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ON THE JETS OF FOLIATION RESPECTING MAPS

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Abstract. Using Weil algebra techniques, we determine all finite dimensional homomorphic images of germs of foliation respecting maps.

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In [2], the second author studied regular homomorphisms with product property on germs of smooth maps and deduced that the classical r-jets are the only possibility. In the present paper, we solve the same problem for germs of a foliation respecting map. In Section 1 we introduce an original concept of the leafwise (k,r)-jet of a foliation respecting map and describe its basic properties. Then we clarify, in Theorem 1, that our main result can be formulated in terms of these jets and of the transversal jets. Section 3 is of auxiliary character. We construct the induced product preserving bundle functors on the category $\mathcal{FM}$ of fibered manifolds. By a result of the third author, [6], each of them is determined by a homomorphism of Weil algebras. In our case, the related Weil algebras have an important additional property, which we call the substitution property. In Proposition 2 we determine all Weil algebras with the substitution property. This supplies us with sufficient background for the proof of Theorem 1 in Section 4.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [4].

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1. Bundles of leafwise jets

A foliated manifold $(M,\mathcal{F})$ is an m-dimensional manifold M endowed with an n-dimensional foliation $\mathcal{F}$, [7]. We write $\mathcal{F}ol$ for the category of foliated manifolds and foliation respecting maps and L_z for the leaf of $\mathcal{F}$ passing through $z \in M$. Given two foliated manifolds $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$, a leafwise r-jet of M_1 into M_2 means an r-jet $j^r_z\gamma$ of a local map γ of L_z into $L_{\gamma(z)}$. All these jets form a bundle

$$\lambda J^r((M_1,\mathcal{F}_1),(M_2,\mathcal{F}_2)), \quad \text{in short}: \ \lambda J^r(M_1,M_2). \tag{1}$$

If x^i, y^p are local leaf coordinates on M_1 and v^t, w^a are local leaf coordinates on M_2, then the induced coordinates on $\lambda J^r(M_1,M_2)$ are

$$w^a_\beta, \quad |\beta| \leqslant r, \tag{2}$$

where β is a multiindex corresponding to y^p. If $(M_3,\mathcal{F}_3)$ is another foliated manifold and $j^r_{\gamma(z)}\delta \in \lambda J^r(M_2,M_3)$, we have the composition

$$j^r_{\gamma(z)}\delta \circ j^r_z\gamma = j^r_z(\delta\circ\gamma) \in \lambda J^r(M_1,M_3).$$

For the product $(M_2\times M_3, \mathcal{F}_2\times\mathcal{F}_3)$, one verifies directly

$$\lambda J^r(M_1, M_2\times M_3) = \lambda J^r(M_1,M_2)\times_{M_1}\lambda J^r(M_1,M_3). \tag{3}$$

The category $\mathcal{FM}$ of fibered manifolds and fibered morphisms is a subcategory of $\mathcal{F}ol$. For two fibered manifolds $p_1\colon Y_1\to B_1$ and $p_2\colon Y_2\to B_2$, we have

$$\lambda J^r(Y_1,Y_2) = \bigcup_{(x_1,x_2)\in B_1\times B_2} J^r(Y_{1x_1}, Y_{2x_2}).$$

Every $\mathcal{F}ol$-morphism $f\colon M_1\to M_2$ defines a section

$$\lambda^r f\colon M_1 \to \lambda J^r(M_1,M_2), \quad (\lambda^r f)(z) = j^r_z(f|L_z).$$

Definition 1. $j^k_z(\lambda^r f)$ is called a leafwise (k,r)-jet of f at z.

We write

$$J^k\lambda J^r(M_1,M_2) = J^k(\lambda J^r(M_1,M_2)\to M_1) \tag{4}$$

for the bundle of these jets. The induced coordinates on (4) are

$$v^t_\alpha, \quad |\alpha|\leqslant k, \qquad w^a_{\alpha,\beta}, \quad |\alpha|\leqslant k, \quad |\alpha|+|\beta|\leqslant k+r, \tag{5}$$

where α is the multiindex corresponding to x^i. For $l < k$, $j^l_z \lambda^{r+k-l} f$ depends on $j^k_z \lambda^r f$ only. This defines a projection

$$\pi^{k,r}_l \colon J^k \lambda J^r(M_1, M_2) \to J^l \lambda J^{r+k-l}(M_1, M_2), \quad l < k. \tag{6}$$

Lemma 1. *Let $g\colon M_2 \to M_3$ be another $\mathcal{F}ol$-morphism. Then the value $j^k_z(\lambda^r(g \circ f))$ depends on $j^k_z(\lambda^r f)$ and $j^k_{f(z)}(\lambda^r g)$ only.*

P r o o f. Consider the coordinate expression of f

$$v^t = f^t(x^i), \quad w^a = f^a(x^i, y^p) \tag{7}$$

and the analogous coordinate expression of g. Then our assertion follows directly from (5). $\square$

This defines the composition of leafwise (k, r)-jets.

