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BOUNDS OF MODULUS OF EIGENVALUES BASED ON STEIN EQUATION

Guang-Da Hu and Qiao Zhu

This paper is concerned with bounds of eigenvalues of a complex matrix. Both lower and upper bounds of modulus of eigenvalues are given by the Stein equation. Furthermore, two sequences are presented which converge to the minimal and the maximal modulus of eigenvalues, respectively. We have to point out that the two sequences are not recommendable for practical use for finding the minimal and the maximal modulus of eigenvalues.

Keywords: eigenvalues, lower and upper bounds, Stein equation

Classification: 65F10, 65F15

1. INTRODUCTION

Spectral radiuses of different types of matrices such as nonnegative matrices [13], nonnegative irreducible matrices [4, 10, 11], H-matrices [8], product of matrices [2] and component-wise product of matrices [12], have been investigated. However, to authors' knowledge, for arbitrary matrix, there are few results presented to estimate the eigenvalues by the lower and upper bounds of their modulus. This is the origin of this paper.

Recently, by applying the relationship between the weighted logarithmic matrix norm and Lyapunov equation [5, 6], a bound of the maximal real part of any real matrix was obtained in [7]. Following the ideas employed in [5], [6] and [7], a upper bound of the spectral radius of any real matrix is given in [14] on the basis of the relationship between the weighted matrix norm and the discrete Lyapunov equation (Stein equation). Furthermore, an iterative scheme to estimate the spectral radius of any real matrix is also obtained in [14]. In this paper, along the line of [14], both lower and upper bounds of modulus of eigenvalues are given by the Stein equation. Furthermore, two sequences are presented which converge to the minimal and the maximal modulus of eigenvalues, respectively. We have to point out that the two sequences are not recommendable for practical use for finding the minimal and the maximal modulus of eigenvalues because they are many times more expensive than the standard method for finding eigenvalues (QR method), see Chapter 7 of [3].

2. PRELIMINARIES

In this paper, $(\cdot,\cdot)$ denotes an inner product on $\mathbf{C}^n$ and $\|\cdot\|$ the corresponding inner product norm. Let H be a positive definite matrix. The function $(\cdot,\cdot)_{(H)}$ defined on $\mathbf{C}^n$ by $(x,y)_{(H)} = y^*Hx$ is said to be weight H inner product in order to distinguish from the standard inner product $(x,y)_{(I)} = y^*x$, where I is the unit matrix. For any matrix F, F^* stands for the conjugate transpose, $\lambda_i(F)$ the $i-$ th eigenvalue, the spectral radius $\rho(F) = \max_i |\lambda_i(F)|$, and $\mathrm{vex}(F)$ the vex-function of matrix F which is the vector formed by stacking the columns of F into one long vector. The symbol $\otimes$ stands for the Kronecker product [9].

In this section, several definitions and lemmas are given. They will be used to prove main results of this paper.

Let matrix $A = [a_{ij}] \in \mathbf{C}^{n\times n}$. Several important examples of matrix norms defined on $\mathbf{C}^{n\times n}$ are as follows. The maximum column sum matrix norm $\|\cdot\|_1$, the maximum row sum matrix norm $\|\cdot\|_\infty$ and the spectral norm $\|\cdot\|_2$ are

$$\begin{aligned}\|A\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|,\\ \|A\|_\infty &= \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|,\end{aligned}$$

and

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)},$$

respectively. The above three formulas can be found in [9].

Lemma 2.1. (Lancaster [9]) If $A \in \mathbf{C}^{n\times n}$, then, for any matrix norm $\|\cdot\|$,

$$\rho(A) \le \|A\|. \tag{1}$$

Definition 2.2. (Lancaster [9]) A matrix $A \in \mathbf{C}^{n\times n}$ is said to be stable with respect to the unit circle if all its eigenvalues $\lambda_i(A)(1 \le i \le n)$ lie inside the unit circle, that is, $|\lambda_i(A)| < 1$ for all $1 \le i \le n$.

