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A HYBRID MEAN VALUE OF THE DEDEKIND SUMS

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Abstract. The main purpose of this paper is to use the M. Toyoizumi's important work, the properties of the Dedekind sums and the estimates for character sums to study a hybrid mean value of the Dedekind sums, and give a sharper asymptotic formula for it.

Keywords: Dedekind sums, Dirichlet L-function, mean value

MSC 2010: 11L40, 11F20, 11M20

1. Introduction

For a positive integer q and an arbitrary integer h, the Dedekind sum $S(h,q)$ is defined as follows:

$$S(h,q)=\sum_{a=1}^{q}\Big(\Big(\frac{a}{q}\Big)\Big)\Big(\Big(\frac{ha}{q}\Big)\Big),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2} & \text{if } x \text{ is not an integer;}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

The various properties of $S(h,q)$ were investigated by many authors. Maybe the most famous property of the Dedekind sums is the reciprocity formula (see references [1], [2], and [3]):

$$S(h,q)+S(q,h)=\frac{h^2+q^2+1}{12hq}-\frac{1}{4} \tag{1}$$

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for all $(h,q)=1$, $q>0$, $h>0$. J.B. Conrey etc. [4] studied the mean value distribution of $S(h,k)$, and first proved the important asymptotic formula

$$(2)\qquad \sum_{h=1}^{k}{}' |S(h,k)|^{2m} = f_m(k)\Big(\frac{k}{12}\Big)^{2m} + O\big((k^{9/5}+k^{2m-1+(1/(m+1))})\log^3 k\big),$$

where $\sum_h{}'$ denotes the summation over all h such that $(k,h)=1$, and

$$\sum_{n=1}^{\infty}\frac{f_m(n)}{n^s} = 2\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s).$$

Especially for $m=1$ and $k=p$, an odd prime, we may immediately get the asymptotic formula

$$(3)\qquad \sum_{h=1}^{p-1}|S(h,p)|^2 = \frac{5}{144}p^2 + O(p^{9/5}\log^3 p).$$

In the spirit of [4] and [5], Wenpeng Zhang [6] obtained a sharper asymptotic formula for (2) with $m=1$ and $k=p^n$. That is,

$$(4)\qquad \sum_{h=1}^{k}{}' |S(h,p)|^2 = \frac{5}{144}k^2\frac{(p^2-1)^2}{p(p^3-1)} + O\Big(k\exp\Big(\frac{3\log k}{\log\log k}\Big)\Big),$$

and

$$(5)\qquad \sum_{h=1}^{k}{}'\frac{S(h,p)}{h} = \frac{\pi^2}{72}k\Big(1-\frac{1}{p^2}\Big) + O(k^{1/2}).$$

In this paper, we use M. Toyoizumi's important work, the properties of the Dedekind sums and the estimates for character sums to study the mean value

$$\sum_{a=1}^{p-1}a^k S^2(a,p),$$

and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any odd prime p and positive integer k, we have the asymptotic formula*

$$\sum_{a=1}^{p-1} a^k S^2(a,p) = Ap^{k+2} + O\Big(p^{k+3/2}\exp\Big(\frac{6\log p}{\log\log p}\Big)\Big),$$

where the constant

$$A = \Bigg(\frac{5}{144(k+1)} + \frac{1}{\pi^4}\sum_{1\leqslant m\leqslant k/2} \frac{\binom{k}{2m-1}(2m-1)!}{(-1)^{m+1}(2\pi)^{2m}} \sideset{}{'}\sum_{n=1}^{\infty}\sideset{}{'}\sum_{v=1}^{\infty} \frac{\sum_{d_1|n}\sum_{d_2|n}\cos(2\pi d_1 v\overline{d_2}/p)}{n^2v^{2m}}\Bigg).$$

For general positive integer q and $m \geqslant 1$, whether there exists an asymptotic formula for the mean value

$$\sideset{}{'}\sum_{a=1}^{q} a^k S^{2m}(a,q)$$

is an open problem.

2. Some lemmas

To prove our theorem, we need several lemmas.