Consider a $(p+1)$-tuple $\varrho = (r_0, r_1, \ldots, r_p)$ of integers such that

$$r_0 > r_1 > \ldots > r_p \geqslant 0. \tag{8}$$

For $f, g \in \mathcal{F}ol(M_1, M_2)$, we define $j^\varrho_z f = j^\varrho_z g$, $z \in M_1$, by

$$\lambda^{r_0} f(z) = \lambda^{r_0} g(z), \quad j^i_z \lambda^{r_i} f = j^i_z \lambda^{r_i} g, \quad i = 1, \ldots, p, \tag{9}$$

and write $J^\varrho(M_1, M_2)$ for the set of all these jets.

Proposition 1. *$J^\varrho(M_1, M_2)$ is a fibered manifold over $M_1 \times M_2$.*

P r o o f. We start with arbitrary quantities v^t_α, $|\alpha| \leqslant p$ and $w^a_{\alpha_p,\beta_p}$, $|\alpha_p| \leqslant p$, $|\alpha_p| + |\beta_p| \leqslant p + r_p$. Since $r_{p-1} > r_p$, we can prescribe the remaining $w^a_{\alpha_{p-1},\beta_{p-1}}$, $|\alpha_{p-1}| \leqslant p-1$, $|\alpha_{p-1}| + |\beta_{p-1}| \leqslant p - 1 + r_{p-1}$, arbitrarily. In the last step of this recurrence procedure we have $w^a_{\alpha_1,\beta_1}$, $|\alpha_1| \leqslant 1$, $|\alpha_1| + |\beta_1| \leqslant 1 + r_1$. Since $r_0 > r_1$, we can prescribe the remaining $w^a_{\beta_0}$, $|\beta_0| \leqslant r_0$, arbitrarily. So we obtain the induced coordinates on $J^\varrho(M_1, M_2)$. $\square$

By (3), we deduce

$$J^\varrho(M_1, M_2 \times M_3) = J^\varrho(M_1, M_2) \times_{M_1} J^\varrho(M_1, M_3). \tag{10}$$

We say that two $\mathcal{F}ol$-morphisms $f, g\colon M_1 \to M_2$ determine the same transversal r-jet $\tau^r_z f = \tau^r_z g$ at $z \in M_1$, if

$$j^r_z(h \circ f) = j^r_z(h \circ g)$$

for every local function h on M_2 constant on the leaves. We write $\tau J^r(M_1, M_2)$ for the bundle of these jets. Clearly,

$$\tau J^r(M_1, M_2 \times M_3) = \tau J^r(M_1, M_2) \times_{M_1} \tau J^r(M_1, M_3). \tag{11}$$

Since the classical definition of $j^r_x f = j^r_x g$ for two smooth maps $f, g\colon N_1 \to N_2$, $x \in N_1$, is equivalent to $j^r_x(h \circ f) = j^r_x(h \circ g)$ for every smooth function h on N_2, we have a canonical projection

$$J^r \mathcal{F}ol(M_1, M_2) \to \tau J^r(M_1, M_2). \tag{12}$$

More generally, we have a projection

$$J^k \lambda J^r(M_1, M_2) \to \tau J^k(M_1, M_2), \quad j^k_z(\lambda^r f) \mapsto \tau^k_z f. \tag{13}$$

In particular, (12) defines the composition of transversal r-jets. For an $\mathcal{FM}$-morphism $f\colon Y_1 \to Y_2$ over $\underline{f}\colon M_1 \to M_2$, $\tau^r_z f$ coincides with $j^r_{p_1(z)}\underline{f}$.

We point out that there is another characterization of $j^k_z \lambda^r f$. We have $j^k f\colon M_1 \to J^k(M_1, M_2)$.

