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LINEAR LIFTINGS OF SKEW SYMMETRIC TENSOR FIELDS OF TYPE $(1,2)$ TO WEIL BUNDLES

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Abstract. The paper contains a classification of linear liftings of skew symmetric tensor fields of type $(1,2)$ on n-dimensional manifolds to tensor fields of type $(1,2)$ on Weil bundles under the condition that $n \geqslant 3$. It complements author's paper "Linear liftings of symmetric tensor fields of type $(1,2)$ to Weil bundles" (Ann. Polon. Math. *92*, 2007, pp. 13–27), where similar liftings of symmetric tensor fields were studied. We apply this result to generalize that of author's paper "Affine liftings of torsion-free connections to Weil bundles" (Colloq. Math. *114*, 2009, pp. 1–8) and get a classification of affine liftings of all linear connections to Weil bundles.

Keywords: natural operator, Weil bundle

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Let A be a Weil algebra inducing the Weil functor T^A (see [15], [6], [12], [8]) and let n be a non-negative integer. We will denote by $\operatorname{Te} M$ the vector space of all tensor fields of type $(1,2)$ on a manifold M and by $\operatorname{SkTe} M$ the subspace of $\operatorname{Te} M$ consisting of all skew symmetric tensor fields.

A linear lifting of skew symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A is, by definition, a family of linear maps $L_M\colon \operatorname{SkTe} M \to \operatorname{Te} T^A M$ indexed by all n-dimensional manifolds and satisfying

$$L_M(\varphi^* t) = (T^A\varphi)^*(L_N t) \tag{1}$$

for all n-dimensional manifolds M, N, every embedding $\varphi\colon M \to N$ and every $t \in \operatorname{SkTe} N$ (see [10] for the general theory of natural operators).

Our purpose is to describe explicitly all such liftings (for classifications of liftings of tensor fields of some other types to Weil bundles see for example [9], [13], [1], [2], [11]).

In [3] we have constructed six kinds of liftings of symmetric tensor fields. Quite similarly we can now construct six kinds of liftings of skew symmetric tensor fields. So, if $E \in A$, $F\colon A \to A$ is $\mathbb{R}$-linear and $G, H\colon A \times A \to A$ are $\mathbb{R}$-bilinear and such that

$$G(u, vw) = G(u, v)w + G(u, w)v, \tag{2}$$

$$H(u, vw) = H(uv, w) + H(uw, v) \tag{3}$$

for all $u, v, w \in A$, then there are unique linear liftings $\overline{E}$, $\overline{F}^{\mathrm{L}}$, $\overline{F}^{\mathrm{R}}$, $\overline{G}$, $\overline{H}^{\mathrm{L}}$, $\overline{H}^{\mathrm{R}}$ of skew symmetric tensor fields of type $(1, 2)$ to tensor fields of type $(1, 2)$ on T^A such that

$$(\overline{E}_U t)^p_X(Y, Z) = E \cdot (T^A t^p_{qr})(X) \cdot Y^q \cdot Z^r, \tag{4}$$

$$(\overline{F}^{\mathrm{L}}_U t)^p_X(Y, Z) = F((T^A t^q_{qr})(X) \cdot Y^r) \cdot Z^p, \tag{5}$$

$$(\overline{F}^{\mathrm{R}}_U t)^p_X(Y, Z) = F((T^A t^q_{qr})(X) \cdot Z^r) \cdot Y^p, \tag{6}$$

$$(\overline{G}_U t)^p_X(Y, Z) = \frac{1}{2} G\Big(\Big(T^A\Big(\frac{\partial t^q_{qs}}{\partial x^r} - \frac{\partial t^q_{qr}}{\partial x^s}\Big)\Big)(X) \cdot Y^r \cdot Z^s, X^p\Big), \tag{7}$$

$$(\overline{H}^L_U t)^p_X(Y, Z) = \frac{1}{2} H\Big(\Big(T^A\Big(\frac{\partial t^q_{qs}}{\partial x^r} - \frac{\partial t^q_{qr}}{\partial x^s}\Big)\Big)(X) \cdot Y^r, X^s\Big) \cdot Z^p, \tag{8}$$

$$(\overline{H}^{\mathrm{R}}_U t)^p_X(Y, Z) = \frac{1}{2} H\Big(\Big(T^A\Big(\frac{\partial t^q_{qs}}{\partial x^r} - \frac{\partial t^q_{qr}}{\partial x^s}\Big)\Big)(X) \cdot Z^r, X^s\Big) \cdot Y^p \tag{9}$$

for every open subset U of $\mathbb{R}^n$, every $t \in \operatorname{SkTe} U$, every $p \in \{1, \dots, n\}$, every $X \in T^A U$ and all $Y, Z \in A^n$.