Lemma 1. *Let $q \geqslant 3$ be an integer and $(h,q)=1$. Then we have the identity*

$$S(h,q) = \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(h)|L(1,\chi)|^2,$$

where $\varphi(d)$ is the Euler function, χ denotes a Dirichlet character modulo d with $\chi(-1)=-1$, and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof. See Lemma 2 of [6]. □

Lemma 2. *Let $q \geqslant 3$ be an integer, let χ denote any primitive character* $\bmod\, q$. *Then we have the identity*

$$\sum_{a=1}^{q} a^n\chi(a) = \begin{cases} 2q^n\tau(\chi)\displaystyle\sum_{1\leqslant m\leqslant n/2}\frac{\binom{n}{2m-1}(2m-1)!\,L(2m,\overline{\chi})}{(-1)^{m+1}(2\pi)^{2m}} & \text{if } \chi(-1)=1,\\ 2q^n\tau(\chi)\displaystyle\sum_{0\leqslant m\leqslant (n-1)/2}\frac{\binom{n}{2m}(2m)!\,L(2m+1,\overline{\chi})}{(-1)^{m+1}(2\pi)^{2m+1}\mathrm{i}} & \text{if } \chi(-1)=-1,\end{cases}$$

where $\tau(\chi)=\sum_{a=1}^{q}\chi(a)e(a/q)$ *is the Gaussian sum,* $e(y)=\mathrm{e}^{2\pi \mathrm{i}y}$, $L(s,\chi)$ *is the Dirichlet* L*-function corresponding to* χ, *and* $\binom{n}{m}=n!/(m!(n-m)!)$.

P r o o f. See [7]. □

Lemma 3. *Let* $q\geqslant 3$ *be an integer. Then we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}|L(1,\chi)|^4=\frac{5\pi^4}{144}\varphi(q)\prod_{p\mid q}\frac{(p^2-1)^3}{p^4(p^2+1)}+O\Big(\frac{\varphi(q)}{q}\exp\Big(\frac{3\ln q}{\ln\ln q}\Big)\Big),$$

where $\prod_{p\mid q}$ *denotes the product over all distinct prime divisors of* q.

P r o o f. See Lemma 3 of [6]. □

Lemma 4. *Let* $p\geqslant 3$ *be a prime, let* χ^0 *denote the principal character* $\bmod p$, *and let* $\chi_1\neq\chi^0$ *be any even character* $\bmod p$. *Then we have the asymptotic formula*

$$\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}|L(1,\chi_1)|^2|L(1,\chi\overline{\chi_1})|^2=\frac{\varphi(p)}{2}{\sum_{n=1}^{\infty}}'\frac{\big|\sum_{d\mid n}\chi(d)\big|^2}{n^2}+O\Big(\exp\Big(\frac{6\log p}{\log\log p}\Big)\Big),$$

where ${\sum_{n=1}^{\infty}}'$ *denotes the summation over all* n *such that* $(n,p)=1$.

P r o o f. Let $r(n)=\sum_{d\mid n}\overline{\chi}(d)$, let $\chi_1\neq\chi^0$ be any even character $\bmod p$. Then for a parameter $N\geqslant p$, applying Abel's identity we have

$$L(1,\chi_1)L(1,\overline{\chi}\chi_1)=\sum_{n=1}^{\infty}\frac{\chi_1(n)r(n)}{n}=\sum_{1\leqslant n\leqslant N}\frac{\chi_1(n)r(n)}{n}+\int_N^{\infty}\frac{A(y,\chi_1)}{y^2}\,\mathrm{d}y \tag{6}$$

where $A(y,\chi_1)=\sum_{N<n\leqslant y}\chi_1(n)r(n)$.