Lemma 2. *The leafwise (k,r)-jets $j^k_z \lambda^r f$ are in bijection with r-jets $j^r_z(j^k f | L_z)$.*

Proof. If (7) is the coordinate expression of f, then $\lambda^r f$ is expressed by $v^t(x^i)$, $D_\beta w^a(x^i, y^p)$, $|\beta| \leqslant r$. Hence $j^k_z \lambda^r f$ is determined by

$$D_\alpha v^t, \quad D_{\alpha_1, \beta_1}(D_\beta w^a), \qquad |\alpha| \leqslant k, \quad |\alpha_1| + |\beta_1| \leqslant k.$$

On the other hand, the coordinate expression of $j^k f$ is

$$D_\alpha v^t, \quad |\alpha| \leqslant k \quad \text{and} \quad D_{\alpha_1, \beta_1} w^a, \quad |\alpha_1| + |\beta_1| \leqslant k,$$

so that $j^r_z(j^k f | L_z)$ is further determined by $D_\beta(D_{\alpha_1,\beta_1} w^a)$, $|\beta| \leqslant r$. $\square$

By (13), we have a projection $J^\varrho(M_1, M_2) \to \tau J^p(M_1, M_2)$. For every $q \geqslant p$ we define

$$J^{\varrho,q}(M_1, M_2) = J^\varrho(M_1, M_2) \times_{\tau J^p(M_1, M_2)} \tau J^q(M_1, M_2). \tag{14}$$

Write $G\mathcal{F}ol(M_1, M_2)$ for the set of all germs of foliation respecting maps of M_1 into M_2. The rule

$$j^{\varrho,q}_{M_1,M_2}(\mathrm{germ}_z f) = (j^\varrho_z f, \tau^q_z f) =: j^{\varrho,q}_z f$$

is a surjective map $G\mathcal{F}ol(M_1, M_2) \to J^{\varrho,q}(M_1, M_2)$. Analogously to (10), we have

$$J^{\varrho,q}(M_1, M_2 \times M_3) = J^{\varrho,q}(M_1, M_2) \times_{M_1} J^{\varrho,q}(M_1, M_3).$$

Example 1. The classical (r, s)-jet, $s \geqslant r$, of an $\mathcal{FM}$-morphism $f\colon Y \to \overline{Y}$ is defined by $j^{r,s}_z f = (j^r_z f, j^s_z(f|Y_{p(z)}))$, [4], [5]. These jets represent a special case of ϱ-jets with $p = r$, $r_0 = s$, $r_i = p - i$ for $i = 1, \dots, p$. For $q \geqslant r$, the classical (r, s, q)-jets are defined as $j^{r,s,q}_z f = (j^{r,s}_z f, j^q_{p(z)} \underline{f})$, where $\underline{f}$ is the base map of f, [4]. They represent the corresponding special case of (ϱ, q)-jets.

2. Jet-like homomorphisms

We are going to apply an abstract viewpoint similar to [2]. Consider a rule F transforming every pair $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$ of foliated manifolds into a fibered manifold $F((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$ (in short: $F(M_1, M_2)$) over $M_1 \times M_2$ and a system of maps $\varphi_{M_1,M_2}\colon G\mathcal{F}ol(M_1, M_2) \to F(M_1, M_2)$ commuting with the projections $G\mathcal{F}ol(M_1, M_2) \to M_1 \times M_2$ and $F(M_1, M_2) \to M_1 \times M_2$ for all M_1, M_2. Analogously to [2], we formulate the following requirements I–IV.

I. Every $\varphi_{M_1,M_2}\colon G\mathcal{F}ol(M_1, M_2) \to F(M_1, M_2)$ is surjective.

II. If $B_1, \overline{B}_1 \in G_{z_1}\mathcal{F}ol(M_1, M_2)_{z_2}$ and $B_2, \overline{B}_2 \in G_{z_2}\mathcal{F}ol(M_2, M_3)_{z_3}$ satisfy $\varphi(B_1) = \varphi(\overline{B}_1)$ and $\varphi(B_2) = \varphi(\overline{B}_2)$, then $\varphi(B_2 \circ B_1) = \varphi(\overline{B}_2 \circ \overline{B}_1)$.