We can now formulate our main result.

Theorem. *If $n \geqslant 3$, then for every linear lifting L of skew symmetric tensor fields of type $(1, 2)$ to tensor fields of type $(1, 2)$ on T^A there are unique $E \in A$, $\mathbb{R}$-linear $F, F'\colon A \to A$, $\mathbb{R}$-bilinear $G\colon A \times A \to A$ satisfying (2) and $\mathbb{R}$-bilinear $H, H'\colon A \times A \to A$ satisfying (3) such that*

$$L = \overline{E} + \overline{F}^{\mathrm{L}} + \overline{F'}^{\mathrm{R}} + \overline{G} + \overline{H}^{\mathrm{L}} + \overline{H'}^{\mathrm{R}}. \tag{10}$$

The above theorem is quite similar to that of [3], but its proof is not. It is based on the lemma below, which differs from that of [3]. Nevertheless, some parts of both the proofs coincide, so we will omit them and focus on those which are essentially different.

Lemma. *Let L and L' be two linear liftings of skew symmetric tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A. If $n \geqslant 2$, then*

$$L_{\mathbb{R}^n}\Big(x^1 \frac{\partial}{\partial x^1}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\Big) = L'_{\mathbb{R}^n}\Big(x^1 \frac{\partial}{\partial x^1}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\Big) \Longrightarrow L = L'.$$

P r o o f. Suppose that

$$L_{\mathbb{R}^n}\Big(x^1(\partial/\partial x^1)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\Big) = 0. \tag{11}$$

Let $\alpha \in \mathbb{N}^n$, where $\mathbb{N}$ denotes the set of all non-negative integers. According to the Peetre theorem (see [10]), it suffices to show that $L_{\mathbb{R}^n}(x^\alpha(\partial/\partial x^p)\, \mathrm{d}x^q \wedge \mathrm{d}x^r) = 0$ for all $p, q, r \in \{1, \dots, n\}$. Since for any permutation σ of $\{1, \dots, n\}$ we can take $\varphi\colon \mathbb{R}^n \ni x \to (x^{\sigma(1)}, \dots, x^{\sigma(n)}) \in \mathbb{R}^n$ in (1), the proof will be completed as soon as we can show that

$$L_{\mathbb{R}^n}\Big(x^\alpha \frac{\partial}{\partial x^1}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\Big) = 0, \tag{12}$$

$$L_{\mathbb{R}^n}\Big(x^\alpha \frac{\partial}{\partial x^3}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\Big) = 0 \tag{13}$$

(the latter, of course, only in the case $n \geqslant 3$).

From (11) and (1) with $\varphi\colon \mathbb{R}^n \ni x \to (x^1 + 1, x^2, \dots, x^n) \in \mathbb{R}^n$ and $t = x^1(\partial/\partial x^1)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that $L_{\mathbb{R}^n}((\partial/\partial x^1)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2) = 0$. If $n \geqslant 3$, then from this and (1) with $\varphi\colon \mathbb{R}^n \ni x \to (x^1, x^2, x^1 + x^3, x^4, \dots, x^n) \in \mathbb{R}^n$ and $t = (\partial/\partial x^1)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that $L_{\mathbb{R}^n}((\partial/\partial x^3)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2) = 0$. Next, from this and (1) with $\varphi\colon U \ni x \to (x^1, x^2, \varphi^3(x), x^4, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^3 > 0, x^4 \neq 0, \dots, x^n \neq 0\}$ and

$$\varphi^3(x) = \begin{cases} \dfrac{(x^3)^{1-\alpha^3}}{(1-\alpha^3)(x^4)^{\alpha^4} \dots (x^n)^{\alpha^n}} & \text{if } \alpha^3 \neq 1, \\[2ex] \dfrac{\ln |x^3|}{(x^4)^{\alpha^4} \dots (x^n)^{\alpha^n}} & \text{if } \alpha^3 = 1, \end{cases}$$