Note the partition identities

$$A(y,\chi_1)=\sum_{n\leqslant\sqrt{y}}\chi_1(n)\sum_{N/n\leqslant m\leqslant y/n}\chi_1(m)\overline{\chi}(m)+\sum_{m\leqslant\sqrt{y}}\chi_1(m)\overline{\chi}(m)\sum_{N/n\leqslant n\leqslant y/n}\chi_1(n)$$
$$-\Big(\sum_{\sqrt{N}\leqslant n\leqslant\sqrt{y}}\chi_1(n)\Big)\Big(\sum_{\sqrt{N}\leqslant n\leqslant\sqrt{y}}\chi_1(n)\overline{\chi}(n)\Big).$$

Applying the Cauchy inequality and the estimates for character sums

$$\sum_{\chi_1\neq\chi^0}\Big|\sum_{N\leqslant n\leqslant M}\chi_1(n)\Big|^2 = \sum_{\chi_1\neq\chi^0}\Big|\sum_{N\leqslant n\leqslant M\leqslant N+p}\chi_1(n)\Big|^2$$

$$= \varphi(p)\sum_{N\leqslant n\leqslant M\leqslant N+p}\chi^0(n) - \Big|\sum_{N\leqslant n\leqslant M\leqslant N+p}\chi^0(n)\Big|^2 \leqslant \frac{\varphi^2(p)}{4}$$

we have

$$(7)\quad \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}|A(y,\chi_1)|^2 \ll \sqrt{y}\sum_{n\leqslant\sqrt{y}}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\Big|\sum_{N/n\leqslant m\leqslant y/n}\chi_1(m)\overline{\chi}(m)\Big|^2$$

$$+\sqrt{y}\sum_{m\leqslant\sqrt{y}}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\Big|\sum_{N/n\leqslant n\leqslant y/n}\chi_1(n)\Big|^2$$

$$+\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\Big|\sum_{\sqrt{N}\leqslant n\leqslant\sqrt{y}}\chi_1(n)\Big|^2\Big|\sum_{\sqrt{N}\leqslant n\leqslant\sqrt{y}}\chi_1(n)\overline{\chi}(n)\Big|^2$$

$$\ll y\varphi^2(p).$$

Thus from (7) and the Cauchy inequality we obtain

$$(8)\quad \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\Big|\int_N^\infty\frac{A(y,\chi_1)}{y^2}\,\mathrm{d}y\Big|^2$$

$$\leqslant \int_N^\infty\int_N^\infty\frac{1}{y^2z^2}\Big(\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}|A(y,\chi_1)||A(z,\chi_1)|\Big)\,\mathrm{d}y\,\mathrm{d}z$$

$$\ll\Big(\int_N^\infty\frac{1}{y^2}\Big(\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}|A(y,\chi_1)|^2\Big)^{1/2}\mathrm{d}y\Big)^2$$

$$\ll\frac{\varphi^2(p)}{N}.$$

Note that for $(ab,p)=1$, the orthogonality relation for character sums modulo p yields

$$\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\chi_1(a)\chi_1(b)=\begin{cases}\dfrac{\varphi(p)}{2} & \text{if } a\equiv b \bmod p;\\ -\dfrac{\varphi(p)}{2} & \text{if } a\equiv -b \bmod p;\\ 0, & \text{otherwise,}\end{cases}$$

so that

$$(9)\qquad \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \Big|\sum_{1\leqslant n\leqslant N} \frac{\chi_1(n)r(n)}{n}\Big|^2$$

$$= \frac{\varphi(p)}{2} \sideset{}{'}\sum_{\substack{1\leqslant a,\ b\leqslant N\\ a\equiv b \pmod p}} \frac{r(a)\overline{r(b)}}{ab} - \frac{\varphi(p)}{2} \sideset{}{'}\sum_{\substack{1\leqslant a,\ b\leqslant N\\ a\equiv -b \pmod p}} \frac{r(a)\overline{r(b)}}{ab}$$

$$= \frac{\varphi(p)}{2} \sideset{}{'}\sum_{1\leqslant a\leqslant N} \frac{|r(a)|^2}{a^2} + O\Big(\varphi(p)\sum_{b=1}^{N}\sum_{l=1}^{[N/p]} \frac{d(b)d(lp+b)}{(lp+b)b}\Big)$$

