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A NOTE ON FUSION BANACH FRAMES

S. K. KAUSHIK AND VARINDER KUMAR

ABSTRACT. For a fusion Banach frame $(\{G_n, v_n\}, S)$ for a Banach space E , if $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* , then $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is called a fusion bi-Banach frame for E . It is proved that if E has an atomic decomposition, then E also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of a fusion bi-Banach frame is given. Finally, a characterization of fusion bi-Banach frames is given.

1. INTRODUCTION

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [5] in 1952 and re-introduced in 1986 by Daubechies, Grossmann and Meyer [4]. Casazza [2] and Benedetto and Fickus [1] have studied frames in finite dimensional spaces which attracted more attention due to their use in signal processing. Frames are now used as a tool in many areas like data compression, sampling theory, optics, filter banks, signal detection, time-frequency analysis etc.

The concept of frames in Hilbert spaces was extended to Banach spaces by Feichtinger and Gröchenig [6] who introduced the concept of atomic decompositions in Banach spaces. This concept was further generalized by Gröchenig [7] who introduced the notion of Banach frames for Banach spaces. Jain et al. [9], generalized Banach frames in Banach spaces and introduced frames of subspaces (Fusion Banach frames) for Banach spaces. They gave the following definition of a fusion Banach frame.

Definition 1.1 ([9]). Let E be a Banach space. Let $\{G_n\}$ be a sequence of non-trivial subspaces of E and $\{v_n\}$ be a sequence of bounded linear projections such that $v_n(E) = G_n$, $n \in \mathbb{N}$. We associate a Banach space \mathcal{A} and an operator $S : \mathcal{A} \rightarrow E$ with the space E . Then $(\{G_n, v_n\}, S)$ is called a *frame of subspaces (fusion Banach frame)* for E with respect to \mathcal{A} if

- (i) $\{v_n(x)\} \in \mathcal{A}$, for all $x \in E$,
- (ii) there exist constants A, B ($0 < A \leq B < \infty$) such that

$$A\|x\|_E \leq \|\{v_n(x)\}\|_{\mathcal{A}} \leq B\|x\|_E, \quad x \in E,$$

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(iii) S is a bounded linear operator such that

$$S(\{v_n(x)\}) = x, \quad x \in E.$$

The following lemma, proved in [9], is used in the sequel

Lemma 1.2. *Let $\{G_n\}$ be a sequence of non-trivial subspaces of E and $\{v_n\}$ be a sequence of bounded linear projections with $v_n(E) = G_n, n \in \mathbb{N}$. If $\{v_n\}$ is total over E , i.e., $\{x \in E : v_n(x) = 0, \text{ for all } n \in \mathbb{N}\} = \{0\}$, then $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ is a Banach space with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E, x \in E$.*

For other related notions on frames in Banach spaces one may refer to [3, 8, 10, 11].

In the present paper, we introduce fusion bi-Banach frames for a Banach space E . We prove that if E has an atomic decomposition, then E also has a fusion bi-Banach frame. Also, a sufficient condition for the existence of fusion bi-Banach frames is given. Finally, a characterization of fusion bi-Banach frames is obtained.

2. MAIN RESULTS

One may observe that, if $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to some associated Banach space \mathcal{A} , then there may not exist a Banach space \mathcal{A}_1 associated with E^* together with an operator $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 .

In this regard, we have the following examples

Example 2.1. Consider the Banach space

$$E = \ell^\infty(X) = \{\{x_n\} : x_n \in X; \sup_{1 \leq n < \infty} \|x_n\|_X < \infty\}$$

equipped with the norm $\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_X, \{x_n\} \in E$, where $(X, \|\cdot\|)$ is a Banach space. For each $n \in \mathbb{N}$, define $G_n = \{\delta_n^x : x \in X\}$ and $v_n(x) = \delta_n^{x_n}, x = \{x_n\} \in E$, where $\delta_n^x = (0, 0, \dots, 0, x, 0, \dots)$ for all $n \in \mathbb{N}$ and $x \in X$. Then

\downarrow
n-th place

by Lemma 1.2, there exist an associated Banach space $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ with norm $\|\{v_n(x)\}\|_{\mathcal{A}} = \|x\|_E, x \in E$ together with an operator $S: \mathcal{A} \rightarrow E$ given by $S(\{v_n(x)\}) = x, x \in E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} . But, there does not exist a Banach space \mathcal{A}_1 associated with E^* together with an operator $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 . For otherwise, $[\bigcup_{n=1}^\infty G_n] = E$, which is not true.