and $t = (\partial/\partial x^3)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that $L_U((x^3)^{\alpha^3} \dots (x^n)^{\alpha^n}(\partial/\partial x^3)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2) = 0$. The same conclusion can be drawn for $V = \{x \in \mathbb{R}^n\colon\ x^3 < 0, x^4 \neq 0, \dots, x^n \neq 0\}$, and so for $\mathbb{R}^n$, because $U \cup V$ is dense in $\mathbb{R}^n$. Finally, this and (1) with $\varphi\colon U \ni x \to \big((x^1)^{\alpha^1+1}/(\alpha^1+1), (x^2)^{\alpha^2+1}/(\alpha^2+1), x^3, \dots, x^n\big) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^1 > 0,\ x^2 > 0\}$, and $t = (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n}(\partial/\partial x^3)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2$ imply (13).

From (11) and (1) with $\varphi\colon U \ni x \to (\varphi^1(x), x^2, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^1 > 0,\ x^3 \neq 0, \dots, x^n \neq 0\}$ and

$$\varphi^1(x) = \begin{cases} x^1 + (x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} & \text{if } \alpha^1 = 0, \\ (x^1)^{\alpha^1}(x^3)^{\alpha^3} \dots (x^n)^{\alpha^n} & \text{if } \alpha^1 \neq 0, \end{cases}$$

and $t = x^1(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that

$$L_U\left(\frac{\varphi^1}{\partial\varphi^1/\partial x^1}\frac{\partial}{\partial x^1}\left(\frac{\partial\varphi^1}{\partial x^1}\,\mathrm{d}x^1 + \sum_{p=3}^{n}\frac{\partial\varphi^1}{\partial x^p}\,\mathrm{d}x^p\right)\wedge \mathrm{d}x^2\right) = 0.$$

But for every $p \in \{3,\dots,n\}$

$$\frac{\varphi^1\dfrac{\partial\varphi^1}{\partial x^p}}{\dfrac{\partial\varphi^1}{\partial x^1}} = \begin{cases} \varphi^1\dfrac{\partial\varphi^1}{\partial x^p} & \text{if } \alpha^1 = 0,\\[2ex] \dfrac{x^1}{\alpha^1}\dfrac{\partial\varphi^1}{\partial x^p} & \text{if } \alpha^1 \neq 0,\end{cases}$$

is a polynomial. Hence from (13), which we have proved for every $\alpha \in \mathbb{N}^n$ with x^3, x^1 replaced by x^1, x^p (using (11) again in the case $\alpha^1 = 0$) we obtain $L_{\mathbb{R}^n}\big((x^1)^{\alpha^1}(x^3)^{\alpha^3}\dots(x^n)^{\alpha^n}(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\big) = 0$. This and (1) with $\varphi\colon U \ni x \to \big(x^1, (x^2)^{\alpha^2+1}/(\alpha^2+1), x^3, \dots, x^n\big) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^2 > 0\}$, and $t = (x^1)^{\alpha^1}(x^3)^{\alpha^3}\dots(x^n)^{\alpha^n}(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ yield (12), which establishes the lemma. □

P r o o f of the theorem. In the same manner as in [3] we can see that there are unique $\mathbb{R}$-trilinear maps $a,b,c,d,e,f,g,h,i,j,k,l\colon\ A\times A\times A \to A$ such that

(14) $$L_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^1_X(Y,Z) = a(X^1,Y^1,Z^2) + b(X^1,Y^2,Z^1) + c(X^2,Y^1,Z^1),$$

(15) $$L_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^2_X(Y,Z) = d(X^1,Y^2,Z^2) + e(X^2,Y^1,Z^2) + f(X^2,Y^2,Z^1),$$

(16) $$\begin{aligned}L_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^p_X(Y,Z) &= g(X^1,Y^2,Z^p) + h(X^1,Y^p,Z^2) + i(X^2,Y^1,Z^p)\\ &\quad + j(X^2,Y^p,Z^1) + k(X^p,Y^1,Z^2) + l(X^p,Y^2,Z^1)\end{aligned}$$

for every $p \in \{3,\dots,n\}$ and all $X,Y,Z \in A^n$. Also similarly to [3] (using (16) with $p = 3$ and (1) with $\varphi\colon U \ni x \to (x^1, x^2, (x^3)^2/2, x^4, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^3 > 0\}$ and $t = x^1(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ and carrying out polarization if necessary) we get for all $w,x,y,z \in A$

