Czechoslovak Mathematical Journal, 60 (135) (2010), 1065–1074

A REMARK ON THE RANGE OF ELEMENTARY OPERATORS

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(Received June 23, 2009)

Abstract. Let $L(H)$ denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space H into itself. Given $A \in L(H)$, we define the elementary operator $\Delta_A\colon L(H) \longrightarrow L(H)$ by $\Delta_A(X) = AXA - X$. In this paper we study the class of operators $A \in L(H)$ which have the following property: $ATA = T$ implies $AT^*A = T^*$ for all trace class operators $T \in C_1(H)$. Such operators are termed generalized quasi-adjoints. The main result is the equivalence between this character and the fact that the ultraweak closure of the range of Δ_A is closed under taking adjoints. We give a characterization and some basic results concerning generalized quasi-adjoints operators.

Keywords: elementary operators, ultraweak closure, weak closure, quasi-adjoint operator

MSC 2010: 47B47, 47A30, 47B20, 47B10

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H into itself. Given $A, B \in L(H)$, we define the elementary operator $\Delta_{A,B}$ as

$$\begin{aligned}\Delta_{A,B}\colon\ & L(H) \longrightarrow L(H),\\ X \longmapsto\ & \Delta_{A,B}(X) = AXB - X.\end{aligned}$$

If $A = B$, we write simply Δ_A for $\Delta_{A,A}$. The properties of elementary operators, their spectrum (see [9], [10], [12]), norm ([15], [17] and [18]) and ranges ([1], [2], [3], [4], [6], [12], [13], [14], and [16]) have been studied intensively, but many problems remain open [12].

In particular, L. Fialkow [12] and Z. Genkai [14] studied the problem of characterizing operators $A, B \in L(H)$ for which $R(\Delta_{A,B})$, the range of $\Delta_{A,B}$, is dense in $L(H)$ in the norm topology.

Our aim in this paper is a modest one. In the first section, we provide a characterization of the case when the range $R(\Delta_{A,B})$ is weakly and ultraweakly dense in $L(H)$. Complementary results related to the range of the elementary operator $\Delta_{A,B}$ are also given.

An operator $A \in L(H)$ is said to be quasi-adjoint if the norm closure of the range of Δ_A is closed under taking adjoint, i.e. $\overline{R(\Delta_A)} = \overline{R(\Delta_{A^*})} = \overline{R(\Delta_A)}^*$. In [4] it is proved that if A is quasi-adjoint, then $ATA = T$ implies $AT^*A = T^*$ for every trace class operator $T \in C_1(H)$. In order to generalize these results, we initiate the study of a more general class of operators A that have the following property: $ATA = T$ implies $AT^*A = T^*$ for all $T \in C_1(H)$. We call such operators generalized quasi-adjoint operators. In the second section, We give a characterization and some basic properties concerning this class of operators. Finally, we pose and mention some open questions suggested by our results.

Notation and definitions

(1) Let $L(H)$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space H, let $K(H)$ denote the ideal of all compact operators on H, and let $B(H)$ be the class of all finite rank operators. Finally, let $\mathcal{C}(H) = L(H)\big|K(H)$ denote the Calkin algebra.

(2) Given $A, B \in L(H)$, $R(\Delta_{A,B})$ will denote the range of the elementary operator $\Delta_{A,B}$ and $\ker(\Delta_{A,B})$ the kernel of $\Delta_{A,B}$.

Let $\overline{R(\Delta_{A,B})}$ be the norm closure, then $\overline{R(\Delta_{A,B})}^{w}$ will denote the weak closure, and $\overline{R(\Delta_{A,B})}^{w^*}$ the ultra-weak closure of the range $R(\Delta_{A,B})$.

(3) Let $C_1(H)$ be the ideal of trace class operators. The ideal $C_1(H)$ admits a complex valued function $\operatorname{tr}(T)$ which has the characteristic properties of the trace of matrices. The trace function is defined by $\operatorname{tr}(T) = \sum\limits_n \langle Te_n, e_n\rangle$, where (e_n) is any complete orthonormal system in H.

(4) As a Banach space, $C_1(H)$ may be identified with the conjugate space of the ideal $K(H)$ of compact operators by means of the linear isometry $T \longmapsto \Phi_T$, where $\Phi_T(X) = \operatorname{tr}(XT)$. Moreover, $L(H)$ is the dual of $C_1(H)$. The ultra-weak continuous linear functionals on $L(H)$ are those of the form Φ_T for some $T \in C_1(H)$, and the weak continuous linear functionals on $L(H)$ are those of the form Φ_T where $T \in B(H)$.

(5) If φ is a linear functional on $L(H)$, then φ^*, the adjoint of φ, is defined by $\varphi^*(X) = \overline{\varphi(X^*)}$ for all $X \in L(H)$.

(6) Recall that for $x, y \in H$, the operator $x \otimes y \in L(H)$ is defined by $(x \otimes y)z = \langle z, y\rangle\, x$ for all $z \in H$.

