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Jitka Dupačová

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STOCHASTIC GEOMETRIC PROGRAMMING WITH AN APPLICATION

Jitka Dupačová

In applications of geometric programming, some coefficients and/or exponents may not be precisely known. Stochastic geometric programming can be used to deal with such situations. In this paper, we shall indicate which stochastic programming approaches and which structural and distributional assumptions do not destroy the favorable structure of geometric programs. The already recognized possibilities are extended for a tracking model and stochastic sensitivity analysis is presented in the context of metal cutting optimization. Illustrative numerical results are reported.

Keywords: stochastic geometric programming, statistical sensitivity analysis, tracking model, metal cutting optimization

Classification: 90C15, 90C31, 90C90

1. INTRODUCTION

Geometric programs (GP), see e.g. [3, 19], are nonlinear programs in which the objective function and/or some constraints are of the form of *posynomials*:

$$\text{minimize } g_0(\boldsymbol{t}) \text{ subject to } g_k(\boldsymbol{t}) \le 1,\ k = 1,\dots,K,\ \boldsymbol{t} \in \mathbb{R}^M_{++} \tag{1}$$

with

$$g_k(\boldsymbol{t}) = \sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} = \sum_{i \in I_k} u_i(\boldsymbol{t}),\ k = 0,\dots,K. \tag{2}$$

The exponents a_{ij} are real numbers and the coefficients c_i are positive. The next section provides basic information about this type of nonconvex constrained optimization problem.

The recently observed growing interest in GP stems from the fact that various practical problems can be reformulated as geometric programs and there are solution methods which solve even very large-scale GPs efficiently and reliably. With a basic interior-point method which exploits sparsity of the generic geometric program (1) – (2), the reported efficiency is close to that of linear programming solvers. We refer to [4] for an up-to-date survey of various applications and an extensive list of references.

In applications, however, some coefficients c_i and/or exponents a_{ij} are not known precisely, and their incomplete knowledge may be modeled as random. This is then the area of stochastic programming and one deals with the distribution problem or focuses on decision problems. The choice of a suitable approach depends on the problem formulation, on the structure of the geometric program and on distributional assumptions. Section 3 discusses stochastic programming approaches which preserve the favorable structure of (generalized) geometric programs. A suggestion is to use a tracking model.

The stochastic sensitivity analysis which allows constructing approximate confidence bounds for the optimal values and the optimal solutions of stochastic geometric programs (SGP) is another possibility. It was initially suggested and applied in [7, 8] to problems with deterministic constraints in (1). This work was motivated by metal cutting problems where some exponents a_{ij} are obtained as statistical estimates of the true values. These metal cutting problems will be briefly introduced in Section 4. The stochastic sensitivity analysis will be detailed in Section 5 for the general structure of the SGP and its extension to the generalized stochastic geometric programs will be delineated. A numerical illustration based on [8] concludes the paper.

2. GEOMETRIC PROGRAMMING

Let Q denote the total number of *monomials* $u_i(\boldsymbol{t}) = c_i \prod_{j=1}^{M} t_j^{a_{ij}}$ in the formulation of the geometric program (1)−(2) and let $\{I_k, k = 0, \dots, K\}$ be a decomposition of $\{1, \dots, Q\}$ into $K + 1$ disjoint index sets. Notice that simple box inequality constraints can be written as inequalities for monomials.

The special structure of the geometric program (1)−(2) allows deriving a numerically tractable dual problem:

$$\max_{\delta,\lambda} v(\delta,\lambda) := \prod_{i=1}^{Q} (c_i/\delta_i)^{\delta_i} \prod_{k=1}^{K} \lambda_k^{\lambda_k} \tag{3}$$

subject to

$$\sum\nolimits_{i \in I_0} \delta_i = 1,\ \delta_i \geq 0,\ i = 1, \dots, Q,$$

$$\sum\nolimits_{i=1}^{Q} a_{ij}\delta_i = 0,\ j = 1, \dots, M, \quad \sum\nolimits_{i \in I_k} \delta_i = \lambda_k,\ k = 1, \dots, K.$$

