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REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE WITH PSEUDO-$\mathbb{D}$-PARALLEL STRUCTURE JACOBI OPERATOR

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Abstract. We introduce the new notion of pseudo-$\mathbb{D}$-parallel real hypersurfaces in a complex projective space as real hypersurfaces satisfying a condition about the covariant derivative of the structure Jacobi operator in any direction of the maximal holomorphic distribution. This condition generalizes parallelness of the structure Jacobi operator. We classify this type of real hypersurfaces.

Keywords: real hypersurface, structure Jacobi operator

MSC 2010: 53C15, 53B25

1. Introduction

Let $\mathbb{C}P^m$, $m \geqslant 2$, be a *complex projective space* endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary. Let J denote the complex structure of $\mathbb{C}P^m$ and N a locally defined unit normal vector field on M. Then $-JN = \xi$ is a tangent vector field to M called the structure vector field on M. We also call $\mathbb{D}$ the maximal holomorphic distribution on M, that is, the distribution on M given by all vectors orthogonal to ξ at any point of M and let (φ, ξ, η, g) be the almost contact metric structure that the Kaehlerian structure of $\mathbb{C}P^m$ induces on M.

The study of real hypersurfaces in nonflat complex space forms is a classical topic in Differential Geometry. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [11], [12], [13], and is given by the following list: A_1:

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Geodesic hyperspheres. A_2: *Tubes over totally geodesic complex projective spaces.* B: *Tubes over complex quadrics* and $\mathbb{R}P^m$. C: *Tubes over the Segre embedding of* $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n+1=m$ and $m \geqslant 5$. D: *Tubes over the Plucker embedding of the complex Grassmann manifold* $G(2,5)$. In this case $m=9$. E: *Tubes over the cannonical embedding of the Hermitian symmetric space* $SO(10)/U(5)$. In this case $m=15$.

Other examples of real hypersurfaces are *ruled real* ones, that were introduced by Kimura, [5]: Take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ there is a unique complex projective hyperplane cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally although globally it will in general have self-intersections and singularities. Equivalently a ruled real hypersurface is such that $\mathbb{D}$ is integrable or, equivalently, $g(A\mathbb{D},\mathbb{D})=0$, where A denotes the shape operator of the immersion, see [5]. For further examples of ruled real hypersurfaces see [6].

Except these real hypersurfaces there are very few examples of real hypersurfaces in $\mathbb{C}P^n$. So, in Section 3, we present some results about non-existence of certain families of real hypersurfaces in complex projective space.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\widetilde{M},\tilde{g})$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called *Jacobi operator*. That is, if $\widetilde{R}$ is the curvature operator of $\widetilde{M}$, and X is any tangent vector field to $\widetilde{M}$, the Jacobi operator (with respect to X) at $p \in M$, $\widetilde{R}_X \in \mathrm{End}(T_p\widetilde{M})$, is defined as $(\widetilde{R}_X Y)(p) = (\widetilde{R}(Y,X)X)(p)$ for all $Y \in T_p\widetilde{M}$, being a selfadjoint endomorphism of the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$. Clearly, each tangent vector field X to $\widetilde{M}$ provides a Jacobi operator with respect to X.

Let M be a real hypersurface in a complex projective space and let ξ be *the structure vector field* on M. We will call the Jacobi operator on M with respect to ξ the *structure Jacobi operator* on M. In [2] the authors classify, under certain additional conditions, real hypersurfaces of $\mathbb{C}P^m$ whose structure Jacobi operator is parallel, in a certain sense, in the direction of ξ, namely, they suppose that $R'_\xi=0$, where $R'_\xi(Y) = (\nabla_\xi R)(Y,\xi)\xi$. They obtain class A_1 or A_2 hypersurfaces and a non-homogeneous real hypersurface. In [3] they classify real hypersurfaces in $\mathbb{C}P^m$ whose structure Jacobi operator commutes both with the shape operator and with the restriction of the complex structure to M.

In [10] we proved the non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geqslant 3$, whose structure Jacobi operator is $\mathbb{D}$-parallel, that is, $\nabla_X R_\xi = 0$, for any $X \in \mathbb{D}$.