$$+ O\Big(\varphi(p)\sum_{a=1}^{p-1} \frac{d(a)d(p-a)}{a(p-a)}\Big)$$

$$+ O\Big(\varphi(p)\sum_{a=1}^{N}\sum_{(1+a/p)\leqslant l\leqslant N/p}^{[N/p]} \frac{d(a)d(lp-a)}{a(lp-a)}\Big)$$

$$= \frac{\varphi(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \frac{|r(n)|^2}{n^2} + O\Big(\exp\Big(\frac{2\log N}{\log\log N}\Big)\Big),$$

where $d(n)$ is the divisor function and $|r(n)| \leqslant d(n) \ll \exp\big((1+\varepsilon)\frac{\log 2 \log n}{\log\log n}\big)$.

Now

$$(10)\qquad \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} \Big(\sum_{1\leqslant n\leqslant N} \frac{\chi_1(n)r(n)}{n}\Big)\Big(\int_N^{\infty} \frac{\overline{A(y,\chi_1)}}{y^2}\,\mathrm{d}y\Big)$$

$$\ll \ln^2 N \int_N^{\infty} \frac{1}{y^2}\Big(\sum_{\substack{\chi_1 \bmod p\\ \chi(-1)=-1}} |A(y,\overline{\chi_1})|\Big)\,\mathrm{d}y$$

$$\ll \varphi^{3/2}(p) N^{-1/2} \ln^2 N.$$

Taking the parameter $N = p^3$, from (6), (7), (8), (9), and (10) we immediately obtain

$$\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} |L(1,\chi_1)|^2 |L(1,\chi\overline{\chi_1})|^2 = \frac{\varphi(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \frac{|\sum_{d|n}\chi(d)|^2}{n^2} + O\Big(\exp\Big(\frac{6\log p}{\log\log p}\Big)\Big).$$

This completes the proof of Lemma 4. □

Lemma 5. *Let $p \geqslant 3$ be a prime and m a positive integer, let χ^0 denote the principal character* $\bmod p$*, and let χ_1 and χ_2 be any two odd characters* $\bmod p$*. Then we have the the asymptotic formula*

$$E := \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2 \neq \chi^0}} \tau(\chi_1\chi_2) L(2m, \overline{\chi_1\chi_2}) |L(1,\chi_1)|^2 |L(1,\chi_2)|^2$$
$$= \frac{\varphi^2(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \sideset{}{'}\sum_{v=1}^{\infty} \frac{\sum_{d_1|n} \sum_{d_2|n} \cos(2\pi d_1 v \overline{d_2}/p)}{n^2 v^{2m}} + O\Big(p^{3/2} \exp\Big(\frac{6\log p}{\log\log p}\Big)\Big).$$

P r o o f. For any odd characters χ_1 and χ_2 $\bmod p$, let $\chi = \chi_1\chi_2$. Note that for $(ab,p)=1$, the orthogonality relation for character sums modulo p implies

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \chi(a)\chi(b) = \begin{cases} \dfrac{\varphi(p)}{2} & \text{if } a \equiv \pm b \bmod p, \\ 0 & \text{otherwise}, \end{cases}$$

and from Lemma 4 we have

$$E := \sum_{\substack{\chi_1 \bmod p \\ \chi(-1)=-1}} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi^0}} \tau(\chi) L(2m, \overline{\chi}) |L(1,\chi_1)|^2 |L(1,\chi\overline{\chi_1})|^2$$
$$= \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi^0}} \tau(\chi) L(2m, \overline{\chi}) \sum_{\substack{\chi_1 \bmod p \\ \chi(-1)=-1}} |L(1,\chi_1)|^2 |L(1,\chi\overline{\chi_1})|^2$$
$$= \frac{\varphi(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi^0}} \tau(\chi) L(2m, \overline{\chi}) \Big|\sum_{d|n} \overline{\chi(d)}\Big|^2 + O\Big(p^{3/2} \exp\Big(\frac{6\log p}{\log\log p}\Big)\Big)$$
$$= \frac{\varphi(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \tau(\chi) L(2m, \overline{\chi}) \Big|\sum_{d|n} \overline{\chi(d)}\Big|^2$$
$$- \frac{\varphi(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \frac{1}{n^2} \tau(\chi^0) L(2m, \chi^0) \Big|\sum_{d|n} \chi^0(d)\Big|^2 + O\Big(p^{3/2} \exp\Big(\frac{6\log p}{\log\log p}\Big)\Big)$$
$$= \frac{\varphi(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{d_1|n} \sum_{d_2|n} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \tau(\chi) L(2m, \overline{\chi}) \overline{\chi}(d_1) \chi(d_2)$$
$$+ O\Big(p^{3/2} \exp\Big(\frac{6\log p}{\log\log p}\Big)\Big)$$