The optimal solutions $\boldsymbol{t}^*$ of (1) and δ^*, λ^* of (3) are related as follows:

$$\delta_i^* = \frac{u_i(\boldsymbol{t}^*)}{g_0(\boldsymbol{t}^*)} = \frac{u_i(\boldsymbol{t}^*)}{v(\delta^*, \lambda^*)} \text{ for } i \in I_0,$$

$$\delta_i^* = \lambda_k^* u_i(\boldsymbol{t}^*) \text{ for } i \in I_k,\ k = 1, \dots, K.$$

Hence $\frac{\delta_i^*}{\lambda_k^*}$, $i \in I_k$ is the proportional contribution of the ith monomial to the value of the posynomial g_k at the optimal solution $\boldsymbol{t}^*$. By means of these duality relations the numerical solution of small size geometric programs can be based on the solution of their relatively simple duals.

The *degree of difficulty* of a geometric program is defined as $\Delta = Q - M^* - 1$ where M^* denotes the rank of the (Q, M) matrix $\boldsymbol{A} = (a_{ij})$. It refers to the dimensionality of the set of feasible solutions of the dual program. For $\Delta = 0$, i. e. for the *zero degree of difficulty geometric programs*, there is a unique solution of the system $\sum_{i \in I_0} \delta_i = 1$, $\sum_{i=1}^{Q} a_{ij}\delta_i = 0$, $j = 1, \ldots, M$. If this solution $\delta_i \geq 0 \, \forall \, i$ then it is the optimal solution of the dual problem and it is possible to get an explicit representation of the optimal value function of (3) in terms of the coefficients c_i. Moreover, its logarithm is a linear function in coefficients c_i.

In general, geometric programs are not convex programs, but they can be reformulated so that they are convex. When $\operatorname{card} I_k = 1$ for all $k = 1, \ldots, K$, the constraints in (1) are defined by monomials and can be linearized. When $\operatorname{card} I_k > 1$, using the exponential substitution $z_j = \log t_j$ the posynomials (2) are transformed to

$$h_k(\boldsymbol{z}) = \sum_{i \in I_k} c_i \exp\left\{\sum_{j=1}^{M} a_{ij} z_j\right\},\ k = 0, \ldots, K. \tag{4}$$

The resulting transformed GP is then the *convex* program

$$\text{minimize } h_0(\boldsymbol{z}) \text{ subject to } h_k(\boldsymbol{z}) \leq 1,\ k = 1, \ldots, K,\ \boldsymbol{z} \in \mathbb{R}^M. \tag{5}$$

An additional log transform of the functions h_k is frequently recommended.

Convexity of $h_k(\boldsymbol{z})$ in (4) extends to functions of the form

$$H_k(\boldsymbol{z}) := \sum_{i \in I_k} c_i \exp\{\phi_i(\boldsymbol{z})\} \tag{6}$$

with convex functions ϕ_i. This observation motivated one type of generalizations of geometric programming. Another generalization is e. g. to replace posynomials $g_k(\boldsymbol{t})$ in (1) by *generalized posynomials*, i. e. functions which are formed from posynomials using operations of addition, multiplication, positive (fractional) power and maximum.

If some of the coefficients c_i in (2) are negative, complexity of the optimization problem increases substantially. Even from (4) it is clear that convexity of functions $h_k(\boldsymbol{z})$ for nonpositive coefficients c_i cannot be expected. Functions of the form (2) but with some negative coefficients c_i are called *signomials*.

3. STOCHASTIC GEOMETRIC PROGRAMMING

In stochastic geometric programming (SGP) one accepts that some coefficients c_i and/or exponents a_{ij} are not known precisely, and their incomplete knowledge is modeled as random. The origins of stochastic geometric programming are connected with paper [2], where the exponents a_{ij} are deterministic and the coefficients c_j are positive random variables. The main results of the paper are numerically tractable bounds for the optimal value of (1) – (2); see also [23] for their further elaboration and application.