Lemma 2.3. (Lancaster [9]) Let $A, V \in \mathbf{C}^{n\times n}$ and let V be positive definite. The matrix A is stable with respect to the unit circle, if and only if there is a positive definite matrix H satisfying Stein equation

$$H - A^*HA = V. \tag{2}$$

Definition 2.4. (Hu and Liu [6]) For any vector x, any matrix A and any positive definite matrix H, the weight H norm of x and weight H norm of A are defined, respectively, by

$$\|x\|_{(H)} = \sqrt{x^*Hx}, \qquad \|A\|_{(H)} = \max_{x\neq 0}\frac{\|Ax\|_{(H)}}{\|x\|_{(H)}}. \tag{3}$$

For a stable matrix, we can obtain a weight H matrix norm which is less than 1 by the following lemma.

Lemma 2.5. If a matrix $A \in \mathbf{C}^{n\times n}$ is stable, then there is a weight H matrix norm such that

$$\|A\|_{(H)} = \sqrt{1 - \frac{1}{\rho(H)}}, \tag{4}$$

where the positive definite matrix H satisfies the following Stein equation

$$H - A^* H A = I. \tag{5}$$

P r o o f. Because of the condition $\rho(A) < 1$, according to Lemma 2.3, there is a positive definite matrix H satisfying $A^*HA - H = -I$. By Definition 2.4, we can get that

$$\begin{aligned} \|A\|^2_{(H)} &= \max_{x\neq 0} \frac{\|Ax\|^2_{(H)}}{\|x\|^2_{(H)}} = \max_{x\neq 0} \frac{x^*A^*HAx}{x^*Hx} \\ &= \max_{x\neq 0} \frac{x^*(H-I)x}{x^*Hx} = \max_{x\neq 0}(1 - \frac{x^*x}{x^*Hx}) = 1 - \frac{1}{\rho(H)} \end{aligned}$$

which implies the assertion (4). □

From the above lemma, we can immediately obtain the following lemma.

Lemma 2.6. For matrix $A \in \mathbf{C}^{n\times n}$, there is a weight H matrix norm such that

$$\|A\|_{(H)} = \gamma\sqrt{1 - \frac{1}{\rho(H)}}. \tag{6}$$

where the constant $\gamma > \rho(A)$ and the positive definite matrix H satisfies the following Stein equation

$$H - \tilde{A}^* H \tilde{A} = I, \tag{7}$$

where $\tilde{A} = A/\gamma$.

Remark 2.7. For $A \neq 0$, we have $\rho(H) > 1$ from formula (6) and $\|A\|_{(H)} \neq 0$. Here H satisfies Eq. (7). Hence we obtain that

$$\|H\|_m \geq \rho(H) > 1,$$

where $m = 1, 2$ and ∞.

Remark 2.8. When A is a real matrix, the above Lemmas 2.5 and 2.6 are given in [14].

3. MAIN RESULTS

In this section, the main results of this paper are presented. From Lemmas 2.5 and 2.6, we have the following results.

Theorem 3.1. Let $A \in \mathbf{C}^{n\times n}$.

1. If A is stable, then there is a weight H matrix norm such that

$$\rho(A) \le \|A\|_{(H)} \le \sqrt{1 - \frac{1}{\|H\|_1}}, \tag{8}$$

where H satisfies Eq. (5) in Lemma 2.5.

2. If A is unstable, and $\rho(A) < \|A\|_1$, then there is a weight H matrix norm such that

$$\rho(A) \le \|A\|_{(H)} \le \|A\|_1\sqrt{1 - \frac{1}{\|H\|_1}}, \tag{9}$$

where H satisfies Eq. (7) and $\gamma = \|A\|_1$ in Lemma 2.6.

P r o o f. From formula (4),

$$\|A\|_{(H)} = \sqrt{1 - \frac{1}{\rho(H)}}.$$

By Lemma 2.1, $\rho(H) \le \|H\|_1$. We obtain that

$$\rho(A) \le \|A\|_{(H)} = \sqrt{1 - \frac{1}{\rho(H)}} \le \sqrt{1 - \frac{1}{\|H\|_1}}.$$

Formula (8) is proved. Similarly, formula (9) can be derived. The proof is completed. □

Remark 3.2. The spectral radius of positive definite matrix H is equal to its maximal eigenvalue.