By I and II, we have a well defined composition (denoted by the same symbol as the composition of germs and maps)

$$X_2 \circ X_1 = \varphi(B_2 \circ B_1)$$

for $X_1 = \varphi(B_1) \in F_{z_1}(M_1, M_2)_{z_2}$ and $X_2 = \varphi(B_2) \in F_{z_2}(M_2, M_3)_{z_3}$. Write $\varphi_z f$ for $\varphi(\mathrm{germ}_z f)$. For another pair $\overline{M_1}$, $\overline{M}_2$ of foliated manifolds, every local $\mathcal{F}ol$-isomorphism $f\colon M_1 \to \overline{M}_1$ and every $\mathcal{F}ol$-morphism $g\colon M_2 \to \overline{M}_2$ induce a map $F(f, g)\colon F(M_1, M_2) \to F(\overline{M}_1, \overline{M}_2)$ by

$$F(f, g)(X) = (\varphi_{z_2} g) \circ X \circ \varphi_{f(z_1)}(f^{-1}), \quad X \in F_{z_1}(M_1, M_2)_{z_2},$$

where f^{-1} is constructed locally. We require

III. Each map $F(f, g)$ is smooth.

IV. (Product property) $F(M_1, M_2 \times M_3) = F(M_1, M_2) \times_{M_1} F(M_1, M_3)$.

Definition 2. A pair (F, φ) satisfying I–IV will be called a jet-like homomorphism on germs of foliation respecting maps.

Remark 1. In [2], the second author introduced such a concept for germs of smooth maps and deduced that the only jet-like homomorphisms on germs of smooth maps are the classical r-jets.

Clearly, $(J^{\varrho,q}, j^{\varrho,q})$ is a jet-like homomorphism, provided the composition is defined componentwise. The main result of the present paper is the following assertion.

Theorem 1. *Every jet-like homomorphism on germs of foliation respecting maps is of the form* $(J^{\varrho,q}, j^{\varrho,q})$.

The proof is postponed to Section 4.

3. The related Weil algebras

A classical result reads that the product preserving bundle functors on the category $\mathcal{M}f$ of smooth manifolds and smooth maps coincide with the Weil functors T^A and the natural transformations $\mu\colon T^C \to T^A$ are in bijection with the algebra homomorphisms (denoted by the same symbol) $\mu\colon C \to A$, [4], [3]. The third author deduced that the product preserving bundle functors H on $\mathcal{FM}$ are in the following bijection with algebra homomorphisms $\mu\colon C \to A$, [6], [1]. Write pt for one point manifold and $\mathrm{pt}_B\colon B \to \mathrm{pt}$ for the unique map. We have two canonical injections $i_1, i_2\colon \mathcal{M}f \to \mathcal{FM}$ defined by $i_1 B = (\mathrm{id}_B\colon B \to B)$, $i_1 f = (f, f)$, $i_2 B = (\mathrm{pt}_B\colon B \to \mathrm{pt})$, $i_2 f = (f, \mathrm{id}_{\mathrm{pt}})$ and a natural transformation $t\colon i_1 \to i_2$, $t_B = (\mathrm{id}_B, \mathrm{pt}_B)\colon i_1 B \to i_2 B$. Applying H, we obtain two product preserving bundle functors $H \circ i_1 = T^C$ and $H \circ i_2 = T^A$ on $\mathcal{M}f$, and a natural transformation $\mu_B := H(t_B)\colon T^C B \to T^A B$. We have $C = H(i_1\mathbb{R})$, $A = H(i_2\mathbb{R})$, $\mu = H(t_{\mathbb{R}})$. For every fibered manifold $p\colon Y \to B$, HY coincides with the pullback of $T^A p\colon T^A Y \to T^A B$ with respect to $\mu_B\colon T^C B \to T^A B$, which will be denoted by $T^C B \times_{T^A B} T^A Y$. For every $\mathcal{FM}$-morphism $f\colon Y \to \overline{Y}$ over $\underline{f}\colon B \to \overline{B}$ we have

$$Hf = T^C \underline{f} \times_{T^A \underline{f}} T^A f\colon T^C B \times_{T^A B} T^A Y \to T^C \overline{B} \times_{T^A \overline{B}} T^A \overline{Y}. \tag{15}$$

For a product fibered manifold $B \times N$, $H(B \times N) = T^C B \times T^A N$.