Construction of confidence bounds for the optimal value of a geometric program, deriving its moments or probability distribution fall under the *distribution problem* of stochastic geometric programming. It was studied first for the zero degree of difficulty geometric programs in connection with a lognormal distribution of coefficients c_i. Then the logarithm of the optimal value function in (3) is an affine linear function in $\log c_i$, hence, a lognormal distribution of c_i yields a lognormal distribution of the optimal value. These results were extended to the zero degree quadratic geometric programs with quadratic positive semidefinite functions ϕ_i in (6), cf. [20], and to problems with degree of difficulty $\Delta > 0$, cf. [9]. For general positive random coefficients c_i [9, 21] suggest to exploit the central limit theorem to get an asymptotically lognormal distribution of the coefficients and subsequently of the optimal value. In general, approximate confidence bounds and moments of the optimal value can be also obtained by simulation and repeated solution of (1); see e. g. [15].

Individual probabilistic constraints replacing constraints of (1) or (5) have been applied under the assumption of deterministic exponents and normally distributed mostly uncorrelated coefficients c_i; see e. g. [12, 19]. Of course, the assumption of normally distributed costs c_i is not in agreement with the required positivity of coefficients in (2). For general probability distributions of coefficients [11] suggests to approximate the probabilistic constraints by the one-sided Chebyshev inequality and discusses assumptions under which the variance appearing in the resulting deterministic constraint is a posynomial.

A possibility which applies to SGP with *coefficients and exponents in the objective function determined by a random parameter* $\beta \in \mathbb{R}^q$ and to a discrete distribution of this parameter is to use a *tracking model.* The problem we face for a realization (scenario) β^ν of β is the geometric program

$$\min_{\boldsymbol{t}} g_0(\boldsymbol{t}, \beta^\nu) \text{ subject to } \boldsymbol{t} \in \mathbb{R}^M_{++} \cap \mathcal{T}$$

with a fixed set $\mathcal{T}$ described by deterministic posynomial constraints. We are not interested in optimal solutions, say $\boldsymbol{t}^\nu$, for each of the scenarios separately but we need to obtain one acceptable decision. Tracking models related with the goal programming offer such possibility: Try to find the universal compromising solution by minimization of the (positively) weighted distance

$$\sum_\nu p_\nu \| g_0(\boldsymbol{t}, \beta^\nu) - g_0(\boldsymbol{t}^\nu, \beta^\nu) \|$$

on the set $\mathbb{R}^M_{++} \cap \mathcal{T}$. With the L_1-distance and using optimality of $\boldsymbol{t}^\nu$ for scenario β^ν the resulting problem to be solved is the geometric program

$$\min \left\{ \sum_\nu p_\nu g_0(\boldsymbol{t}, \beta^\nu) \; : \; \boldsymbol{t} \in \; \mathbb{R}^M_{++} \cap \mathcal{T} \right\}. \tag{7}$$

The weights $p_\nu > 0$, $\sum_\nu p_\nu = 1$ can be interpreted as scenario probabilities. Sensitivity of the optimal solution of (7) to changes in these weights can be studied by the parametric programming techniques which will be explained in Section 5.

An extension of this idea to generalized stochastic geometric programs is straightforward.

For random costs and exponents *both* in the objective function and in the constraints a *penalization or two-stage approach* was suggested by [13]. First of all, using an additional constraint and an additional variable t_0, the geometric program (1) can be rewritten to have a nonrandom linear objective function:

$$\min \left\{ t_0 \,:\, t_0^{-1} g_0(\boldsymbol{t},\beta) \leq 1,\, g_k(\boldsymbol{t},\beta) \leq 1,\, k=1,\ldots,K, t_0>0, \boldsymbol{t}\in\mathbb{R}^M_{++} \right\}. \tag{8}$$

The constraints of (8) can be further split to

$$u_i(\boldsymbol{t},\beta)\theta_{ik}^{-1} \leq 1,\, i=1,\ldots,Q, k=0,\ldots,K$$

with $\theta_{ik}>0$, $\sum_{i\in I_k}\theta_{ik}=1$ interpreted as the proportional contribution of the ith monomial to the value of the kth posynomial.

The first stage decisions are $t_0>0$, $\boldsymbol{t}\in\mathbb{R}^M_{++}$, $\theta_{ik}>0$, $i\in I_k\,\forall\, k$, and $\sum_{i\in I_k}\theta_{ik} = 1 \forall\, k$ are the first stage constraints. After observing realizations of random coefficients and exponents, a possible violation of constraints $u_i(\boldsymbol{t},\beta)\theta_{ik}^{-1}\leq 1$ can be corrected for an additional cost. The logarithmic penalty function is applied and the case of the multivariate discrete or normal distribution of parameters c_i, a_{ij} is discussed.