Example 2.2. Let E be a Banach space defined as

$$E = c_0(X) = \{\{x_n\} : x_n \in X; \lim_{n \rightarrow \infty} \|x_n\|_X = 0\}$$

equipped with the norm given by

$$\|\{x_n\}\|_E = \sup_{1 \leq n < \infty} \|x_n\|_X, \quad \text{where } (X, \|\cdot\|) \text{ is a Banach space.}$$

Define a sequence $\{G_n\}$ of subspaces of E by

$$G_{2n-1} = \{\delta_{2n-1}^x - 2^{n-1}\delta_{2n}^x : x \in X\}$$

$$G_{2n} = \{\delta_{2n}^x : x \in X\}.$$

Also define operators v_n on E by

$$v_{2n-1}(x) = \delta_{2n-1}^{x_{2n-1}} - 2^{n-1}\delta_{2n}^{x_{2n-1}}$$

$$v_{2n}(x) = \delta_{2n}^{2^{n-1}x_{2n-1}+x_{2n}} \quad \text{for all } x = \{x_n\} \in E \text{ and } n \in \mathbb{N}.$$

Then by Lemma 1.2 there exist an associated Banach space \mathcal{A} and an operator $S: \mathcal{A} \rightarrow E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} .

If $[\bigcup_{n=1}^{\infty} G_n] \neq E$, then there exists $0 \neq f = \{f_i\} \in E^*$ such that $f(y) = 0$ for all $y \in G_n, n \in \mathbb{N}$. This would imply $f_n = 0$ for all $n \in \mathbb{N}$ and hence $f = 0$. Therefore, by Lemma 1.2 again, there exist a Banach space \mathcal{A}_1 associated to E^* and an operator $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 .

In view of the above discussion, we define the following

Definition 2.3. Let E be a Banach space. Let $\{G_n\}$ be a sequence of non-trivial subspaces of E and $\{v_n\}$ be a sequence of bounded linear projections such that $v_n(E) = G_n, n \in \mathbb{N}$. If there exist Banach spaces \mathcal{A} and \mathcal{A}_1 associated with E and E^* respectively and operators $S: \mathcal{A} \rightarrow E$ and $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} and $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 , then we call the system $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ a *fusion bi-Banach frame* for E

In view of Remark 3.2.1 in [9], we have

Every reflexive Banach space has a fusion bi-Banach frame.

Recall that if E is a Banach space and E_d is an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}, \{x_n\}$ is a sequence in E and $\{f_n\}$ is a sequence in E^* , then the pair $(\{f_n\}, \{x_n\})$ is called an *atomic decomposition* for E with respect to E_d if

- (i) $\{f_n(x)\} \in E_d, x \in E;$
- (ii) there exist constants A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E;$$

- (iii) $x = \sum_{n=1}^{\infty} f_n(x)x_n, x \in E.$

The next result is regarding the existence of fusion bi-Banach frames for a Banach space having an atomic decomposition.

Theorem 2.4. *Let E be a Banach space. If E has an atomic decomposition, then it also has a fusion bi-Banach frame.*

Proof. Let $(\{f_n\}, \{x_n\})$ be an atomic decomposition for E with respect to E_d . Define $G_n = [x_n]$, $n \in \mathbb{N}$ and $v_n(x) = f_n(x)x_n$, $n \in \mathbb{N}$. Then there exist an associated Banach space $\mathcal{A} = \{\{v_n(x)\} : x \in E\}$ together with an operator $S: \mathcal{A} \rightarrow E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} . Further $[\bigcup_{n=1}^{\infty} G_n] = E$ (as $[x_n] = E$). So, $v_n^*(f) = 0$ for all $n \in \mathbb{N}$ imply $f = 0$, where $f \in E^*$. Thus, $\{v_n^*\}$ is total over E^* and so by Lemma 1.2, there exist an associated Banach space \mathcal{A}_1 and an operator $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 . Hence, $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for E . \square

Next, we observe that if E be a Banach space and $\{G_n\}$ be a sequence of non-trivial subspaces of E with associated sequence of projections $\{v_n\}$ with $v_n(E) = G_n$, $n \in \mathbb{N}$, then it is possible that there exist a Banach space \mathcal{A}_1 associated with E^* together with a bounded linear operator $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 and there may not exist any Banach space \mathcal{A} associated with E together with an operator $S: \mathcal{A} \rightarrow E$ such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} . Indeed, let

$$E = \ell^2(X) = \left\{ \{x_n\} : x_n \in X; \sum_{n=1}^{\infty} \|x_n\|_X^2 < \infty \right\},$$

where $(X, \|\cdot\|)$ is a Banach space, equipped with the norm given by

$$\|\{x_n\}\|_E = \left(\sum_{n=1}^{\infty} \|x_n\|_X^2 \right)^{1/2}.$$

Define for $n \in \mathbb{N}$, $G_n = \{\delta_1^x + \delta_{n+1}^x : x \in X\}$ and $v_n(x) = \delta_1^{x_{n+1}} + \delta_{n+1}^{x_{n+1}}$, $x = \{x_n\} \in E$, where $\delta_n^x = (0, 0, \dots, 0, x, 0, \dots)$, $x \in X$.