Consider the product fibered manifold $\mathbb{R}^{k,l} = (\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k)$. For a jet-like homomorphism (F, φ) we define $F_{k,l} Y = F_0(\mathbb{R}^{k,l}, Y)$ and

$$F_{k,l} f\colon F_{k,l} Y \to F_{k,l} \overline{Y}, \quad F_{k,l} f(\varphi_0(g)) = \varphi_0(f \circ g). \tag{16}$$

Then $F_{k,l}$ is a product preserving bundle functor on $\mathcal{FM}$, corresponding to some $\mu_{k,l}\colon C_{k,l} \to A_{k,l}$. Every smooth function $g\colon \mathbb{R}^{k+l} \to \mathbb{R}$ can be interpreted as an

$\mathcal{FM}$-morphism $\widetilde{g}\colon \mathbb{R}^{k,l} \to i_2\mathbb{R}$. According to the general theory, [3], [4], $A_{k,l}$ is the factor algebra of the algebra $E(k+l)$ of germs of smooth functions on $\mathbb{R}^{k+l}$ at 0 with respect to the ideal $\mathcal{N}_{k,l}$ of all $\operatorname{germ}_0 g$ satisfying $\varphi_0(\widetilde{g}) = \varphi_0(\widetilde{\nu})$, where ν is the zero function on $\mathbb{R}^{k+l}$. Write

$$G_0(\mathbb{R}^{\overline{k},\overline{l}}, \mathbb{R}^{k,l})_0$$

for the set of all germs at 0 of $\mathcal{FM}$-morphisms $f\colon \mathbb{R}^{\overline{k},\overline{l}} \to \mathbb{R}^{k,l}$ satisfying $f(0) = 0$. By construction, ideals $\mathcal{N}_{k,l}$ have the following *substitution property*:

$$X \in \mathcal{N}_{k,l} \quad \text{and} \quad S \in G_0(\mathbb{R}^{\overline{k},\overline{l}}, \mathbb{R}^{k,l})_0 \quad \text{implies} \quad X \circ S \in \mathcal{N}_{\overline{k},\overline{l}}. \tag{17}$$

Denote by $\mathfrak{m} = \mathfrak{m}(k+l)$ and $\mathfrak{m}_B = \mathfrak{m}(k) \subset \mathfrak{m}(k+l)$ the maximal ideals.

In the case of J^{ϱ}, we write $A^{\varrho}_{k,l}$ and $C^{\varrho}_{k,l}$.

Lemma 3. *We have $A^{\varrho}_{k,l} = E(k+l)/\mathcal{N}^{\varrho}_{k,l}$, where*

$$\mathcal{N}^{\varrho}_{k,l} = \left\langle \mathfrak{m}^{r_0+1}, \mathfrak{m}^1_B\mathfrak{m}^{r_1+1}, \ldots, \mathfrak{m}^p_B\mathfrak{m}^{r_p+1}, \mathfrak{m}^{p+1}_B \right\rangle \tag{18}$$

and $C^{\varrho}_{k,l} = E(k)/\mathfrak{m}^{p+1}_B$.

P r o o f. Consider the situation from the proof of Proposition 1 with $M_1 = \mathbb{R}^{k,l}$, $M_2 = i_2\mathbb{R}$ and $\operatorname{germ}_0 g(x^i, y^p) \in \mathfrak{m}(k+l)$. Ideals $\mathfrak{m}^{p+1}_B$ and $\mathfrak{m}^p_B\mathfrak{m}^{r_p+1}$ forbid the derivatives $D_{\alpha_p,\beta_p}g(0)$ with $|\alpha_p| > p$ or $|\alpha_p| \geqslant p$ and $|\alpha_p| + |\beta_p| > p + r_p$ to enter $A^{\varrho}_{k,l}$. Further, the ideal $\mathfrak{m}^{p-1}_B\mathfrak{m}^{r_{p-1}+1}$ forbids the derivatives $D_{\alpha_{p-1},\beta_{p-1}}g(0)$ with $|\alpha_{p-1}| \geqslant p-1$ and $|\alpha_{p-1}| + |\beta_{p-1}| > p-1+r_{p-1}$. In the last but one step of this recurrence procedure, we have excluded the derivatives $D_{\alpha_1,\beta_1}g(0)$ with $|\alpha_1| \geqslant 1$ and $|\alpha_1| + |\beta_1| > 1 + r_1$. Finally, the ideal $\mathfrak{m}^{r_0+1}$ forbids the derivatives $D_{\beta_0}g(0)$ with $|\beta_0| > r_0$. Thus, what remains are the coordinates $w_{\alpha,\beta}$ from the proof of Proposition 1.