In [7, 8] a technique for construction of confidence bounds for the optimal value and the optimal solution based on sensitivity results for deterministic geometric programs [17] and on the stochastic sensitivity analysis of [6] was proposed. It was motivated by metal cutting problems where some of the exponents a_{ij} in the objective function of (1) are obtained as statistical estimates of the true values. The technique applies also to other GP problems with estimated coefficients and/or exponents both in the objective function and constraints of (1). Its general form will be presented in Section 5.

4. METAL CUTTING PROBLEMS

Many typical problems in optimization of cutting conditions in machining can be formulated as (generalized) geometric programs; several examples are given e. g. in [7, 14, 19], see also [5, 18].

The most popular optimality criterion is minimization of the total machining costs which equal the sum of the machining time costs, the costs of tool changing time per component and the tool cost per component:

$$C = xT_c + xT_d\frac{T_{ac}}{T} + y\frac{T_{ac}}{T} = xT_c + \frac{T_{ac}}{T}(xT_d+y) \tag{9}$$

where x is the labor plus overhead cost rate, y is the tool cost, T_c is the machining time, T_{ac} the actual cutting time, T_d the tool changing time and T the tool life. Whereas x, y and T_d are understood as a fixed input, the tool life, the cutting time and the machining time depend on the cutting conditions, such as the depth of the cut d, the feed f, and the speed v which is proportional to the number r of revolutions per minute. The machining time

$$T_c = Lr^{-1}f^{-1} \tag{10}$$

where L is the length of the workfeed motion between the tool and the workpiece.

There are surface finish requirements, limitations on the machine tool dynamics and box constraints on variables mostly formulated as monomial or posynomial constraints. For example, the machine power constraint has the form

$$k_{Fz} f^{y_{Fz}} d^{x_{Fz}} v \le 60 P_{\text{eff}} \tag{11}$$

where P_{eff} (in W) denotes the effective power and k_{Fz}, x_{Fz} and y_{Fz} are empirical constants that appear in the cutting force function. Prescribed upper and lower bounds on the product rf restrain the feed per minute, etc. Similar models apply also to other metal working procedures, e. g. to turning, drilling and milling.

Due to the nonhomogeneity of the machined and cutting material, a variability of the tool life is observed even at fixed machining conditions. From the technical point of view, this is the main source of uncertainty in the machining costs which asks for an analysis of the precision of the optimally settled cutting conditions, of the corresponding tool life, and of the machining costs.

According to the common technical practice, the tool life T is related to the cutting conditions via the Taylor equation [22]

$$vT^n f^m d^p = A \tag{12}$$

where n, m, p and A are empirical constants which correspond to the tool and the workpiece material.

Substituting (10), (12) for T_c, T in (9), accepting the common assumption that $\frac{T_{ac}}{T_c} \sim 1$ and using the proportionality of the speed v and the number r of revolutions per minute, the objective function (9) can be written in the form

$$k_1 v^{-1} f^{-1} + k_2 v^{1/n-1} f^{m/n-1} d^{p/n}$$

or

$$C_1 r^{-1} f^{-1} + C_2 r^{1/n-1} f^{m/n-1} d^{p/n} \tag{13}$$

to be minimized with respect to v, f, d or r, f, d subject to constraints on monomials or posynomials such as (11).

Optimization of the multiple tools turning operation exploits a time diagram according to which various tools are supposed to be active at a given instant. It means that in the GP problem, the power constraints have to be considered separately at all particular points in time where the pattern changes and it is necessary to evaluate the contribution of all tools to the total machining costs per component including the various tool lives. This is the idea behind optimization of the cutting conditions on automatic production lines e. g. [16]. In the case of the *multipass turning operations* [18], the new feature is that there are signomials, i. e. *differences* of posynomials, in the objective function and the optimization problem cannot be transformed into a convex program.