(17) $$wg(x,y,z) = g(x,y,wz),$$

(18) $$wh(x,y,z) = h(x,wy,z),$$

(19) $$wi(x,y,z) = i(x,y,wz),$$

(20) $$wj(x,y,z) = j(x,wy,z),$$

(21) $$wk(x,y,z) + xk(w,y,z) = k(wx,y,z).$$

From (16) with $p = 3$ and (1) with $\varphi\colon U \ni x \to \left(x^1, \frac{1}{2}(x^2)^2, x^3, \ldots, x^n\right) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^2 > 0\}$, and $t = x^1(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that

$$(22)\quad L_{\mathbb{R}^n}\Big(x^1x^2\frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^3_X(Y,Z)$$

$$= g(X^1, X^2Y^2, Z^3) + h(X^1, Y^3, X^2Z^2) + i\Big(\frac{(X^2)^2}{2}, Y^1, Z^3\Big)$$

$$+ j\Big(\frac{(X^2)^2}{2}, Y^3, Z^1\Big) + k(X^3, Y^1, X^2Z^2) + l(X^3, X^2Y^2, Z^1).$$

On the other hand, from (16) with $p = 3$ and (1) with $\varphi\colon U \ni x \to (x^1x^2, x^2, \ldots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^2 \neq 0\}$ and $t = x^1(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that

$$(23)\quad L_{\mathbb{R}^n}\Big(x^1x^2\frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^3_X(Y,Z)$$

$$= g(X^1X^2, Y^2, Z^3) + h(X^1X^2, Y^3, Z^2)$$

$$+ i(X^2, X^2Y^1 + X^1Y^2, Z^3) + j(X^2, Y^3, X^2Z^1 + X^1Z^2)$$

$$+ k(X^3, X^2Y^1 + X^1Y^2, Z^2) + l(X^3, Y^2, X^2Z^1 + X^1Z^2).$$

Comparing (22) with (23) and carrying out polarization if necessary we get

$$(24)\quad g(x, wy, z) = g(xw, y, z) + i(w, xy, z),$$

$$(25)\quad h(x, y, wz) = h(xw, y, z) + j(w, y, xz),$$

$$(26)\quad i(wx, y, z) = i(w, xy, z) + i(x, wy, z),$$

$$(27)\quad j(wx, y, z) = j(w, y, xz) + j(x, y, wz),$$

$$(28)\quad k(x, y, wz) = k(x, wy, z),$$

$$(29)\quad 0 = k(x, wy, z) + l(x, y, wz).$$

Define $G(u, v) = 4k(v, u, 1)$ for all $u, v \in A$. (21) shows that G satisfies (2). From (28) we deduce that $k(x, y, z) = \frac{1}{4}G(yz, x)$ for all $x, y, z \in A$. In addition, by (29), $l = -k$. Therefore $k(X^p, Y^1, Z^2) + l(X^p, Y^2, Z^1) = \frac{1}{4}G(Y^1Z^2 - Y^2Z^1, X^p)$ for every $p \in \{1, \ldots, n\}$ and all $X, Y, Z \in A^n$. But, on account of (7),

$$(30)\quad \overline{G}_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^p_X(Y,Z) = \frac{1}{4}G(Y^1Z^2 - Y^2Z^1, X^p),$$

too. Consequently, by (16), we may replace L by $L - \overline{G}$ and assume that $k = 0$ and $l = 0$ from now on.

Define $H(u, v) = 4i(v, u, 1)$ and $H'(u, v) = 4j(v, 1, u)$ for all $u, v \in A$. (26) and (27) show that H and H' satisfy (3). From (19) and (20) we deduce that $i(x, y, z) =$

$\frac{1}{4}H(y,x)z$ and $j(x,y,z)=\frac{1}{4}H'(z,x)y$ for all $x,y,z\in A$. Therefore $i(X^2,Y^1,Z^p)=\frac{1}{4}H(Y^1,X^2)Z^p$ and $j(X^2,Y^p,Z^1)=\frac{1}{4}H'(Z^1,X^2)Y^p$ for every $p\in\{1,\dots,n\}$ and all $X,Y,Z\in A^n$. But

$$\overline{H}^{\mathrm{L}}_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)^p_X(Y,Z)=\frac{1}{4}(H(Y^1,X^2)-H(Y^2,X^1))Z^p, \tag{31}$$

$$\overline{H'}^{\mathrm{R}}_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)^p_X(Y,Z)=\frac{1}{4}(H'(Z^1,X^2)-H'(Z^2,X^1))Y^p \tag{32}$$

on account of (8) and (9). Consequently, by (16), we may replace L by $L-\overline{H}^{\mathrm{L}}-\overline{H'}^{\mathrm{R}}$ and assume that $i=0$ and $j=0$ from now on (note that the conditions $k=0$ and $l=0$ remain valid).