(7) For any subset $\mathcal{S}$ of $L(H)$, we denote the polar of $\mathcal{S}$ by

$$\mathcal{S}^\circ = \{\Phi \in L'(H)\colon \ \Phi(x) = 0 \text{ for all } x \in \mathcal{S}\}.$$

2. The range of the elementary operator $\Delta_{A,B}$

Lemma 2.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subspaces of $L(H)$. Then $\mathcal{S}_1^\circ \subset \mathcal{S}_2^\circ$ if and only if $\mathcal{S}_2 \subset \overline{\mathcal{S}_1}$.*

Proof. This is an easy consequence of the bipolar theorem. $\square$

Theorem 2.2. *Let $A, B \in L(H)$, then*

$$R(\Delta_{A,B})^\circ \simeq R(\Delta_{A,B})^\circ \cap K(H)^\circ \oplus \ker(\Delta_{B,A}) \cap C_1(H).$$

Proof. Let $\Phi = \Phi_T + \Phi_\circ$ be the canonical decomposition of a continuous linear functional $\Phi \in L'(H)$ into a trace form part and a functional vanishing on $K(H)$ [5]. Then we have $\Phi \in R(\Delta_{A,B})^\circ$ if and only if $\Phi_\circ, \Phi_T \in R(\Delta_{A,B})^\circ$ and we have $\Phi_T \in R(\Delta_{A,B})^\circ$ if and only if $T \in \ker(\Delta_{B,A}) \cap C_1(H)$.

Indeed, let $x, y \in H$, then we have

$$\Phi(A(x \otimes y)B) = \Phi_T(A(x \otimes y)B) = \operatorname{tr}(TAx \otimes B^*y) = \langle TAx, B^*y\rangle$$

and

$$\Phi(x \otimes y) = \Phi_T(x \otimes y) = \operatorname{tr}(T(x \otimes y)) = \langle Tx, y\rangle\,.$$

It follows that

$$\langle TAx, B^*y\rangle = \langle Tx, y\rangle\,,$$

for all $x, y \in H$ and hence

$$\Phi_T(AXB) = \Phi_T(X)$$

for all finite rank operators X. Since the class of finite rank operators is dense in $L(H)$ relative to the ultra-weak operator topology, it follows that $\Phi_T \in R(\Delta_{A,B})^\circ$. This implies that

$$\Phi_\circ = \Phi - \Phi_T \in R(\Delta_{A,B})^\circ.$$

Conversely, the preceding computation shows that if $BTA = T$ and $T \in C_1(H)$, then $\Phi_T \in R(\Delta_{A,B})^\circ$. The proof is complete. $\square$

Corollary 2.3. *Let $A, B \in L(H)$. Then the following statements are equivalent:*
(1) $\overline{R(\Delta_{A,B})}^{w^*} = L(H)$.
(2) $K(H) \subset \overline{R(\Delta_{A,B})}$.
(3) $\ker(\Delta_{B,A}) \cap C_1(H) = \{0\}$.

Proof. The negation of (1) and (3) is equivalent to the fact that there exists a nonzero ultraweakly continuous linear form Φ_T such that $\Phi_T \in R(\Delta_{A,B})^\circ$. By Theorem 2.2 this occurs if and only if $R(\Delta_{A,B})^\circ \not\subset K(H)^\circ$. It follows from Lemma 2.1 that the last condition is equivalent to $K(H) \not\subset \overline{R(\Delta_{A,B})}$. $\square$

Corollary 2.4. *Let $A, B \in L(H)$, then*

$$\overline{R(\Delta_{A,B})} \cap K(H) = \overline{R(\Delta_{A,B})}^{w^*} \cap K(H).$$

Proof. Setting $S := R(\Delta_{A,B})$, we have trivially $\overline{S}^{w^*} \cap K(H) \supset \overline{S} \cap K(H)$ where

$$\overline{S} \cap K(H) = \bigcap \{\ker(\psi) \cap K(H) \colon \ \psi \in L'(H), \psi(S) = 0\},$$

and

$$\overline{S}^{w^*} \cap K(H) = \bigcap \{\ker(\varphi_T) \cap K(H) \colon \ T \in C_1(H), \varphi_T(S) = 0\}.$$

To establish the converse inclusion, we consider any $K \in \overline{S}^{w^*} \cap K(H)$ and $\varphi \in L'(H)$ such that $\varphi(S) = 0$ and prove that $\varphi(K) = 0$. By Theorem 2.2, the canonical decomposition $\varphi = \varphi_T + \varphi_\circ$ satisfies $\varphi_T(S) = \varphi_\circ(S) = 0$. Since $K \in K(H)$, we have $\varphi_\circ(K) = 0$. On the other hand,

$$K \in \overline{S}^{w^*} \cap K(H) = \bigcap \{\ker(\varphi_T) \cap K(H) \colon \ T \in C_1(H), \varphi_T(S) = 0\},$$

which entails $\varphi_T(K) = 0$. Thus indeed $\varphi(K) = \varphi_T(K) + \varphi_\circ(K) = 0$. $\square$

Theorem 2.5. *Let $A, B \in L(H)$. Then*
(1) *every finite rank operator in $\overline{R(\Delta_{A,B})}^{w} \cap \ker(\Delta_{A^*,B^*})$ vanishes,*
(2) *every trace class operator in $\overline{R(\Delta_{A,B})}^{w^*} \cap \ker(\Delta_{A^*,B^*})$ vanishes.*

Proof. (1) Let T be a finite rank operator in $\overline{R(\Delta_{A,B})}^{w} \cap \ker(\Delta_{A^*,B^*})$, then $T^* \in \ker(\Delta_{B,A}) \cap B(H)$. It follows that Φ_{T^*} vanishes on the range of $\Delta_{B,A}$. In particular, $\Phi_{T^*}(T) = \operatorname{tr}(T^*T) = 0$, that is $T^*T = 0$, thus $T = 0$.