Now we discuss the lower bound of modulus of eigenvalues. Assume that the eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_n$. If $\det A \ne 0$, the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots$ $\dots, \frac{1}{\lambda_n}$, respectively. This result implies that we can get the lower bound of modulus of eigenvalues of A by estimating the spectral radius of A^{-1}.

Theorem 3.3. Let $A \in \mathbf{C}^{n\times n}$. Assume that $\delta < \min_{1\le i\le n} |\lambda_i(A)|$ for some $\delta > 0$, then there is a weight L matrix norm such that

$$\|A^{-1}\|_{(L)} = \frac{1}{\delta}\sqrt{1 - \frac{1}{\rho(L)}} \le \frac{1}{\delta}\sqrt{1 - \frac{1}{\|L\|_1}}, \tag{10}$$

where the positive definite matrix L satisfies the following Stein equation

$$A^*LA - \delta^2 L = A^*A. \tag{11}$$

Furthermore,

$$\min_{1\le i\le n} |\lambda_i(A)| \ge \frac{\delta}{\sqrt{1-\frac{1}{\rho(L)}}} \ge \frac{\delta}{\sqrt{1-\frac{1}{\|L\|_1}}}. \tag{12}$$

P r o o f. Since $|\lambda_i(A)| > \delta$, then $1/|\lambda_i(A)| < 1/\delta$. Since $1/\lambda_i(A)$ is the eigenvalues of A^{-1} for $i = 1, 2, \ldots, n$, we get

$$\frac{1}{|\min_{1\le i\le n} \lambda_i(A)|} = \rho(A^{-1}) < \frac{1}{\delta}, \tag{13}$$

From Lemma 2.6, we have

$$\rho(A^{-1}) \le \|A^{-1}\|_{(L)} = \frac{1}{\delta}\sqrt{1-\frac{1}{\rho(L)}}, \tag{14}$$

where L satisfies the Stein equation

$$L - \tilde{A}^*L\tilde{A} = I, \tag{15}$$

here $\tilde{A} = A^{-1}\delta$ is stable.

By Eq. (15), we have the Stein equation

$$A^*LA - \delta^2 L = A^*A. \tag{16}$$

By Eqs. (13) and (14), we have

$$\min_{1\le i\le n} |\lambda_i(A)| \ge \frac{\delta}{\sqrt{1-\frac{1}{\rho(L)}}} \ge \frac{\delta}{\sqrt{1-\frac{1}{\|L\|_1}}}. \tag{17}$$

□

By Lemma 2.6 and Theorems 3.1 and 3.3, we have the following result to estimate modulus of eigenvalues.

Theorem 3.4. For matrix $A \in \mathbf{C}^{n\times n}$, if the conditions of Lemma 2.6 and Theorem 3.3 are satisfied, then

$$\frac{\delta}{\sqrt{1-\frac{1}{\|L\|_1}}} \le |\lambda_i(A)| \le \gamma\sqrt{1-\frac{1}{\|H\|_1}}, \quad i = 1, 2, \ldots, n, \tag{18}$$

where $\gamma = 1$ if A is stable.

Remark 3.5. In [14], the lower bound of modulus of eigenvalues is not considered. By the lower bound, more information on eigenvalues can be known.

In the following, two sequences are presented which converge to the minimal and the maximal modulus of eigenvalues, respectively. Before the proof of this result, we discuss briefly how to solve the Stein equation. For a stable matrix A, the Stein equation (4) has a unique solution [9] which will be used in the sequel. An efficient computational method for solving the Stein equation is provided in [1].

Theorem 3.6. Let $A \in \mathbf{C}^{n\times n}$, the following two sequences can be used to estimate the upper and lower bounds of modulus of its eigenvalues, respectively.