\downarrow
 nth place

Then $[\bigcup_{n=1}^{\infty} G_n] = E$ and $v_i v_j = 0$ for all $i \neq j$.

But, since for any $0 \neq x \in X$, $\delta_1^x = (x, 0, 0, \dots) \in E$ is such that $v_n(\delta_1^x) = 0$, for all $n \in \mathbb{N}$, there exist no associated Banach space \mathcal{A} such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} . However, there exist a Banach space \mathcal{A}_0 and an operator $T: \mathcal{A}_0 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_0 .

In view of the above discussion, we prove the following result

Theorem 2.5. *Let E be a Banach space and $\{G_n\}$ be a sequence of subspaces of E with $[\bigcup_{n=1}^{\infty} G_n] = E$. Let $\{v_n\}$ be a sequence of projections on E satisfying $v_n(E) = G_n$, $n \in \mathbb{N}$ and $v_i v_j = 0$ for all $i \neq j$. Then there exist Banach spaces \mathcal{A} and \mathcal{A}_1 associated with E and E^* , respectively, and operators $S: \mathcal{A} \rightarrow E$ and $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame*

for E if every sequence $\{x_n\} \subset E$ such that $x_n \in G_n$ and $x_n \neq 0, n \in \mathbb{N}$ satisfies $\bigcap_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \dots] = \{0\}$.

Proof. Since $[\bigcup_{n=1}^{\infty} G_n] = E$, there exist an associated Banach space \mathcal{A}_1 and a bounded linear operator $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 . Let, if possible, there exist no Banach space \mathcal{A} associated with E such that $(\{G_n, v_n\}, S)$ is a fusion Banach frame for E with respect to \mathcal{A} where $S: \mathcal{A} \rightarrow E$ is a bounded linear operator. Now, since $[\bigcup_{n=1}^{\infty} G_n] = E$ and $v_i v_j = 0$ for all $i \neq j, u_n = \sum_{i=1}^n v_i$ is a bounded linear projection of E onto $[\bigcup_{i=1}^n G_i]$ along $[\bigcup_{i=n+1}^{\infty} G_i], n \in \mathbb{N}$. Write $E = [\bigcup_{i=1}^n G_i] \oplus [\bigcup_{i=n+1}^{\infty} G_i], n \in \mathbb{N}$. Then

$$\{x \in E : v_i(x) = 0, i = 1, 2, \dots, n\} = [\bigcup_{i=n+1}^{\infty} G_i], \quad n \in \mathbb{N}.$$

Since $(\{G_n, v_n\}, S)$ is not a fusion Banach frame for E with respect to any associated Banach space, there exists $0 \neq x \in \bigcap_{n=1}^{\infty} [\bigcup_{i=n+1}^{\infty} G_i]$. So, there exists

$y_1 = \sum_{i=1}^{m_1} z_i$ where $z_i \in G_i (1 \leq i \leq m_1)$ such that $\text{dist}(x, y_1) < 1$, that is,

$\text{dist}(x, [\bigcup_{i=1}^{m_1} G_i]) < 1$. Also, $x \in [\bigcup_{i=m_1+1}^{\infty} G_i]$. So, we can choose $m_2 > m_1$ and

$y_2 = \sum_{i=m_1+1}^{m_2} z_i$, where $z_i \in G_i (m_1 + 1 \leq i \leq m_2)$ such that $\text{dist}(x, [\bigcup_{i=m_1+1}^{m_2} G_i]) <$

$\frac{1}{2}$. Proceeding like this, for each $n \in \mathbb{N}$, we get a sequence $\{z_n\} \subset E$ and an increasing sequence $\{m_n\}$ of positive integers such that $z_n \in G_n, n \in \mathbb{N}$ and $\text{dist}(x, [\bigcup_{i=m_{n-1}+1}^{m_n} G_i]) < \frac{1}{n}$.