In this paper, we introduce the notion of pseudo-$\mathbb{D}$-parallelness of the structure Jacobi operator for real hypersurfaces in $\mathbb{C}P^m$. It generalizes $\mathbb{D}$-parallelness of the structure Jacobi operator. The structure Jacobi operator of a real hypersurface of $\mathbb{C}P^m$ is pseudo-$\mathbb{D}$-parallel if it satisfies

$$(\nabla_X R_\xi)Y = c\{\eta(Y)\varphi AX + g(\varphi AX, Y)\xi\} \tag{1.1}$$

where c is a nonzero constant, $X \in \mathbb{D}$ and $Y \in TM$. We obtain the following

Theorem. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geqslant 3$, with pseudo-$\mathbb{D}$-parallel structure Jacobi operator. If $c \neq -1$ then $c < 0$ and M is locally congruent to a geodesic hypersphere of radius r such that $\cot^2(r) = -c$.*

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geqslant 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \varphi X + \eta(X)N$, and $-JN = \xi$. Then (φ, ξ, η, g) is an almost contact metric structure on M, see [1]. That is, we have

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\varphi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \varphi AX \tag{2.4}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ & - 2g(\varphi X, Y)\varphi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi \tag{2.6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M.

In the sequel we need the following results:

Theorem 2.1 [8]. *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geqslant 2$. Then the following are equivalent:*

1. *M is locally congruent to one of the homogeneous hypersurfaces of class A_1 or A_2.*
2. *$\varphi A = A\varphi$.*

Theorem 2.2 [8]. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geqslant 2$. If ξ is principal with principal curvature α, given a principal vector field $X \in \mathbb{D}$ with principal curvature λ, φX is also principal with principal curvature $(\alpha\lambda + 2)/(2\lambda - \alpha)$.*

3. Some previous results

Proposition 3.1. *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geqslant 4$, whose shape operator is given by $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$, $A\varphi U = 0$ and there exist two nonnull holomorphic distributions $\mathbb{D}_0$ and $\mathbb{D}_c$ such that $\mathbb{D}_0 \oplus \mathbb{D}_c = \operatorname{span}\{\xi, U, \varphi U\}^\perp$, for any $Z \in \mathbb{D}_0$, $AZ = 0$, for any $W \in \mathbb{D}_c$, $AW = -(c+1)\alpha^{-1}W$, where U is a unit vector field in $\mathbb{D}$, α and β are nonvanishing smooth functions defined on M, $(\varphi U)(\beta) = 0$ and c is a constant $c \neq 0, -1$.*

P r o o f. For any $W \in \mathbb{D}_c$, Codazzi equation yields $(\nabla_W A)\varphi W - (\nabla_{\varphi W} A)W = -2\xi$. Taking its scalar product with ξ we get

$$\beta g([\varphi W, W], U) = 2\Big(\frac{c+1}{\alpha}\Big)^2 + 2c. \tag{3.1}$$

The scalar product with U gives

$$g([\varphi W, W], U) = 2\beta. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\alpha^2(\beta^2 - c) = (c+1)^2. \tag{3.3}$$

As c is constant and we suppose $(\varphi U)(\beta) = 0$, from (3.3) $(\beta^2 - c)(\varphi U)(\alpha) = 0$. From (3.3) $\beta^2 - c \neq 0$. Thus $(\varphi U)(\alpha) = 0$. The Codazzi equation also gives $(\nabla_{\varphi U} A)\xi - (\nabla_\xi A)\varphi U = U$. If we develop it, as $(\varphi U)(\beta) = (\varphi U)(\alpha) = 0$ we obtain

$$\beta \nabla_{\varphi U} U + A \nabla_\xi \varphi U = U. \tag{3.4}$$

Taking its scalar product with U we get $1 = g(\nabla_\xi \varphi U, \beta\xi) = -\beta g(\varphi U, \varphi A\xi) = -\beta^2$. This is impossible and finishes the proof. □

With a proof similar to the proof of Proposition 3.2 in [10] we obtain

Proposition 3.2. *Let M be a ruled real hypersurface in $\mathbb{C}P^m$, $m \geqslant 2$. Then M does not satisfy* (1.1) *for any $X \in \mathbb{D}$, $Y \in TM$.*