1. There are a sequence of positive definite matrices $\{H_k\}$ and a sequence of numbers $\{\alpha_k\}$ as follows:

$$\alpha_0 = \|A\|_1, \tag{19}$$

$$\alpha_0^2 H_1 - A^* H_1 A = \alpha_0^2 I, \tag{20}$$

$$\alpha_1 = \alpha_0\sqrt{1 - \tfrac{1}{\|H_1\|_1}}, \tag{21}$$

$$\vdots$$

$$\alpha_k^2 H_{k+1} - A^* H_{k+1} A = \alpha_k^2 I, \tag{22}$$

$$\alpha_{k+1} = \alpha_k\sqrt{1 - \tfrac{1}{\|H_{k+1}\|_1}}, \tag{23}$$

such that for $i = 1, \dots, n$,

$$|\lambda_i(A)| \le \alpha_k, \quad \text{and} \quad \lim_{k\to\infty} \alpha_k = \rho(A) = \max_{1\le i\le n} |\lambda_i(A)|. \tag{24}$$

2. If A is nonsingular, there are a sequence of positive definite matrices $\{L_k\}$ and a sequence of numbers $\{\beta_k\}$ as follows.

$$\beta_0 = 1/\|A^{-1}\|_1, \tag{25}$$

$$A^* L_1 A - \beta_0^2 L_1 = A^* A, \tag{26}$$

$$\beta_1 = \beta_0/\sqrt{1 - \tfrac{1}{\|L_1\|_1}}, \tag{27}$$

$$\vdots$$

$$A^* L_{k+1} A - \beta_k^2 L_{k+1} = A^* A, \tag{28}$$

$$\beta_{k+1} = \beta_k/\sqrt{1 - \tfrac{1}{\|L_{k+1}\|_1}}, \tag{29}$$

such that for $i = 1, \dots, n$,

$$|\lambda_i(A)| \ge \beta_k \quad \text{and} \quad \lim_{k\to\infty} \beta_k = \min_{1\le i\le n} |\lambda_i(A)|. \tag{30}$$

Proof. It is known that $\alpha_0 = \|A\|_1 \ge \rho(A)$. If $\alpha_0 = \rho(A)$, then the maximum of modulus of eigenvalues is obtained and the theorem is proved. Otherwise, $\alpha_0 > \rho(A)$ and $\rho(\frac{A}{\alpha_0}) < 1$. Let $\gamma = \alpha_0$ in Lemma 2.6, we obtain

$$\|A\|_{(H_1)} = \alpha_0\sqrt{1 - \frac{1}{\rho(H_1)}} \le \alpha_0\sqrt{1 - \frac{1}{\|H_1\|_1}},$$

since $\rho(H_1) \leq \|H_1\|_1$. Here H_1 satisfies

$$\alpha_0^2 H_1 - A^* H_1 A = \alpha_0^2 I$$

which is equivalent to Eq. (7) in Lemma 2.6 for $\gamma = \alpha_0$.

From the properties of the matrix norm, we have

$$\rho(A) \leq \|A\|_{(H_1)} \leq \alpha_0 \sqrt{1 - \frac{1}{\|H_1\|_1}} = \alpha_1.$$

If $\alpha_1 = \rho(A)$, then the maximal module of eigenvalues is obtained and the theorem is obtained. Otherwise, we obtain the positive definite matrix H_2 such that

$$\|A\|_{H_2} = \alpha_1 \sqrt{1 - \frac{1}{\rho(H_2)}} \leq \alpha_0 \sqrt{1 - \frac{1}{\|H_2\|_1}} = \alpha_2,$$

where H_2 satisfies the Eq. (7) in Lemma 2.6 for $\gamma = \alpha_1$.

We can repeat the above process for $k \geq 2$, then the theorem is proved if $\rho(A) = \alpha_k$, or Eqs. (22) and (23) hold if $\rho(A) < \alpha_k$. Thus, we only need to prove $\lim_{k\to\infty} \alpha_k \to \rho(A)$ under the condition $\rho(A) < \alpha_k$ for all $k \geq 0$. In the case, $\{\alpha_k\}$ is a monotone decreasing sequence and has a limit since it is bounded. Let α be the limit of α_k as $k \to \infty$. Assume that $\alpha \neq \rho(A)$, then we can get $\rho(A) < \alpha$ since $\rho(A) < \alpha_k$ for $k \geq 0$. Thus, the Stein equation

$$H - \tilde{A}^* H \tilde{A} = I$$

where $\tilde{A} = A/\alpha$, and its equivalent one

$$G_\infty x = c$$

have unique solutions $H > 0$ and x, respectively, where

$$G_\infty = I - \tilde{A}^T \otimes \tilde{A}^*, \quad x = \text{vex}(H), \quad c = \text{vex}(I).$$