To find $C^{\varrho}_{k,l}$, we take into account that an $\mathcal{FM}$-morphism $g\colon \mathbb{R}^{k,l} \to i_1\mathbb{R}$ is determined by the base map $\underline{g}\colon \mathbb{R}^k \to \mathbb{R}$. So the derivatives v^t_α, $|\alpha| \leqslant p$, in the proof of Proposition 1 imply $C^{\varrho}_{k,l} = E(k)/\mathfrak{m}^{p+1}_B$. □

Clearly, in the case of $J^{\varrho,q}$ we have $A_{k,l} = A^{\varrho}_{k,l} = E(k+l)/\mathcal{N}^{\varrho}_{k,l}$ and $C_{k,l} = E(k)/\mathfrak{m}^{q+1}_B$.

Given k and l, we say that an ideal $I \subset \mathfrak{m}(k+l) = \mathfrak{m}$ has the substitution property, if (17) holds for every $X \in I$ and $S \in G_0(\mathbb{R}^{k,l}, \mathbb{R}^{k,l})_0$. Our main tool is the following algebraic result.

Proposition 2. *Let $I \subset \mathfrak{m}$ be an ideal of finite codimension with the substitution property. Then there exists a $(p+1)$-tuple ϱ satisfying (8) such that $I = \mathcal{N}^{\varrho}_{k,l}$.*

P r o o f. Since I is of finite codimension, there exists a minimal p such that $(x^1)^{p+1} \in I$. By the substitution property, $\mathfrak{m}_B^{p+1} \subset I$. Further, there exists a minimal r_0 satisfying $(y^1)^{r_0+1} \in I$, so that $\mathfrak{m}^{r_0+1} \subset I$. If $p = 0$, we have $I = \langle \mathfrak{m}^{r_0+1}, \mathfrak{m}_B \rangle$. If not, there is a minimal r_1 satisfying $(x^1)(y^1)^{r_1+1} \in I$, so that $\mathfrak{m}_B \mathfrak{m}^{r_1+1} \subset I$. Then $x^1(y^1)^{r_0} \in \mathfrak{m}^{r_0+1}$ implies $r_1 < r_0$. Assume that in the ith step of this procedure we have deduced $\mathfrak{m}_B^i \mathfrak{m}^{r_i+1} \subset I$. If $p > i$, there is a minimal r_{i+1} satisfying $(x^1)^{i+1}(y^1)^{r_{i+1}+1} \in I$, so that $\mathfrak{m}_B^{i+1}\mathfrak{m}^{r_{i+1}+1} \subset I$. Then $(x^1)^{i+1}(y^1)^{r_i} \in I$ implies $r_i > r_{i+1}$. In the last step, we obtain $\mathfrak{m}_B^p \mathfrak{m}^{r_p+1} \subset I$. □

In other words, for every k, l there exist $p(k,l)$ and $\varrho(k,l)$ such that $\mathcal{N}_{k,l}$ is of the form (18).

Lemma 4. *The numbers $p(k,l)$ and $\varrho(k,l)$ are independent of k and l.*

P r o o f. If $k \geqslant 1$ and $l \geqslant 1$, the construction of $p(k,l)$ and $\varrho(k,l)$ in the proof of Proposition 2 depends on x^1 and y^1 only. If we have another $\mathcal{N}_{\overline{k},\overline{l}}$ with $\overline{k} \geqslant 1$ and $\overline{l} \geqslant 1$, we consider the canonical projection $\mathfrak{m}(\overline{k}+\overline{l}) \to \mathfrak{m}(1+1)$ determined by $f(x^i, y^p) \mapsto f(x^1, 0, \ldots, 0, y^1, 0, \ldots, 0)$ and the canonical injection of $\mathfrak{m}(1+1)$ into $\mathfrak{m}(k+l)$. Then the substitution property implies our claim. In the case $l = 0$ or $k = 0$, we consider x^1 or y^1 only. □

$\mathcal{FM}$-morphisms of $\mathbb{R}^{k,l}$ into $i_1\mathbb{R}$ are determined by the base maps. Hence the Weil algebra $C_{k,l}$ of $F_{k,l}$ is independent of l. Then the substitution property implies $C_{k,l} = E(k)/\mathfrak{m}_B^{q+1} = \mathbb{D}_k^q$ for an integer q. In fact, this is the manifold result from [2].