Thus $x \in [z_{n+1}, z_{n+2}, \dots], n \in \mathbb{N}$. Consider a sequence $\{x_n\} \subset E$ with $0 \neq x_n \in G_n, n \in \mathbb{N}$ such that $x_n = z_n$ whenever $z_n \neq 0$. Then $x \in [x_{n+1}, x_{n+2}, \dots], n \in \mathbb{N}$.

Hence $\bigcap_{n=1}^{\infty} [x_{n+1}, x_{n+2}, \dots] \neq \{0\}$. □

Finally, we give a characterization of fusion bi-Banach frames in terms of a sequence in bv_0 , where bv_0 is the linear space of all sequences $\{\alpha_n\}$ of scalars with

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \text{ and for which the norm } \|\{\alpha_n\}\| = \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| \text{ is finite.}$$

Theorem 2.6. *Let E be a Banach space and $(\{G_n, v_n\}, S)$ be a fusion Banach frame for E , where the projections $\{v_n\}$ on E are such that $v_i v_j = 0$ for all $i \neq j$. Then $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for E if and only*

if for every $x \in E$, there exist $\{\alpha_j\} \in bv_0$ and $z \in E$ such that $v_n(x) = \alpha_n v_n(z)$, $n \in \mathbb{N}$ and $\sup_{1 \leq n < \infty} \left\| \sum_{i=1}^n v_i(z) \right\| < \infty$.

Proof. Let $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ be a fusion bi-Banach frame for E . For each $k \in \mathbb{N}$, write $u_k = \sum_{i=1}^k v_i$. Then $\lim_{k \rightarrow \infty} u_k(x) = x$, $x \in E$. Therefore, there exists a sequence $\{m_n\}$ of positive integers such that

$$\|x - u_k(x)\| < \frac{1}{4^{n+1}}, \quad k \geq m_n, \quad n \in \mathbb{N}.$$

Take $y_n = \sum_{i=m_{n-1}+1}^{m_n} v_i(x)$, $n \in \mathbb{N}$. Then $\|y_n\| \leq \frac{2}{4^n}$, $n \in \mathbb{N}$.

So, $\sum_{n=1}^{\infty} 2^{n-1} \|y_n\| \leq \sum_{n=1}^{\infty} 2^{-n}$. Thus, the series $\sum_{n=1}^{\infty} 2^{n-1} y_n$ converges.

Put $z = \sum_{n=1}^{\infty} 2^{n-1} y_n$ and $\alpha_j = 2^{1-n}$, $m_{n-1} + 1 \leq j \leq m_n$, $n \in \mathbb{N}$.

Therefore, $\{\alpha_j\} \in bv_0$. Also, we have

$$v_j(z) = 2^{n-1} v_j(x), \quad m_{n-1} + 1 \leq j \leq m_n, \quad n \in \mathbb{N}.$$

Hence, $v_j(x) = \alpha_j v_j(z)$, $j \in \mathbb{N}$.

Conversely, for integers $p < q$, we have

$$\begin{aligned} \left\| \sum_{i=p}^q v_i(x) \right\| &= \left\| \sum_{i=p}^q \alpha_i \left(\sum_{j=1}^i v_j(z) - \sum_{j=1}^{i-1} v_j(z) \right) \right\| \\ &\leq \left(|\alpha_p| + \sum_{i=p}^{q-1} |\alpha_i - \alpha_{i+1}| + |\alpha_q| \right) \sup_{1 \leq n < \infty} \left\| \sum_{j=1}^n v_j(z) \right\| \end{aligned}$$

Since, $\{\alpha_j\} \in bv_0$, $\left\{ \sum_{i=1}^n v_i(x) \right\}$ is a Cauchy sequence and hence converges.

Also, since $\{v_n\}$ is total on E and

$$v_j \left(x - \lim_{n \rightarrow \infty} \sum_{i=1}^n v_i(x) \right) = 0, \quad \text{for all } j \in \mathbb{N},$$

it follows that $x = \lim_{n \rightarrow \infty} \sum_{i=1}^n v_i(x)$. Therefore, $\left[\bigcup_{n=1}^{\infty} G_i \right] = E$. Thus, $\{v_n^*\}$ is total over E^* and so by Lemma 1.2, there exist a Banach space \mathcal{A}_1 associated with E^* and an operator $T: \mathcal{A}_1 \rightarrow E^*$ such that $(\{v_n^*(E^*), v_n^*\}, T)$ is a fusion Banach frame for E^* with respect to \mathcal{A}_1 . Hence, $(\{G_n, v_n\}, S; \{v_n^*(E^*), v_n^*\}, T)$ is a fusion bi-Banach frame for E . □

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