1}(U) \subset E$, $r_1 = r'_{/V}\colon V \to U$ and $i\colon U \to V$ be an inclusion. We define:

$r\colon V \to X$ given by $r = h \circ r_1$,

$Z = K$ and $q\colon Z \to V$ given by $q = i \circ j \circ s$.

We observe that for each $z \in Z$ $r(q(z)) = z$ and the proof is complete. □

Theorem 3.11 *Let* $Y_n \in \mathrm{ANMR}$ *for each n (not necessarily compact) and let* $Y = \prod_{n=1}^{\infty} Y_n$. *Then for any compact set $A \subset Y$ there exists a compact set $K \subset Y$ such that $A \subset K$ and* $K \in \mathrm{AANMC}(Y)$.

Proof Let $A \subset Y$ be a compact set. Let $\pi_n\colon Y \to Y_n$ be a map given by

$$\pi_n(y_1, y_2, \ldots, y_n, \ldots) = y_n$$

for each $(y_1, y_2, \ldots, y_n, \ldots) \in \prod_{n=1}^{\infty} Y_n$, $n = 1, 2, \ldots$ We define the compact set $K \subset Y$:

$$K = \prod_{n=1}^{\infty} \pi_n(A).$$

It is clear that $A \subset K$. From 3.4.10 and 3.4.4 we get that $K \in \mathrm{AANMC}(Y)$. □

Theorem 3.12 *Let* $X \in \mathrm{AANMR}$ *and* $Y \in \mathrm{ANMR}$ *(not necessarily compact). Then for any compact set $A \subset X \times Y$ there exists a compact set $K \subset X \times Y$ such that $A \subset K$ and* $K \in \mathrm{AANMC}(X \times Y)$.

Proof Let $\pi\colon X \times Y \to Y$ be a map given by $\pi(x, y) = y$ for each $(x, y) \in X \times Y$ and let $A \subset X \times Y$ be a compact set. We define $K = X \times \pi(A)$. It is clear that $A \subset K$. From 3.4.10 and 3.4.3 we get that $K \in \mathrm{AANMC}(X \times Y)$ and the proof is complete. □

Theorem 3.13 *Let $X = \bigcup_{n=1}^{\infty} U_n$, where $U_n \subset U_{n+1}$ and U_n is an open set in X for each n. Assume that for each n* $U_n \in \mathrm{ANMR}$. *Then* $K(X) = \mathrm{ANMC}(X)$.

Proof Let $K \subset X$ be a compact set. We observe that there exists n such that $K \subset U_n$. From 3.4.10 and 3.4.8 we get that $K \in \mathrm{ANMC}(X)$ and the proof is complete. □

We shall now present an example for a metric space that is neither of ANMR type nor of AANMR type but its every compact subset is either of AMC(X) or ANMC(X), or AANMC(X).

Example 3.14 Let $X = \{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\} \cup (2,3)$ where $(2,3) \subset \mathbb{R}$ is an open interval. We observe that $X \notin \mathrm{AANMR}$, since X is not a compact space.

From 3.4.10 and 3.9.2 $X \notin \mathrm{ANMR}$. If $K \subset X$ is a compact set, then $K = K_1 \cup K_2$, where $K_1 \subset (\{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\})$, $K_2 \subset (2,3)$, $K_1 \cap K_2 = \emptyset$ and K_1, K_2 are compact.

From 3.4.10 if $K_2 \neq \emptyset$ then $K_2 \in \mathrm{AMC}(X)$. The set $K_1 = \{\frac{1}{n}\}$, $n = 1, 2, \ldots$, then $K_1 \in \mathrm{AMC}(X)$ (see 3.5) or $K_1 = \{\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_k}\}$, $k > 1$, then $K_1 \in \mathrm{ANMC}(X)$ (see 3.6) or $K_1 = \{\frac{1}{n_k}\}_{k=1}^{\infty} \cup \{0\}$, then $K_1 \in \mathrm{AANMC}(X)$ (see 3.7).

From 3.4.6 the set $K \in \mathrm{ANMC}(X)$ or $K \in \mathrm{AANMC}(X)$. In particular, if $K_1 = \emptyset$ and K_2 is any compact and nonempty subset of the interval $(2,3)$ or $K_1 = \{\frac{1}{n}\}$, $n = 1, 2, \ldots$ and $K_2 = \emptyset$ then $K \in AMC(X)$.

4 Fixed point result

In this part of the paper we shall present a few applications of multicores in the fixed point theory.

Theorem 4.1 *Let X be a metric space and let $\varphi\colon X \multimap X$ be an admissible and compact map. Assume that*

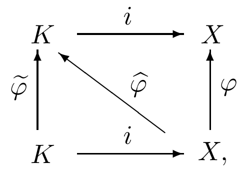
$$K = \overline{\varphi(X)} \in \mathrm{ANMC}(X).$$

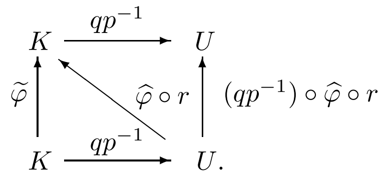
Then φ is a Lefschetz map.