Proposition 3.3. *There exist no real hypersurfaces M in $\mathbb{C}P^m$, $m \geqslant 3$, whose shape operator is given by $A\xi = (c+1)\xi + \beta U$, $AU = \beta\xi + (\beta^2/(c+1) - 1)U$, $A\varphi U = -\varphi U$, $AZ = -Z$, for any tangent vector Z orthogonal to* $\operatorname{span}\{\xi, U, \varphi U\}$, *where U is a unit vector field in $\mathbb{D}$, β is a nonvanishing smooth function defined on M and c is a constant $c \neq 0, -1$.*

P r o o f. Let us call $\mathbb{D}_U = \operatorname{span}\{\xi, U, \varphi U\}^\perp$ and take $Z \in \mathbb{D}_U$. Codazzi equation gives $(\nabla_Z A)U - (\nabla_U A)Z = 0$. If we take its scalar product with U we get

$$g(\nabla_U U, Z) = \frac{2Z(\beta)}{\beta}, \tag{3.5}$$

and its scalar product with ξ yields

$$g(\nabla_U U, Z) = \frac{Z(\beta)}{\beta}. \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$Z(\beta) = 0. \tag{3.7}$$

If we develop $(\nabla_U A)\xi - (\nabla_\xi A)U = -\varphi U$ and take its scalar product with U we have $U(\beta) = 2\beta(c+1)^{-1}\xi(\beta)$, and its scalar product with ξ gives $\xi(\beta) = 0$. Thus we get

$$U(\beta) = \xi(\beta) = 0. \tag{3.8}$$

If we now develop $(\nabla_{\varphi U} A)\xi - (\nabla_\xi A)\varphi U = U$ and take its scalar product with U we obtain

$$(c+1) + (\varphi U)(\beta) - \frac{\beta^2}{c+1} - \beta^2 + \frac{\beta^2}{c+1} g(\nabla_\xi \varphi U, U) = 0. \tag{3.9}$$

If we take its scalar product with ξ, it follows

$$g(\nabla_\xi \varphi U, U) = 4 + c. \tag{3.10}$$

From (3.9) and (3.10) we have $(\varphi U)(\beta) = -(2\beta^2 + (c+1)^2)/(c+1)$. Bearing in mind (3.7) and (3.8) this yields

$$\operatorname{grad}(\beta) = \omega \varphi U, \tag{3.11}$$

where $\omega = -(2\beta^2 + (c+1)^2)/(c+1)$. Then $\nabla_X \operatorname{grad}(\beta) = X(\omega)\varphi U + \omega \nabla_X \varphi U$, for any $X \in TM$. As $g(\nabla_X \operatorname{grad}(\beta), Y) = g(\nabla_Y \operatorname{grad}(\beta), X)$, for any $X, Y \in TM$ we obtain $X(\omega)g(\varphi U, Y) + \omega g(\nabla_X \varphi U, Y) = Y(\omega)g(\varphi U, X) + \omega g(\nabla_Y \varphi U, X)$, for any $X, Y \in TM$. If we suppose $\omega \neq 0$, taking $Y = \xi$ we have $-g(U, AX) = g(\nabla_\xi \varphi U, X)$, for any $X \in TM$. If we take $X = U$ we get $g(\nabla_\xi \varphi U, U) = -g(AU, U) = 1 - \beta^2/(c+1)$. From (3.10) we obtain $\beta^2 = -(c+3)(c+1)$. Then β is constant. Thus anyway $\omega = 0$, which means $2\beta^2 + (c+1)^2 = 0$, which is impossible, finishing the proof. □

4. Proof of the theorem

We suppose $c \neq 0, -1$.

Suppose firstly that M is not Hopf. Thus, at least locally, we can write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$ and β is a nonvanishing function. From (1.1) we get

$$\begin{aligned} &- g(Y, \varphi AX)\xi - \eta(Y)\varphi AX + g(\nabla_X A\xi, \xi)AY + g(A\xi, \varphi AX)AY \\ &+ \alpha(\nabla_X A)Y - g(Y, \nabla_X A\xi)A\xi - g(AY, \xi)\nabla_X A\xi \\ &\qquad = c\{\eta(Y)\varphi AX + g(\varphi AX, Y)\xi\}, \end{aligned} \tag{4.1}$$