It is accepted that the Taylor equation (12) represents the tool life for commonly used materials. For special alloys, however, the logarithm of the tool life does not exhibit linear but quadratic behavior, cf. [14, 20]. It means that the objective function h_0 in the transformed GP (5) changes to

$$H_0(\boldsymbol{z}) = \sum_{i \in I_0} c_i \exp\left\{\sum_{j=1}^{M} a_{ij} z_j + \sum_{j=1}^{M}\sum_{l=1}^{M} b_{ijl} z_j z_l\right\}.$$

The resulting metal cutting problem is a generalized geometric program with a special choice of functions ϕ_i, $i \in I_0$ in (6). Randomness of its parameters was considered and the distribution problem for the optimal total machining cost was solved in [20] under simplifying assumptions of deterministic exponents and random independent coefficients c_i with normal or lognormal distributions for the zero degree of difficulty problem.

5. SENSITIVITY ANALYSIS

It is convenient to apply the sensitivity analysis results, cf. [10, 17], to the convex transformed GP of the form (5) whose coefficients and exponents are differentiable functions of a parameter $\beta \in \mathbb{R}^q$:

$$\text{minimize } h_0(\boldsymbol{z};\beta) \quad \text{subject to} \quad h_k(\boldsymbol{z};\beta) \le 1, k = 1,\dots,K. \tag{14}$$

Assume that (14) was solved with coefficients and exponents determined by a specific parameter value, say, β^*. The classical sensitivity analysis results for the optimal solution $\boldsymbol{z}^* := \boldsymbol{z}(\beta^*)$ and for the corresponding multipliers $\mu^* := \mu(\beta^*)$ hold true under linear independence of gradients of the binding constraints, the strict complementarity condition and the second order sufficient condition. These conditions guarantee, inter alia, that the set of active constraints does not change for small changes of parameter values and that the derivatives of the optimal solution and of the multipliers with respect to parameters can be obtained from the first order necessary conditions by the implicit function theorem, cf. [10]. Moreover, except for the strict complementarity condition, for geometric programs these assumptions reduce to a simple rank condition on the matrix $\boldsymbol{A}(\beta^*)$ of exponents determined by β^*; see [17] for details. Among others, we shall assume that the matrix $\boldsymbol{A}(\beta^*)$ is of a full rank, i. e. $M^* = M$.

Without loss of generality assume that $h_k(\boldsymbol{z};\beta) \le 1, k = 1,\dots,K_0 \le K$, are the constraints in (14) that are *active* at $\boldsymbol{z}^*$ for $\beta = \beta^*$ and that they are defined by means of Q_0 monomials. Denote $\boldsymbol{A}_0(\beta)$ the corresponding (Q_0, M) submatrix of exponents $a_{ij}(\beta)$ in the objective and in the active constraints and put $\boldsymbol{A}_0^* := \boldsymbol{A}_0(\beta^*)$. The Lagrange function for the reduced parametrized problem (14) is

$$L(\boldsymbol{z},\mu;\beta) = h_0(\boldsymbol{z};\beta) + \sum_{k=1}^{K_0} \mu_k (h_k(\boldsymbol{z};\beta) - 1).$$

To express the second order derivatives of the Lagrange function in a transparent way, we use an auxiliary substitution $\boldsymbol{w}(\beta) = \boldsymbol{A}_0(\beta)\boldsymbol{z}$ and denote

$$\tilde{h}_k(\boldsymbol{w};\beta) = \sum_{i \in I_k} c_i(\beta) \exp(w_i(\beta)), k = 0,\dots,K_0.$$

Let $\boldsymbol{C}^*$ be the (K_0, Q_0) matrix of gradients $\nabla_{\boldsymbol{w}} \tilde{h}_k(\boldsymbol{w}^*; \beta^*)^\top$ of $\tilde{h}_k, k = 1, \ldots, K_0$, at $[\boldsymbol{w}^* = \boldsymbol{A}_0^* \boldsymbol{z}^*, \beta^*]$, $\boldsymbol{H}^*$ the diagonal matrix

$$\boldsymbol{H}^* = \mathrm{diag}\{c_i(\beta^*)\exp(w_i^*), i \in I_0,\, \mu_k^* c_i(\beta^*)\exp(w_i^*), i \in I_k, k = 1, \ldots, K_0\},$$