Proof From the assumption we get a locally convex space E, an open set $U \subset E$, a map $r\colon U \to X$, a metric space Z, a map $q\colon Z \to U$ such that the map $p\colon Z \to K$ given by $p(z) = r(q(z))$ for each $z \in Z$ is a Vietoris map. Let $\widehat{\varphi}\colon X \multimap K$ be a multivalued map given by $\widehat{\varphi}(x) = \varphi(x)$ for each $x \in X$, $\widetilde{\varphi}\colon K \multimap K$ be a multivalued map given by $\widetilde{\varphi}(x) = \varphi(x)$ for each $x \in K$ and let $i\colon K \to X$ be an inclusion.

We have the following commutative diagrams:

$$\begin{array}{ccc} Z & \xrightarrow{r \circ q} & X \\ & \searrow^{p} & \uparrow{\scriptstyle i} \\ & & K, \end{array}$$





We observe that

$$(\widehat{\varphi} \circ r) \circ (q \circ p^{-1}) =$$
$$= \widehat{\varphi} \circ (r \circ q) \circ p^{-1} = \widehat{\varphi} \circ (i \circ p) \circ p^{-1} = (\widehat{\varphi} \circ i) \circ (p \circ p^{-1}) = \widetilde{\varphi} \circ \mathrm{Id}_K = \widetilde{\varphi}.$$

The map $\psi \equiv (qp^{-1}) \circ \widehat{\varphi} \circ r$ is admissible and compact. From 2.24 ψ is a Lefschetz map. Using the above diagrams and applying a method of proving commonly known in mathematical literature (see [2, 7, 8, 9, 10, 11, 19, 20, 21]), it can be proved that the map φ is a Lefschetz map. □

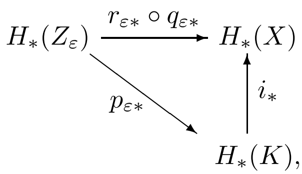
The following theorem is the simple consequence of the above theorem.

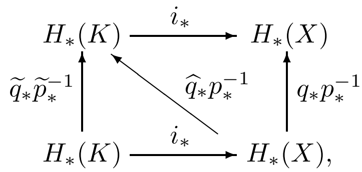
Theorem 4.2 *Let X be a metric space and let $\varphi\colon X \multimap X$ be an admissible and compact map. Assume that $K = \overline{\varphi(X)} \in AMC(X)$. Then φ has a fixed point.*

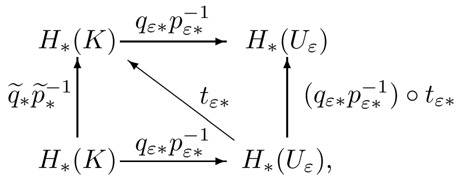
Theorem 4.3 *Let X be a metric space and let $\varphi\colon X \multimap X$ be an admissible and compact map. Assume that there exists a compact set $K \subset X$ such that K is of finite type, $\varphi(X) \subset K$ and $K \in \mathrm{AANMC}(X)$. Then φ is a Lefschetz map.*

Proof From the assumption we have for each $\varepsilon > 0$ a locally convex space E_ε, an open set $U_\varepsilon \subset E_\varepsilon$, a map $r_\varepsilon\colon U_\varepsilon \to X$, a metric space Z_ε, a Vietoris map $p_\varepsilon\colon Z_\varepsilon \to K$ such that the map $s_\varepsilon\colon Z_\varepsilon \to K$ given by $s_\varepsilon(z) = r_\varepsilon(q_\varepsilon(z))$ for each $z \in Z$ satisfied the condition $d(s_\varepsilon(z), p_\varepsilon(z)) < \varepsilon$ for each $z \in Z_\varepsilon$. Let $\widehat{\varphi}\colon X \multimap K$ be a multivalued map given by $\widehat{\varphi}(x) = \varphi(x)$ for each $x \in X$, $\widetilde{\varphi}\colon K \multimap K$ be a multivalued map given by $\widetilde{\varphi}(x) = \varphi(x)$ for each $x \in K$ and let $i\colon K \to X$ be an inclusion. Let $(p, q) \subset \varphi$. Then $(p, \widehat{q}) \subset \widehat{\varphi}$ and $(\widetilde{p}, \widetilde{q}) \subset \widetilde{\varphi}$ where $\widetilde{p}, \widetilde{q}, \widehat{q}$ are respective contractions of maps p, q. From 2.26 there exists $\varepsilon_1 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_1$ $s_{\varepsilon *} = p_{\varepsilon *}$.