for any $X \in \mathbb{D}$, $Y \in TM$. Taking $Y = \varphi U$ in (4.1) we obtain $-(c+1)g(U, AX)\xi + g(\nabla_X A\xi, \xi)A\varphi U + g(A\xi, \varphi AX)A\varphi U + \alpha(\nabla_X A)\varphi U - g(\varphi U, \nabla_X A\xi)A\xi = 0$. Its scalar product with ξ gives $(c+1)g(U, AX) + \alpha g(A\varphi U, \varphi AX) = 0$. And taking $X = \varphi U$ we have $(c+1)g(AU, \varphi U) = 0$. Thus

$$g(AU, \varphi U) = 0. \tag{4.2}$$

Take $Y = U$ in (4.1). Its scalar product with ξ yields

$$-(\beta^2 - (c+1))g(A\varphi U, X) + \alpha g(A\varphi AU, X) = 0 \tag{4.3}$$

for any $X \in \mathbb{D}$. Thus $-(\beta^2 - (c+1))A\varphi U + \alpha A\varphi AU$ has no component in $\mathbb{D}$. From (4.2) we get

$$-(\beta^2 - (c+1))A\varphi U + \alpha A\varphi AU = 0. \tag{4.4}$$

On the other hand, $-(c+1)g(AU, X) + \alpha g(A\varphi A\varphi U, X) = 0$, for any $X \in \mathbb{D}$. Thus $-(c+1)AU + \alpha A\varphi A\varphi U$ has no component in $\mathbb{D}$. This yields

$$(c+1)AU - \alpha A\varphi A\varphi U = \beta(c+1+\alpha g(A\varphi U, \varphi U))\xi. \tag{4.5}$$

From (4.5) we have $(c+1)g(AU, U) = \alpha g(A\varphi A\varphi U, U)$ and from (4.4), $-(\beta^2 - (c+1))g(A\varphi U, \varphi U) + \alpha g(A\varphi AU, \varphi U) = 0$. From these equalities we get

$$(c+1)g(AU, U) = (c+1-\beta^2)g(A\varphi U, \varphi U). \tag{4.6}$$

Then we take $Y \in \mathbb{D}_U$, $X \in \mathbb{D}$ in (4.1) and the scalar product with ξ. We obtain $(c+1)g(\varphi AX, Y) = -\alpha g(AY, \varphi AX)$. Taking $X = Y$ we have

$$g(\varphi X, AX) = 0 \tag{4.7}$$

for any $X \in \mathbb{D}_U$. Moreover $(c+1)A\varphi X + \alpha A\varphi AX$ has no component in $\mathbb{D}$.Thus

$$(c+1)A\varphi X + \alpha A\varphi AX = \alpha\beta g(\varphi AX, U)\xi \tag{4.8}$$

for any $X \in \mathbb{D}_U$. If we change X by φX in (4.8) we obtain $-(c+1)AX + \alpha A\varphi A\varphi X = \alpha\beta g(\varphi A\varphi X, U)\xi$. Its scalar product with X yields $-(c+1)g(AX, X) + \alpha g(A\varphi A\varphi X, X) = 0$. The scalar product of (4.8) with φX gives $(c+1)g(A\varphi X, \varphi X) - \alpha g(A\varphi A\varphi X, X) = 0$. From these expressions we get

$$g(AX, X) = g(A\varphi X, \varphi X) \tag{4.9}$$

for any $X \in \mathbb{D}_U$. Taking $Y \in \mathbb{D}_U$, $X = \varphi U$ in(4.1) and its scalar product with ξ we have $(c+1)g(\varphi A\varphi U, Y) = -\alpha g(AY, \varphi A\varphi U)$. As for any $X \in \mathbb{D}$, $(c+1)g(AX, U) = -\alpha g(A\varphi U, \varphi AX)$, we get $g(AU, Y) = g(A\varphi U, \varphi Y)$, for any $Y \in \mathbb{D}_U$. Changing Y by φY, for any $Y \in \mathbb{D}_U$, $-g(Y, A\varphi U) = g(AU, \varphi Y)$. Thus $A\varphi U - \varphi AU$ has no component in $\mathbb{D}_U$, and from (4.6)

$$A\varphi U - \varphi AU = \frac{\beta^2}{c+1} g(A\varphi U, \varphi U)\varphi U. \tag{4.10}$$