Define $F(u)=2g(u,1,1)$ and $F'(u)=2h(u,1,1)$ for every $u\in A$. From (17) and (24) we deduce that $g(x,y,z)=\frac{1}{2}F(xy)z$ and from (18) and (25) that $h(x,y,z)=\frac{1}{2}F'(xz)y$ for all $x,y,z\in A$. Therefore $g(X^1,Y^2,Z^p)=\frac{1}{2}F(X^1Y^2)Z^p$ and $h(X^1,Y^p,Z^2)=\frac{1}{2}F'(X^1Z^2)Y^p$ for every $p\in\{1,\dots,n\}$ and all $X,Y,Z\in A^n$. But, on account of (5) and (6),

$$\overline{F}^{\mathrm{L}}_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)^p_X(Y,Z)=\frac{1}{2}F(X^1Y^2)Z^p, \tag{33}$$

$$\overline{F'}^{\mathrm{R}}_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)^p_X(Y,Z)=\frac{1}{2}F'(X^1Z^2)Y^p, \tag{34}$$

too. Consequently, by (16), we may replace L by $L-\overline{F}^{\mathrm{L}}-\overline{F'}^{\mathrm{R}}$ and assume that also $g=0$ and $h=0$ from now on.

But now $d=0$, $e=0$, $f=0$ as well. Indeed, from (16) with $p=3$, (15) and (1) with $\varphi\colon\ \mathbb{R}^n\ni x\to(x^1,x^2,x^2+x^3,x^4,\dots,x^n)\in\mathbb{R}^n$ and $t=x^1(\partial/\partial x^1)\,\mathrm{d}x^1\wedge\mathrm{d}x^2$ it follows that $0=L_{\mathbb{R}^n}\big(x^1(\partial/\partial x^1)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\big)^3_X(Y,Z)=-d(X^1,Y^2,Z^2)-e(X^2,Y^1,Z^2)-f(X^2,Y^2,Z^1)$, which yields the desired conclusion.

From (14), (16) with $p=3$ and (1) with $\varphi\colon\ \mathbb{R}^n\ni x\to(x^1+x^3,x^2,\dots,x^n)\in\mathbb{R}^n$ and $t=x^1(\partial/\partial x^1)\,\mathrm{d}x^1\wedge\mathrm{d}x^2$ it follows that

$$\begin{aligned}L_{\mathbb{R}^n}\Big((x^1+x^3)\frac{\partial}{\partial x^1}(\mathrm{d}x^1+\mathrm{d}x^3)\wedge\mathrm{d}x^2\Big)^1_X(Y,Z)&\\ =a(X^1+X^3,Y^1+Y^3,Z^2)+b(X^1+X^3,Y^2,Z^1+Z^3)&\\ +c(X^2,Y^1+Y^3,Z^1+Z^3).&\end{aligned} \tag{35}$$

From (14), (16) with $p=3$ and (1) with $\varphi\colon\ \mathbb{R}^n\ni x\to(x^1,x^2,x^1+x^3,x^4,\dots,x^n)\in\mathbb{R}^n$ and $t=x^1(\partial/\partial x^1)\,\mathrm{d}x^1\wedge\mathrm{d}x^2$ it follows that $L_{\mathbb{R}^n}\big(x^1\big(\partial/\partial x^1-\partial/\partial x^3\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\big)^3_X(Y,Z)=-a(X^1,Y^1,Z^2)-b(X^1,Y^2,Z^1)-c(X^2,Y^1,Z^1)$. This, after using (16) with $p=3$ again and then interchanging x^1 and x^3, yields

$$L_{\mathbb{R}^n}\Big(x^3\frac{\partial}{\partial x^1}\,\mathrm{d}x^3\wedge\mathrm{d}x^2\Big)^1_X(Y,Z) = a(X^3,Y^3,Z^2)+b(X^3,Y^2,Z^3)+c(X^2,Y^3,Z^3). \tag{36}$$