$\boldsymbol{B}^* = \nabla^2_{\boldsymbol{z},\beta} L(\boldsymbol{z}^*, \mu^*; \beta^*)$ the (M, q) matrix of the second order derivatives of the Lagrange function at the point $[\boldsymbol{z}^*, \mu^*, \beta^*]$ and $\boldsymbol{D}^*$ the (K_0, q) matrix of gradients $\nabla_\beta \tilde{h}_k(\boldsymbol{w}^*; \beta^*)^\top$, $k = 1, \ldots, K_0$, at the point $[\boldsymbol{w}^* = \boldsymbol{A}_0^* z^*, \beta^*]$.

Under strict complementarity conditions, the (M, q) matrix $\frac{\partial \boldsymbol{z}(\beta^*)}{\partial \beta}$ of derivatives $\nabla_\beta z_j(\beta^*)^\top$, $j = 1, \ldots, M$, of the optimal solution and the (K_0, q) matrix of derivatives of multipliers $\mu_k^* = \mu_k(\beta^*)$ of the constraints active at β^* are uniquely determined by the following system of equations

$$\begin{pmatrix} \boldsymbol{A}_0^{*\top} \boldsymbol{H}^* \boldsymbol{A}_0^* & (\boldsymbol{C}^* \boldsymbol{A}_0^*)^\top \\ \boldsymbol{C}^* \boldsymbol{A}_0^* & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \frac{\partial \boldsymbol{z}(\beta^*)}{\partial \beta} \\ \frac{\partial \mu(\beta^*)}{\partial \beta} \end{pmatrix} = - \begin{pmatrix} \boldsymbol{B}^* \\ \boldsymbol{D}^* \end{pmatrix} \tag{15}$$

provided that the matrix of the system (15) is nonsingular.

The gradient of the optimal value function is

$$\nabla_\beta \varphi(\beta^*) = \nabla_\beta L(\boldsymbol{z}(\beta^*), \mu(\beta^*); \beta^*) \tag{16}$$

and this assertion holds true even in the case that the strict complementarity conditions are not fulfilled. A simple formula is obtained when the constraints do not depend on the parameter β : we have then the gradient of the optimal value function

$$\nabla_\beta \varphi(\beta^*) = \nabla_\beta h_0(\boldsymbol{z}^*, \beta^*) = \nabla_\beta \tilde{h}_0(\boldsymbol{w}^*; \beta^*) \boldsymbol{A}_0^*; \tag{17}$$

moreover, in (15), the matrix $\boldsymbol{D}^* = \boldsymbol{0}$ and $\boldsymbol{B}^* = \nabla^2_{\boldsymbol{z}\beta} h_0(\boldsymbol{z}^*; \beta^*)$.

Assume now that the parameter value β^* is an *asymptotically normal estimate* of the true parameter value $\hat{\beta}$,

$$\sqrt{N}(\beta^* - \hat{\beta}) \sim \mathcal{N}(0, \sigma^2 \boldsymbol{M}).$$

Using the above sensitivity results and the Delta theorem, we get an asymptotically normal distribution of the optimal solution of (14) and of the optimal value function $\varphi(\beta^*)$ computed with the estimated coefficient values: if the matrix of derivatives $\frac{\partial \boldsymbol{z}(\beta^*)}{\partial \beta} \neq 0$ then according to [6], the optimal solution is asymptotically normal

$$\sqrt{N}(\boldsymbol{z}(\beta^*) - \boldsymbol{z}(\hat{\beta})) \sim \mathcal{N}\left(0,\, \sigma^2 (\frac{\partial \boldsymbol{z}(\hat{\beta})}{\partial \beta}) \boldsymbol{M} (\frac{\partial \boldsymbol{z}(\hat{\beta})}{\partial \beta})^\top\right). \tag{18}$$

The rank of the asymptotic distribution equals the rank of $(\frac{\partial \boldsymbol{z}(\hat{\beta})}{\partial \beta}) \boldsymbol{M} (\frac{\partial \boldsymbol{z}(\hat{\beta})}{\partial \beta})^\top$. The optimal value $\varphi(\beta^*)$ is asymptotically normal as well,

$$\sqrt{N}(\varphi(\beta^*) - \varphi(\hat{\beta})) \sim \mathcal{N}(0,\, \sigma^2 \nabla_\beta \varphi(\hat{\beta})^\top \boldsymbol{M} \nabla_\beta \varphi(\hat{\beta})). \tag{19}$$

Whereas the distribution (19) can be used directly to obtain an asymptotic confidence interval for the optimal value, (18) applies to *logarithms* of the optimal solution. For application of these results, it is essential that for the sample size N large enough, the covariance matrix in (18) and the variance in (19) can be evaluated at the estimated values of parameters β and σ^2.