$$= \frac{\varphi^2(p)}{4} \sideset{}{'}\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{d_1|n} \sum_{d_2|n} \sum_{\substack{u=1 \\ ud_2 \equiv \pm v d_1 \pmod p}}^{p-1} e\Big(\frac{u}{p}\Big) \sideset{}{'}\sum_{v=1}^{\infty} \frac{1}{v^{2m}} + O\Big(p^{3/2} \exp\Big(\frac{6\log p}{\log\log p}\Big)\Big)$$

$$= \frac{\varphi^2(p)}{2} \sideset{}{'}\sum_{n=1}^{\infty} \sideset{}{'}\sum_{v=1}^{\infty} \frac{\sum_{d_1|n}\sum_{d_2|n} \cos(2\pi d_1 v \overline{d_2}/p)}{n^2 v^{2m}} + O\Big(p^{3/2} \exp\Big(\frac{6\log p}{\log\log p}\Big)\Big).$$

This completes the proof of Lemma 5. □

3. Proof of theorem

In this section, we complete the proof of our theorem. From Lemma 1, Lemma 2, Lemma 3, Lemma 4, and Lemma 5 we have

$$\sum_{a=1}^{p-1} a^k S^2(a,p) = \sum_{a=1}^{p-1} a^k \Bigg(\frac{p}{\pi^2 \varphi(p)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(a) |L(1,\chi)|^2 \Bigg)^2$$

$$= \frac{p^2}{\pi^4 \varphi^2(p)} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} |L(1,\chi_1)|^4 \sum_{a=1}^{p-1} a^k \chi^0(a)$$

$$+ \frac{p^2}{\pi^4 \varphi^2(p)} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2 \neq \chi^0}} |L(1,\chi_1)|^2 |L(1,\chi_2)|^2 \sum_{a=1}^{p-1} a^k \chi_1\chi_2(a)$$

$$= \frac{p^2}{\pi^4 \varphi^2(p)} \Big(\frac{5\pi^4 p}{144} + O\Big(\exp\Big(\frac{3\ln p}{\ln\ln p}\Big)\Big)\Big) \Big(\frac{(p-1)^{k+1}}{k+1} + O((p-1)^k) \Big)$$

$$+ \frac{2p^{k+2}}{\pi^4 \varphi^2(p)} \sum_{1 \leqslant m \leqslant k/2} \frac{\binom{k}{2m-1}(2m-1)!}{(-1)^{m+1}(2\pi)^{2m}}$$

$$\times \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1\chi_2 \neq \chi^0}} \tau(\chi_1\chi_2) L(2m, \overline{\chi_1\chi_2}) |L(1,\chi_1)|^2 |L(1,\chi_2)|^2$$

$$= Ap^{k+2} + O\Big(p^{k+3/2} \exp\Big(\frac{6\log p}{\log\log p}\Big)\Big),$$

where

$$A = \Bigg(\frac{5}{144(k+1)} + \frac{1}{\pi^4} \sum_{1 \leqslant m \leqslant k/2} \frac{\binom{k}{2m-1}(2m-1)!}{(-1)^{m+1}(2\pi)^{2m}} \sideset{}{'}\sum_{n=1}^{\infty} \sideset{}{'}\sum_{v=1}^{\infty} \frac{\sum_{d_1|n}\sum_{d_2|n} \cos(2\pi d_1 v \overline{d_2}/p)}{n^2 v^{2m}} \Bigg).$$

This completes the proof. □

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