From (14), (16) with $p=3$ and (1) with $\varphi\colon U\ni x\to(x^1,x^2,x^1x^3,x^4,\ldots,x^n)\in\mathbb{R}^n$, where $U=\{x\in\mathbb{R}^n\colon x^1\neq 0\}$, and $t=x^1(\partial/\partial x^1)\,\mathrm{d}x^1\wedge\mathrm{d}x^2$ it follows that $L_{\mathbb{R}^n}\big(x^1\big(\partial/\partial x^1-(x^3/x^1)\partial/\partial x^3\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\big)^3_X(Y,Z)=-(X^3/X^1)(a(X^1,Y^1,Z^2)+b(X^1,Y^2,Z^1)+c(X^2,Y^1,Z^1))$. This, after using (16) with $p=3$ again and then interchanging x^1 and x^3, yields

$$L_{\mathbb{R}^n}\Big(x^1\frac{\partial}{\partial x^1}\,\mathrm{d}x^3\wedge\mathrm{d}x^2\Big)^1_X(Y,Z) = \frac{X^1}{X^3}(a(X^3,Y^3,Z^2)+b(X^3,Y^2,Z^3)+c(X^2,Y^3,Z^3)). \tag{37}$$

Combining (35), (36), (37) with (14) we see that

$$\begin{aligned} L_{\mathbb{R}^n}\Big(x^3\frac{\partial}{\partial x^1}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)^1_X(Y,Z) &= a(X^1,Y^3,Z^2)+a(X^3,Y^1,Z^2)+b(X^1,Y^2,Z^3)\\ &\quad+b(X^3,Y^2,Z^1)+c(X^2,Y^1,Z^3)+c(X^2,Y^3,Z^1)\\ &\quad-\frac{X^1}{X^3}(a(X^3,Y^3,Z^2)+b(X^3,Y^2,Z^3)+c(X^2,Y^3,Z^3)). \end{aligned} \tag{38}$$

On the other hand, (38) and (1) with $\varphi\colon U\ni x\to\big(\frac12(x^1)^2,x^2,\ldots,x^n\big)\in\mathbb{R}^n$, where $U=\{x\in\mathbb{R}^n\colon x^1>0\}$, and $t=x^3(\partial/\partial x^1)\,\mathrm{d}x^1\wedge\mathrm{d}x^2$ implies that

$$\begin{aligned} L_{\mathbb{R}^n}\Big(x^3\frac{\partial}{\partial x^1}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)^1_X(Y,Z) &= \frac{1}{X^1}\bigg(a\Big(\frac{(X^1)^2}{2},Y^3,Z^2\Big)+a(X^3,X^1Y^1,Z^2)+b\Big(\frac{(X^1)^2}{2},Y^2,Z^3\Big)\\ &\quad+b\Big(X^3,Y^2,X^1Z^1)+c(X^2,X^1Y^1,Z^3\Big)+c(X^2,Y^3,X^1Z^1)\bigg)\\ &\quad-\frac{X^1}{2X^3}(a(X^3,Y^3,Z^2)+b(X^3,Y^2,Z^3)+c(X^2,Y^3,Z^3)). \end{aligned} \tag{39}$$

Comparing (38) with (39) (both multiplied by $2X^1X^3$) we get

$$2wxa(x,y,z)-2x^2a(w,y,z)=wa(x^2,y,z)-x^2a(w,y,z), \tag{40}$$

$$2wxa(x,y,z)=2xa(x,wy,z), \tag{41}$$

$$-2w^2c(x,y,z)=-w^2c(x,y,z). \tag{42}$$

From (42) we see at once that $c = 0$.

Taking $x = 1$ in (40) we deduce that a is A-linear with respect to the first variable. By (41), it is also A-linear with respect to the second.