We refer to [7] for a detailed discussion of the sensitivity analysis for optimization of a single pass single tool turning operation where all constraints are linearizable. Then the set of feasible solutions is convex, polyhedral (described by constraints $\log c_k + \sum_j a_{kj} z_j \le 0 \,\forall\, k$) and nondegeneracy of its vertices implies that the linear independence condition is satisfied.

These ideas can be extended to the stability analysis of *generalized geometric programs* that allow transformation of the objective and constraints to the form (6) with functions ϕ_i convex and differentiable. Assume for simplicity that the random parameters appear only in the exponents of the objective function and let $\tilde{\varphi}(\beta)$ denote the optimal value of the program

$$\min\{H_0(\boldsymbol{z};\beta) \,:\, H_k(\boldsymbol{z}) \le 1,\, k = 1,\dots,K,\, \boldsymbol{z} \in \mathbb{R}^M\}.$$

Then the gradient of the optimal value function at $\beta = \beta^*$ is

$$\nabla_\beta \tilde{\varphi}(\beta^*) = \nabla_\beta H_0(\boldsymbol{z}(\beta^*);\beta^*) = \sum\nolimits_i c_i \exp\{\phi(z(\beta^*);\beta^*)\} \nabla_\beta \phi(z(\beta^*);\beta^*),$$

compare with (17); the asymptotic distribution of $\tilde{\varphi}(\beta^*)$ follows the pattern (19).

6. NUMERICAL ILLUSTRATION

The obtained results can be used for constructing confidence regions for the optimal cutting conditions and for the minimal cost of the machining process. The numerical illustration below is based on [8].

A carbon steel workpiece ($D = 100\,\mathrm{mm}$) is to be rough turned in the length $L = 80\mathrm{mm}$ by using a sintered carbide tool under optimal cutting conditions that minimize the total machining costs (13). The maximum and minimum speed capacity is 4000 resp. 112 revolutions per minute, the maximum and minimum available feed per minute fr equals 6000 resp. 1. From the point of view of chip formation and of the strength of the cutting tool the upper and lower bounds on the feed f (0.45 resp. 0.05 mm per revolution) and on the depth of the cut d (5 resp. 0.5 mm) are prescribed. The machine power constraint (11) is split into two constraints that correspond to different ranges of the cutting speed:

$$k_{F_z} f^{y_{F_z}} d^{x_{F_z}} \le 2M_{kp} D^{-1} \tag{20}$$

$$k_{F_z} f^{y_{F_z}} d^{x_{F_z}} r^{z_M} \le 2k_p D^{-1}. \tag{21}$$

Moreover, a constraint on the *torque for high revolution chuck* is taken into account

$$k_{F_z} f^{y_{F_z}} d^{x_{F_z}} \le 3\mu D_u D^{-1} \left(F_{u0} - k_u r^2\right). \tag{22}$$

Notice that the last constraint (22) cannot be linearized.

In the corresponding GP, $I_0 = \{1,2\}$, $M = 3$ and the transformed decision variables are $z_1 = \log v$ (or $\log r$), $z_2 = \log f, z_3 = \log d$. The unit costs needed for evaluation of coefficients C_1, C_2 in (13) were calculated with the tool cost per cutting edge $y = 1.5$ and tool changing time $T_d = 1$. See [8] for numerical values of coefficients and exponents in (20) – (22).

The standard values of parameters in the Taylor equation (12) are: A=293, n=0.36, m=0.39, p=0.11. The optimal solution of the GP described above was obtained by GAMS CONOPT:

$$r = 1169.448\,(v = 367.393),\ f = 0.180,\ d = 0.500$$

with the minimal unit costs $\varphi = 3.397$ and the corresponding tool life $T = 4.219$.