We want to prove that AU and $A\varphi U$ have no component in $\mathbb{D}_U$. Thus we can suppose $AU = \beta\xi + g(AU, U)U + \mu Z$, $A\varphi U = g(A\varphi U, \varphi U)\varphi U + \varepsilon W$, where μ, ε are smooth functions on M and Z, W unit vector fields in $\mathbb{D}_U$. Then, $\varphi AU = g(AU, U)\varphi U + \mu\varphi Z$. Thus $A\varphi U - \varphi AU = \beta^2(c+1)^{-1}g(A\varphi U, \varphi U)\varphi U + \varepsilon W - \mu\varphi Z$. From (4.10) we have

$$\varepsilon W = \mu\varphi Z. \tag{4.11}$$

Taking $Y = \varphi Z$, $X = U$ in (4.1) and its scalar product with ξ we get

$$-(c+1)g(AU, Z) = \alpha g(A\varphi Z, \varphi AU).$$

This gives

$$(c+1)\mu + \alpha\mu g(AU, U) + \alpha\mu g(A\varphi Z, \varphi Z) = 0. \tag{4.12}$$

Taking $Y = Z$, $X = \varphi U$ in (4.1) and its scalar product with ξ, we have similarly

$$(c+1)\mu + \alpha\mu g(A\varphi U, \varphi U) + \alpha\mu g(AZ, Z) = 0. \tag{4.13}$$

From (4.12) and (4.13), if $\mu \neq 0$, it follows $\alpha \neq 0$ and $g(A\varphi U, \varphi U) + g(AZ, Z) = g(AU, U) + g(A\varphi Z, \varphi Z)$. From (4.9) we should have $g(A\varphi U, \varphi U) = g(AU, U)$. From (4.6), $\beta^2 g(A\varphi U, \varphi U) = 0$. This yields $AU = \beta\xi + \mu Z$, $A\varphi U = \mu\varphi Z$, $g(AZ, Z) = g(A\varphi Z, \varphi Z) = -(c+1)/\alpha$. From (4.5) we obtain $(c+1)AU - \alpha A\varphi A\varphi U = (c+1)\beta\xi$. This gives $(c+1)\beta\xi + (c+1)\mu Z + \alpha\mu AZ = (c+1)\beta\xi$. Thus $(c+1)\mu Z + \alpha\mu AZ = 0$. Taking its scalar product with U we get $\alpha\mu^2 = 0$, which is impossible. We have obtained that μ must be zero, and if we write $A\varphi U = \delta\varphi U$, we have $AU = \beta\xi + (1 - \beta^2/(c+1))\delta U$. Moreover, $\mathbb{D}_U$ is A-invariant. Take a unit $Z \in \mathbb{D}_U$ such that $AZ = \lambda Z$. From (4.8) it follows that $(c+1)A\varphi Z + \alpha\lambda A\varphi Z = 0$. If $A\varphi Z = 0$, taking $X = \varphi Z$ in (4.8) we get $-(c+1)AZ = 0$. Thus $\lambda = 0$. Thus the unique eigenvalues of A that could appear in $\mathbb{D}_U$ are either 0 or $-(c+1)/\alpha$. We also can conclude that the corresponding eigenspaces are holomorphic, that is, they are invariant by φ.

Suppose firstly that there exists $Z \in \mathbb{D}_U$ such that $AZ = A\varphi Z = 0$. The Codazzi equation gives $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\varphi Z$. Developing this equation and taking its scalar product with φZ we get

$$g(\nabla_Z U, \varphi Z) = -\frac{1}{\beta}. \tag{4.14}$$

On the other hand, $(\nabla_Z A)\varphi U - (\nabla_{\varphi U} A)Z = 0$. This implies $Z(\delta)\varphi U + \delta\nabla_Z\varphi U - A\nabla_Z\varphi U + A\nabla_{\varphi U}Z = 0$. Taking its scalar product with Z, and bearing in mind (2.3), we obtain

$$\delta g(\nabla_Z U, \varphi Z) = 0. \tag{4.15}$$

If $\delta \neq 0$, (4.14) and (4.15) give a contradiction. Thus $\delta = 0$. In this case, if for any $Z \in \mathbb{D}_U$, $AZ = 0$, recall that we have $AU = \beta\xi$, $A\varphi U = 0$. Thus we obtain a ruled real hypersurface. Proposition 3.2 implies that this case does not occur.