From (14) and (1) with $\varphi\colon U \ni x \to \big(x^1, \frac{1}{2}(x^2)^2, x^3, \dots, x^n\big) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^2 > 0\}$ and $t = x^1(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that

$$\text{(43)} \qquad L_{\mathbb{R}^n}\Big(x^1 x^2 \frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^1_X (Y, Z) = a(X^1, Y^1, X^2 Z^2) + b(X^1, X^2 Y^2, Z^1).$$

On the other hand, from (14), (15) and (1) with $\varphi\colon U \ni x \to (x^1 x^2, x^2, \dots, x^n) \in \mathbb{R}^n$, where $U = \{x \in \mathbb{R}^n\colon\ x^2 \neq 0\}$ and $t = x^1(\partial/\partial x^1)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ it follows that

$$\text{(44)} \qquad L_{\mathbb{R}^n}\Big(x^1 x^2 \frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^1_X (Y, Z) = \frac{1}{X^2}\big(a(X^1 X^2, X^2 Y^1 + X^1 Y^2, Z^2) + b(X^1 X^2, Y^2, X^2 Z^1 + X^1 Z^2)\big).$$

Comparing (43) with (44) (both multiplied by X^2) we get

$$\text{(45)} \qquad w a(x, y, wz) = a(wx, wy, z),$$

$$\text{(46)} \qquad 0 = a(wx, wy, z) + b(wx, y, wz).$$

Since a is A-linear with respect to the first and second variables, from (45) we deduce that it is also A-linear with respect to the third. Thus $a(x, y, z) = \frac{1}{2}Exyz$ for all $x, y, z \in A$, where $E = 2a(1, 1, 1)$. In addition, by (46), $b = -a$. Therefore $a(X^1, Y^1, Z^2) + b(X^1, Y^2, Z^1) = \frac{1}{2}EX^1(Y^1 Z^2 - Y^2 Z^1)$ for all $X, Y, Z \in A^n$. But, on account of (4),

$$\text{(47)} \qquad \overline{E}_{\mathbb{R}^n}\Big(x^1 \frac{\partial}{\partial x^1}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\Big)^p_X (Y, Z) = \begin{cases} \frac{1}{2}EX^1(Y^1 Z^2 - Y^2 Z^1) & \text{if } p = 1, \\ 0 & \text{if } p \in \{2, \dots, n\}. \end{cases}$$

Consequently, by (14)–(16) and the lemma we conclude that $L = \overline{E}$ (where L actually stands for $L - \overline{G} - \overline{H}^{\mathrm{L}} - \overline{H'}^{\mathrm{R}} - \overline{F}^{\mathrm{L}} - \overline{F'}^{\mathrm{R}}$ with the original L), which completes the proof that E, F, F', G, H, H' exist.

Analysis of formulas (30)–(34) and (47) makes it obvious that they are uniquely determined, and the theorem is proved. □

Remark. The author does not know whether every linear lifting can be represented in the form (10) when $n = 2$, but it is easy to see that then this representation need not be unique. Indeed, for every $C \in A$ and every derivation D of the algebra A we may put $E = C$, $F(u) = Cu$, $F'(u) = -Cu$, $G(u,v) = -uD(v)$, $H(u,v) = uD(v)$, $H'(u,v) = -uD(v)$ for all $u, v \in A$. Then (by the lemma, (30)–(34) and (47)) the right-hand side of (10) equals zero in the case $n = 2$.

The remainder of the paper will be devoted to some corollaries of the theorem.

If L is given by (10), then from (4)–(9) it is evident that the two liftings $\overline{E} + \overline{\frac{1}{2}(F - F')}^{\mathrm{L}} + \overline{\frac{1}{2}(-F + F')}^{\mathrm{R}} + \overline{G} + \overline{\frac{1}{2}(H - H')}^{\mathrm{L}} + \overline{\frac{1}{2}(-H + H')}^{\mathrm{R}}$ and $\overline{\frac{1}{2}(F + F')}^{\mathrm{L}} + \overline{\frac{1}{2}(F + F')}^{\mathrm{R}} + \overline{\frac{1}{2}(H + H')}^{\mathrm{L}} + \overline{\frac{1}{2}(H + H')}^{\mathrm{R}}$ transform all skew symmetric tensor fields into skew symmetric and symmetric ones, respectively, and the sum of them equals L. This yields the two corollaries.

Corollary 1. *If $n \geqslant 3$, then for every linear lifting L of skew symmetric tensor fields of type $(1,2)$ to skew symmetric tensor fields of type $(1,2)$ on T^A there are unique $E \in A$, $\mathbb{R}$-linear $F\colon A \to A$, $\mathbb{R}$-bilinear $G\colon A \times A \to A$ satisfying (2) and $\mathbb{R}$-bilinear $H\colon A \times A \to A$ satisfying (3) such that*

$$L = \overline{E} + \overline{F}^{\mathrm{L}} - \overline{F}^{\mathrm{R}} + \overline{G} + \overline{H}^{\mathrm{L}} - \overline{H}^{\mathrm{R}}.$$