Remark 2. Ideal I from Proposition 2 was studied by the first two authors in [1]. However, formula (16) from [1] is not correct because of a gap in the proof. Thus, in the special case of fibered manifolds, Theorem 1 corrects the result concerning fibered (r,s,q)-jets from [1].

4. Proof of Theorem 1

Since φ is defined on germs, it suffices to discuss fibered manifolds only. We determine the homomorphism $\mu_{k,l}\colon C_{k,l} = \mathbb{D}_k^q \to A_{k,l}^{\varrho}$. Every $\mathcal{FM}$-morphism $\mathbb{R}^{k,l} \to i_1\mathbb{R}$ is determined by its base map $g\colon \mathbb{R}^k \to \mathbb{R}$ and is of the form $\overline{g} = (g \circ p_{k,l}, g)$, where $p_{k,l}\colon \mathbb{R}^{k,l} \to \mathbb{R}^k$ is the bundle projection. Hence $C_{k,l}$ is the factor algebra of $E(k)$ with respect to the ideal of all $\mathrm{germ}_0\, g$ satisfying $\varphi_0(\overline{g}) = \varphi_0(\overline{\nu})$, where ν is the zero function on $\mathbb{R}^k$. Then $T^{C_{k,l}}B = J_0^q(\mathbb{R}^k, B)$ with classical q-jets and (because of the general definition of μ at the beginning of Section 3) $\mu_{k,l}$ is of the form

$$\mu_{k,l}(j_0^q g) = j_0^{\varrho}(\widetilde{g \circ p_{k,l}}). \tag{19}$$

Since (19) is a correct definition, we have $q \geqslant p$. Thus, we have deduced the algebra version of the formula

$$F_{k,l}Y = J_0^{\varrho,q}(\mathbb{R}^{k,l}, Y). \tag{20}$$

For a fibered manifold $\overline{p}\colon \overline{Y} \to \overline{B}$ with $\dim \overline{Y} = k + l$, $\dim \overline{B} = k$, we define

$$P^{\varrho,q}\overline{Y} = \operatorname{inv} J_0^{\varrho,q}(\mathbb{R}^{k,l}, \overline{Y}),$$

where inv indicates that we consider (ϱ, q)-jets of local $\mathcal{FM}$-isomorphisms. This is a principal bundle over $\overline{Y}$ with the structure group

$$G_{k,l}^{\varrho,q} = \operatorname{inv} J_0^{\varrho,q}(\mathbb{R}^{k,l}, \mathbb{R}^{k,l})_0.$$

Then one verifies directly that $F(\overline{Y}, Y)$ coincides with the associated bundle

$$F(\overline{Y}, Y) = P^{\varrho,q}\overline{Y}\left[J_0^{\varrho,q}(\mathbb{R}^{k,l}, Y)\right].$$

Indeed, for $X = \operatorname{germ}_z f \in G_{\overline{z}}\mathcal{F}ol(\overline{Y}, Y)_z$, if we take an arbitrary $u = j_0^{\varrho,q}\psi \in P_{\overline{z}}^{\varrho,q}\overline{Y}$, we have

$$\varphi(X) = \{j_0^{\varrho,q}\psi, j_0^{\varrho,q}(f \circ \psi)\}.$$

One verifies directly that this definition is independent of the choice of u.

This implies $(F, \varphi) = (J^{\varrho,q}, j^{\varrho,q})$.

Remark 3. The form of our description of all jet-like homomorphisms on $\mathcal{F}ol$ is properly related with the technique used in the proof of Theorem 1. In a concrete geometric situation, the consequences of the underlying projections $\pi_l^{k,r}$ from (6) need not be indicated. For example, in the case $(J^k\lambda J^r, j^k\lambda^r)$ of leafwise (k, r)-jets, we have $p = k$ and $\varrho = (r + k, r + k - 1, \ldots, r)$, but all $j_z^i\lambda^{r+k-i}f$, $i = 0, \ldots, k - 1$, are the canonical projections of $j_z^k\lambda^r f$.

Remark 4. We remark that our approach can be modified to the case of 2-fibered manifolds $Z \to Y \to B$, i.e. pairs of surjective submersions $Z \to Y$ and $Y \to B$. A section $Y \to Z$ can be viewed as a base-preserving $\mathcal{FM}$-morphism of $Y \to B$ into $Z \to B$ and the ideas of Section 1 can be directly applied to this situation.

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