Now we suppose that there exists $Z \in \mathbb{D}_U$ such that $AZ = 0$, that is $Z \in \mathbb{D}_0$ and there exists $W \in \mathbb{D}_U$ such that $AW = -(c+1)/\alpha W$. From Proposition 3.1 we have $(\varphi U)(\beta) \neq 0$. From (1.1) $(\nabla_{\varphi U} R_\xi)U = -c\delta\xi$. On the other hand, $(\nabla_{\varphi U} R_\xi)U = (\varphi U)(\beta^2)U - \beta^2\nabla_{\varphi U}U + \delta\xi - \alpha A\nabla_{\varphi U}U + \alpha^2\delta\xi + \alpha\beta\delta U$. Its scalar product with U yields $(\varphi U)(\beta^2) = 0$. Thus this kind of real hypersurfaces does not satisfy our condition.

Therefore we must suppose that $AU = \beta\xi + \delta(1 - \beta^2/(c+1))U$, $A\varphi U = \delta\varphi U$, $AZ = -(c+1)\alpha^{-1}Z$ for any $Z \in \mathbb{D}_U$. From the Codazzi equation $(\nabla_Z A)\varphi Z - (\nabla_{\varphi Z} A)Z = -2\xi$. Developing it and taking its scalar product with ξ we get

$$\beta g([\varphi Z, Z], U) = 2c + 2\Big(\frac{c+1}{\alpha}\Big)^2. \tag{4.16}$$

and its scalar product with U yields

$$\Big(\frac{c+1}{\alpha} + \delta\Big(1 - \frac{\beta^2}{c+1}\Big)\Big)g([\varphi Z, Z], U) - 2\beta\frac{c+1}{\alpha} = 0. \tag{4.17}$$

From (4.16) and (4.17) we have

$$[(c+1)^2 + \alpha\delta(c+1-\beta^2)][(c+1)^2 + c\alpha^2] = \alpha^2\beta^2(c+1)^2. \tag{4.18}$$

From (1.1), $(\nabla_{\varphi U} R_\xi)U = -c\delta\xi$. Taking its scalar product with ξ we obtain

$$-c\delta = \delta\Big[\alpha\Big(1 - \frac{\beta^2}{c+1}\Big)\delta - \beta^2 + 1\Big]. \tag{4.19}$$

If we suppose that $\delta = 0$, $A\xi = \alpha\xi + \beta U$, $A\varphi U = 0$, $AZ = -(c+1)\alpha^{-1}Z$, for any $Z \in \mathbb{D}_U$. From Codazzi equation we have $(\nabla_{\varphi U} A)\xi - (\nabla_\xi A)\varphi U = U$. Taking its scalar product with U we have $(\varphi U)(\beta) = \beta^2 + 1$. But as from (1.1) $(\nabla_{\varphi U} R_\xi)U = 0$, taking its scalar product with U we get $(\varphi U)(\beta^2) = 0$, giving a contradiction. Thus $\delta \neq 0$. From (4.18) and (4.19) we obtain

$$\alpha^2 = (c+1)^2. \tag{4.20}$$

Thus changing, if necessary, ξ by $-\xi$, we can suppose that $\alpha = c+1$. From (4.18) and (4.20) we have

$$(\alpha - \beta^2)(1+\delta) = 0. \tag{4.21}$$

From Proposition 3.3 $\delta \neq -1$. Then from (4.21), $\beta^2 = \alpha = c+1$. That is, $A\xi = (c+1)\xi + \beta U$, $AU = \beta\xi$, $A\varphi U = \delta\varphi U$, $AZ = -Z$, for any $Z \in \mathbb{D}_U$.