Similarly, since $\rho(A) < \alpha_k$ for $k \geq 0$, the following equivalent equations

$$H_{k+1} - \tilde{A_k}^* H_{k+1} \tilde{A}_k = I$$

and

$$G_k x_k = c$$

have unique solutions $H_{k+1} > 0$ and x_k, respectively, where

$$\tilde{A}_k = A/\alpha_k, \quad G_k = I - \tilde{A_k}^T \otimes \tilde{A_k}^*, \quad x_k = \text{vex}(H_{k+1}).$$

Since $\lim_{k\to\infty} \alpha_k = \alpha$, $\lim_{k\to\infty} G_k = G_\infty$. It follows that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} H_k = H$. From (23), we have

$$\alpha = \alpha \sqrt{1 - \frac{1}{\rho(H)}}.$$

which contradicts the assumption $\alpha \neq \rho(A)$. Clearly, the assertion 1 is proved.

The proof of the assertion 2 is as follows. Notice that

$$\min_{1\leq i\leq n} |\lambda_i(A)| \geq 1/\|A^{-1}\|_1.$$

Let $\delta = 1/\|A^{-1}\|_1$ in Theorem 3.3. The proof of the assertion 2 is similar to the assertion 1. Thus, the proof is completed. □

Remark 3.7. For computing the modulus of eigenvalues, the computational effort of the sequences in Theorem 3.6 is many times more expensive than the standard method (QR method) for finding eigenvalues, see Chapter 7 of [3]. Hence the two sequences in Theorem 3.6 are not recommendable for practical use for finding the minimal and the maximal modulus of eigenvalues. In this paper, we only emphasize the theoretical aspect of the results derived.

The following result shows the partial order relations among the weight matrices in Theorem 3.6. The partial order relation $H_{k+1} > H_k$ means that the matrix $(H_{k+1} - H_k)$ is positive definite.

Theorem 3.8. Let $\det A \neq 0$. For the two sequences of positive definite matrices $\{H_k\}$ and $\{L_k\}$ in Theorem 3.6, if $\alpha_{k+1} < \alpha_k$ and $\beta_{k+1} > \beta_k$, then for any $k \geq 0$,

$$H_{k+1} > H_k \tag{31}$$

and

$$L_{k+1} > L_k. \tag{32}$$

P r o o f. We only give the proof of (31), since the proof of (32) is similar. From Theorem 3.6, we see that

$$\begin{aligned} \alpha_{k-1}^2 H_k - A^* H_k A &= \alpha_{k-1}^2 I, \\ \alpha_k^2 H_{k+1} - A^* H_{k+1} A &= \alpha_k^2 I, \end{aligned}$$

they can be rearranged to

$$\begin{aligned} H_k - \frac{1}{\alpha_{k-1}^2} A^* H_k A &= I, \\ H_{k+1} - \frac{1}{\alpha_k^2} A^* H_{k+1} A &= I. \end{aligned}$$

From above two equations, we have that

$$(H_{k+1} - H_k) - \frac{A^*}{\alpha_k}(H_{k+1} - H_k)\frac{A}{\alpha_k} = \left(\frac{1}{\alpha_k^2} - \frac{1}{\alpha_{k-1}^2}\right) A^* H_k A. \tag{33}$$

Since $\alpha_k < \alpha_{k-1}$ and $\det A \neq 0$, then

$$\left(\frac{1}{\alpha_k^2} - \frac{1}{\alpha_{k-1}^2}\right) A^* H_k A > 0. \tag{34}$$

By Lemma 2.3 and (34), since the matrix $\frac{A}{\alpha_k}$ is stable, (33) implies

$$H_{k+1} - H_k > 0.$$

The proof is completed. □

4. CONCLUSION

Both lower and upper bounds of modulus of eigenvalues are given by the Stein equation. Furthermore, two sequences are presented which converge to the minimal and the maximal modulus of eigenvalues, respectively. We only emphasize the theoretical aspect of the results derived. The results are not recommended for practical computations.

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