Corollary 2. *If $n \geqslant 3$, then for every linear lifting L of skew symmetric tensor fields of type $(1,2)$ to symmetric tensor fields of type $(1,2)$ on T^A there are unique $\mathbb{R}$-linear $F\colon A \to A$ and $\mathbb{R}$-bilinear $H\colon A \times A \to A$ satisfying (3) such that*

$$L = \overline{F}^{\mathrm{L}} + \overline{F}^{\mathrm{R}} + \overline{H}^{\mathrm{L}} + \overline{H}^{\mathrm{R}}.$$

Combining the theorem of this paper with that of [3] we easily obtain

Corollary 3. *If $n \geqslant 3$, then for every linear lifting L of tensor fields of type $(1,2)$ to tensor fields of type $(1,2)$ on T^A there are unique $E, E' \in A$, $\mathbb{R}$-linear $F, F', F'', F'''\colon A \to A$, $\mathbb{R}$-bilinear $G, G'\colon A \times A \to A$ satisfying (2) and $\mathbb{R}$-bilinear $H, H', H'', H'''\colon A \times A \to A$ satisfying (3) such that*

$$\begin{aligned} L = {} & (\overline{E} + \overline{F}^{\mathrm{L}} + \overline{F'}^{\mathrm{R}} + \overline{G} + \overline{H}^{\mathrm{L}} + \overline{H'}^{\mathrm{R}}) \circ \mathrm{sk} \\ & + (\overline{E'} + \overline{F''}^{\mathrm{L}} + \overline{F'''}^{\mathrm{R}} + \overline{G'} + \overline{H''}^{\mathrm{L}} + \overline{H'''}^{\mathrm{R}}) \circ \mathrm{sy}, \end{aligned}$$

where sk *and* sy *denote the alternation and symmetrization of tensor fields of type $(1,2)$ and the liftings of symmetric tensor fields are also given by formulas* (4)–(9).

We end the paper with an application of our result in classification of all affine liftings of linear connections to Weil bundles (the contrast between these liftings and those of general connections to some non-Weil bundles is worth pointing out, see for instance [5]). Namely, combining our theorem with that of [4] we easily obtain

Corollary 4. *If $n \geqslant 3$, then for every affine lifting L of linear connections to linear connections on T^A there are unique $E \in A$, $\mathbb{R}$-linear $F, F'\colon A \to A$, $\mathbb{R}$-bilinear $G, G'\colon A \times A \to A$ satisfying* (2) *and $\mathbb{R}$-bilinear $H, H', H'', H'''\colon A \times A \to A$ satisfying* (3) *such that*

$$L_M\nabla = \nabla^A + \overline{E}_M T + \overline{F}^{\mathrm{L}}_M T + \overline{F'}^{\mathrm{R}}_M T + \overline{G}_M T + \overline{H}^{\mathrm{L}}_M T + \overline{H'}^{\mathrm{R}}_M T + \widetilde{G'}_M(\operatorname{tr} R) + \widetilde{H''}^{\mathrm{L}}_M(\operatorname{tr} R) + \widetilde{H'''}^{\mathrm{R}}_M(\operatorname{tr} R)$$

for every n-dimensional manifold M and every linear connection ∇ on M, where ∇^A denotes the complete lift of ∇ (see [14], [7]*), T and R the torsion and curvature tensors of ∇ and $\widetilde{G'}$ $\widetilde{H''}^{\mathrm{L}}$, $\widetilde{H'''}^{\mathrm{R}}$ are the unique liftings of 2-forms to tensor fields of type $(1,2)$ on T^A such that*

$$(\widetilde{G'}_U t)^p_X(Y,Z) = G'((T^A t_{qr})(X) \cdot Y^q \cdot Z^r, X^p),$$
$$(\widetilde{H''}^{\mathrm{L}}_U t)^p_X(Y,Z) = H''((T^A t_{qr})(X) \cdot Y^q, X^r) \cdot Z^p,$$
$$(\widetilde{H'''}^{\mathrm{R}}_U t)^p_X(Y,Z) = H'''((T^A t_{qr})(X) \cdot Z^q, X^r) \cdot Y^p$$

for every open subset U of $\mathbb{R}^n$, every 2-form t on U, every $p \in \{1,\ldots,n\}$, every $X \in T^A U$ and all $Y, Z \in A^n$.

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