From (1.1) we get $(\nabla_U R_\xi)\varphi U = (\nabla_{\varphi U} R_\xi)\varphi U = (\nabla_Z R_\xi)\varphi U = 0$, for any $Z \in \mathbb{D}_U$. Taking the corresponding scalar products with φU we get

$$U(\delta) = (\varphi U)(\delta) = Z(\delta) = 0, \tag{4.22}$$

for any $Z \in \mathbb{D}_U$. Codazzi equation implies $(\nabla_U A)\varphi U - (\nabla_{\varphi U} A)U = -2\xi$. This yields $\delta g(\nabla_{\varphi U} U, \varphi U) = 0$. On the other hand, as $(\nabla_\xi A)\varphi U - (\nabla_{\varphi U} A)\xi = -U$, we obtain $\xi(\delta) = \beta g(\nabla_{\varphi U} U, \varphi U)$. From these equations we get

$$\xi(\delta) = 0. \tag{4.23}$$

From (4.22) and (4.23) we conclude that δ is constant.

Suppose $\delta = 0$. Codazzi equation yields $(\nabla_{\varphi U} A)\xi - (\nabla_\xi A)\varphi U = U$. Its scalar product with U gives $\beta^2 = -1$, which is impossible. Thus $\delta \neq 0$.

From the Codazzi equation $(\nabla_U A)\varphi U - (\nabla_{\varphi U} A)U = -2\xi$, and its scalar product with U yields

$$g(\nabla_U \varphi U, U) = -2\beta. \tag{4.24}$$

From (1.1), $(\nabla_U R_\xi)\varphi U = 0$. Taking its scalar product with U we get

$$(\delta+1) g(\nabla_U \varphi U, U) = 0. \tag{4.25}$$

From (4.24) and (4.25) we should have $\delta = -1$ and we arrive to a contradiction. So we conclude that M must be Hopf, that is $A\xi = \alpha\xi$. Now α is locally constant and (4.1) changes to

$$\begin{aligned} -& g(Y,\varphi AX)\xi - \eta(Y)\varphi AX + \alpha(\nabla_X A)Y - \alpha^2 g(Y,\varphi AX)\xi - \alpha^2\eta(Y)\varphi AX \\ &= c\{\eta(Y)\varphi AX + g(Y,\varphi AX)\xi\} \end{aligned} \tag{4.26}$$

for any $X \in \mathbb{D}$, $Y \in TM$. Taking its scalar product with ξ we obtain

$$(c+1) g(\varphi AX, Y) + \alpha g(A\varphi AX, Y) = 0$$

for any $X \in \mathbb{D}$, $Y \in TM$. Thus

$$(c+1)\varphi AX + \alpha A\varphi AX = 0 \tag{4.27}$$

for any $X \in \mathbb{D}$. Then for any $X, Y \in \mathbb{D}$, $g((c+1)\varphi AX + \alpha A\varphi AX, Y) = 0 = -g(X, (c+1)A\varphi Y + \alpha A\varphi AY)$. Thus

$$(c+1)A\varphi Y + \alpha A\varphi AY = 0 \tag{4.28}$$

for any $Y \in \mathbb{D}$. From (4.27) and (4.28) we have $\varphi AY = A\varphi Y$ for any $Y \in \mathbb{D}$. As also $A\varphi\xi = \varphi A\xi = 0$, we conclude that $A\varphi = \varphi A$. Now from Theorem 2.1, M must be locally congruent to a real hypersurface of type either A_1 or A_2.

If M is of type A_1, for any $X \in \mathbb{D}$, $AX = \cot(r)X$. From (1.1) $(\nabla_X R_\xi)\xi = \cot(r)c\varphi X = -\cot^3(r)\varphi X$. Thus $c = -\cot^2(r)$ and M is locally congruent to a geodesic hypersphere of radius r such that $\cot^2(r) = -c$. It is easy to see that this geodesic hypersphere satisfies (1.1)

If M is of type A_2 we write $A\xi = 2\cot(2r)\xi$. In $\mathbb{D}$ we have two nonzero holomorphic distributions corresponding, respectively to the eigenvalues $\cot(r)$ and $-\tan(r)$. Take a unit X such that $AX = \cot(r)X$. From (1.1), if we develop $(\nabla_X R_\xi)\xi$ we obtain $c = -\cot^2(r)$. The same procedure applied to a unit Y such that $AY = -\tan(r)Y$ yields $c = -\tan^2(r)$. This gives $c = -1$, which is impossible.

This finishes the proof. $\square$

Remark. It is easy to see that geodesic hypersphere of radius $\frac{1}{4}\pi$ satisfies (1.1) for the case $c = -1$, although we have been not able to find a complete classification in this case.

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