We have the following commutative diagrams:







where $t_{\varepsilon *} \equiv (\widehat{q}_*p_*^{-1}) \circ r_{\varepsilon *}$. Hence the homomorphism $q_*p_*^{-1}$ is a Leray endomorphism. Assume that $\Lambda(\varphi) \neq \{0\}$ then there exists $(p,q) \subset \varphi$ such that $\Lambda(q_*p_*^{-1}) \neq 0$. Hence for each $0 < \varepsilon \leq \varepsilon_1$ there exists $x_\varepsilon \in ((q_\varepsilon p_\varepsilon^{-1}) \circ \widehat{\psi} \circ r_\varepsilon)(x_\varepsilon)$, so $r_\varepsilon(x_\varepsilon) \in (r_\varepsilon \circ (q_\varepsilon p_\varepsilon^{-1}) \circ \widehat{\psi})(r_\varepsilon(x_\varepsilon))$, where $\widehat{\psi} = \widehat{q}p^{-1}$ (see the above diagrams). We have $z_\varepsilon \in (p_\varepsilon^{-1} \circ \widehat{\psi})(r_\varepsilon(x_\varepsilon))$ such that $r_\varepsilon(q_\varepsilon(z_\varepsilon)) = r_\varepsilon(x_\varepsilon)$, $p_\varepsilon(z_\varepsilon) \in \widehat{\psi}(r_\varepsilon(x_\varepsilon))$ and $d(r_\varepsilon(x_\varepsilon), p_\varepsilon(z_\varepsilon)) = d(r_\varepsilon(q_\varepsilon(z_\varepsilon)), p_\varepsilon(z_\varepsilon)) < \varepsilon$. We observe that for each $\varepsilon > 0$ $r_\varepsilon(x_\varepsilon) \in K$ is the ε-fixed point of the map $\widetilde{\psi} = \widetilde{q}\widetilde{p}^{-1}$ (see the above diagrams). The set K is compact, so $\widetilde{\psi}$ has a fixed point. It is clear that $\mathrm{Fix}(\widetilde{\psi}) \subset \mathrm{Fix}(\varphi)$ and the proof is complete. □

We recall that an admissible map $\varphi_X : X \multimap X$ is a compact absorbing contraction (we write $\varphi_X \in CAC(X)$), provided that there exists an open set $U \subset X$ such that the following conditions are satisfied:

(i) the map $\varphi_U : U \to U$ given by $\varphi_U(x) = \varphi_X(x)$ for each $x \in U$ is compact and $\overline{\varphi_U(U)} \subset U$,

(ii) for each $x \in X$ there exists a natural number n such that $\varphi_X^n(x) \subset U$, where $\varphi_X^n = \varphi_X \circ \varphi_X \circ \ldots \circ \varphi_X$, ($n$-iterate).

Theorem 4.4 *Let X be a metric space and $\varphi_X : X \multimap X$ be an admissible map. Assume that $\varphi_X \in CAC(X)$ and $\overline{\varphi_U(U)} \in$ ANMC(X), then φ_X is a Lefschetz map.*

Proof Let $\varphi\colon (X,U) \multimap (X,U)$ given by $\varphi(x) = \varphi_X(x)$ for each $x \in X$ and let $(p,q) \subset \varphi_X$. Then there exists a space Z such that $p\colon Z \to X$ is a Vietoris map and $q\colon Z \to X$ is a continuous map. Let $\widetilde{p}\colon p^{-1}(U) \to U$ given by $\widetilde{p}(x) = p(x)$ for each $x \in p^{-1}(U)$ and $q\colon p^{-1}(U) \to U$ given by $\widetilde{q}(x) = q(x)$

for each $x \in p^{-1}(U)$. Then $(\widetilde{p},\widetilde{q}) \subset \varphi_U$. From 3.4.5, 3.4.7 and 4.1 we get that the homomorphism $\widetilde{q}_*\widetilde{p}_*^{-1}\colon H_*(U) \to H_*(U)$ is a Leray endomorphism. Let $\widehat{p},\widehat{q}\colon (Z, p^{-1}(U)) \to (X,U)$ given by $\widehat{p}(x) = p(x)$ and $\widehat{q}(x) = q(x)$ for each $x \in Z$. Then $(\widehat{p},\widehat{q}) \subset \varphi$. The homomorphism $\widehat{q}_*\widehat{p}_*^{-1}\colon H_*(X,U) \to H_*(X,U)$ is weakly nilpotent (see [8, 21]). Hence, $q_*p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_*p_*^{-1}) = \Lambda(\widetilde{q}_*\widetilde{p}_*^{-1})$ (see Lemma 2.6 in [21] and 2.7). Assume that $\Lambda(\varphi_X) \neq \{0\}$ then there exists $(p,q) \subset \varphi_X$ such that $\Lambda(q_*p_*^{-1}) \neq 0$. The above deduction, 4.1 and 3.4.7 implicate that $\mathrm{Fix}(\varphi_U) \neq \emptyset$ and the proof is complete. □

Remark 4.5 Generally, the last theorem can be proven with the assumption that $\varphi_X \in GCAC(X)$ (see [10, 11, 21]).

Open problem 4.6 *Let X be a metric space that is not compact. Assume that* $\mathrm{ANMC}(X) = K(X)$. *Is the space* $X \in \mathrm{ANMR}$*?*

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