Estimates of the empirical constants in the Taylor equation (12) may be obtained via regression analysis based on the linearized assumed empirical relationship between the tool life and the cutting conditions

$$\log T = \log A^{1/n} - 1/n \log v - m/n \log f - p/n \log d + \epsilon \tag{23}$$

with ϵ independent of $\log T$, with zero mean value and with a fixed unknown variance $\sigma^2 > 0$. Such an approach is needed e. g. for nonstandard materials.

Let $\beta_0, \beta_1, \beta_2, \beta_3$ denote the regression coefficients in (23) rewritten to the standard form of the linear regression model with regressors $z_1 = \log v$, $z_2 = \log f$, $z_3 = \log d$

$$\log T = \beta_0 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 + \epsilon.$$

Their estimates β^* enter as parameters into the second posynomial of the objective function (13), i. e. into the second term of $h_0(\boldsymbol{z};\beta)$. We have

$$c_2(\beta) = C_2 \exp(-\beta_0), a_{21}(\beta) = -\beta_1 - 1, a_{22}(\beta) = -\beta_2 - 1, a_{23}(\beta) = -\beta_3. \tag{24}$$

For fitting the regression (23), the data from [7] were used: Normality of the residuals was observed and the estimate β^* was obtained by the Least squares method, with the estimate of the correspondingly indexed variance matrix

$$\sigma^2 \boldsymbol{M} = \begin{pmatrix} 0.016 & 0.004 & 0.004 & -0.075 \\ 0.004 & 0.004 & 0.001 & -0.016 \\ 0.004 & 0.001 & 0.004 & -0.019 \\ -0.075 & -0.016 & -0.019 & 0.367 \end{pmatrix}.$$

The resulting estimates of coefficients in the Taylor equation (12) are n=0.3532, m=0.3816, p=0.1103, A=298.3155. The optimal solution of the GP with estimated parameters provided by GAMS CONOPT is

$$r^* = 1138.122\,(v^* = 357.551),\ f^* = 0.180,\ d^* = 0.500$$

and the optimal value $\varphi(\beta^*) = 3.453$. This means that the only difference in the optimal cutting conditions is due to the cutting speed. At this solution, the constraint (22) and the lower bound on d are active and the tool life $T(\beta^*) = 4.336$.

For the sensitivity analysis we start with

$$z_1(\beta^*) = \log r^* = 7.037\,(\log v^* = 5.879),\ z_2(\beta^*) = \log f^* = -1.714,$$

$$z_3(\beta^*) = \log d^* = -0.693.$$

Solving the system (15) we obtain

$$\frac{\partial \boldsymbol{z}(\beta^*)}{\partial \beta} = \begin{pmatrix} 0.52259 & 3.97761 & -0.90567 & -0.36216 \\ -0.01817 & -0.13831 & 0.03149 & 0.01259 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}.$$

The variance matrix of logarithms of the estimated optimal solution $\boldsymbol{z}(\beta^*)$, see (18), equals

$$\begin{pmatrix} 0.19093 & -0.00664 & 0.0 \\ -0.00664 & 0.00023 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}$$

and the variance of the estimated minimal costs, see (19), is 0.04695.

Hence, the 2σ confidence interval for the mimimal cost is $(2.9632, 3.8298)$, and 2σ confidence intervals for the optimal speed and feed can be constructed similarly as in [7], whereas the depth of the cut d is kept on its lower bound. The 2σ confidence interval for logarithm of the tool life computed for the optimal setup of cutting conditions follows from the regression analysis connected with the Taylor equation, see (23).

The differences in the optimal cutting conditions and in the optimal cost obtained for standard and estimated parameter values seem to be negligible. Notice, however, that the bounds of the confidence interval for the optimal value relate to the optimal cost of cutting one part of a large series, say several thousands of pieces, which makes the confidence interval width information an important ingredient in the analysis of economic features of the metal cutting problem.

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Jitka Dupačová, Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics – Charles University, Sokolovská 83, 186 75 Praha 8. Czech Republic.
e-mail: dupacova@karlin.mff.cuni.cz