(2) It suffices to replace $B(H)$ with $C_1(H)$ in the above proof. $\square$

Theorem 2.6. *Let $A, B \in L(H)$. Then*

(1) $\overline{R(\Delta_{A,B})}^{w} = L(H)$ *if and only if* $\ker(\Delta_{B,A}) \cap B(H) = \{0\}$;

(2) $\overline{R(\Delta_{A,B})}^{w^*} = L(H)$ *if and only if* $\ker(\Delta_{B,A}) \cap C_1(H) = \{0\}$.

Proof. (1) Suppose that $\overline{R(\Delta_{A,B})}^{w} = L(H)$ and $T \in \ker(\Delta_{B,A}) \cap B(H)$. It follows that $T^* \in \overline{R(\Delta_{A,B})}^{w} \cap \ker(\Delta_{A^*,B^*})$, hence $T = 0$ by Theorem 2.5.

Conversely, assume that there exists $T \in L(H) \big| \overline{R(\Delta_{A,B})}^{w}$. It follows that there is an operator $S \in B(H)$ such that $\operatorname{tr}(ST) \neq 0$ and $\operatorname{tr}(SX) = 0$ for each $X \in R(\Delta_{A,B})$. Hence, we obtain that $S \in \ker(\Delta_{B,A}) \cap B(H)$ and $S \neq 0$.

(2) It suffices to replace $B(H)$ with $C_1(H)$ in the preceding proof.

Remark 2.7. If $A, B \in L(H)$ are such that $\|A\|\|B\| < 1$, then Corollary 2.3 and Theorem 2.6 show that $\overline{R(\Delta_{A,B})}^{w} = \overline{R(\Delta_{A,B})}^{w^*} = L(H)$.

Theorem 2.8. *Let $A, B \in L(H)$. Then*

1) $\overline{R(\Delta_{B})}^{w} \subset \overline{R(\Delta_{A})}^{w}$ *if and only if* $\ker(\Delta_A) \cap B(H) \subset \ker(\Delta_B) \cap B(H)$;

2) $\overline{R(\Delta_{B})}^{w^*} \subset \overline{R(\Delta_{A})}^{w^*}$ *if and only if* $\ker(\Delta_A) \cap C_1(H) \subset \ker(\Delta_B) \cap C_1(H)$.

Proof. (1) Assume that $\ker(\Delta_A) \cap B(H) \subset \ker(\Delta_B) \cap B(H)$. Let Φ_T be a weakly continuous linear form that vanishes on $R(\Delta_A)$. Then it is easy to see that

$$\Phi_T(AXA - X) = \operatorname{tr}[T(AXA - X)] = \operatorname{tr}[(ATA - T)X] = 0$$

for all $X \in L(H)$, hence $ATA = T$ and $T \in \ker(\Delta_A) \cap B(H) \subset \ker(\Delta_B) \cap B(H)$. Observe that

$$\Phi_T(BXB - X) = \operatorname{tr}[T(BXB - X)] = 0,$$

thus Φ_T annihilates $R(\Delta_B)$. It follows that $\overline{R(\Delta_{B})}^{w} \subset \overline{R(\Delta_{A})}^{w}$. For the converse implication we reverse the above argument.

(2) It suffices to replace $B(H)$ with $C_1(H)$ in the preceding proof. □

Remark 2.9. Let $a = (A_1, A_2, \ldots, A_n)$ and $b = (B_1, B_2, \ldots, B_n)$ be n-tuples of operators in $L(H)$, let $R_{a,b}$ denote the generalized elementary operator on $L(H)$ defined by $R_{a,b}(X) = \sum\limits_{i=1}^{n} A_i X B_i$. Notice that the above results still hold for the elementary operator $R_{a,b}$.

3. Generalized quasi-adjoint operators

Definition 3.1. Let $A \in L(H)$. We say that the operator A is quasi-adjoint if

$$\overline{R(\Delta_A)} = \overline{R(\Delta_{A^*})}.$$

Remark 3.2. Let $A \in L(H)$, then A is quasi-adjoint if and only if $\overline{R(\Delta_A)}$ is a self adjoint subspace of $L(H)$. Equivalently, $R(\Delta_A)^\circ$, the annihilator of $R(\Delta_A)$, is a self adjoint subspace of $L'(H)$ in the sense that $\Phi \in R(\Delta_A)^\circ$ implies $\Phi^* \in R(\Delta_A)^\circ$.

Theorem 3.3. *If $A \in L(H)$ the following statements are equivalent:*
(1) *A is quasi-adjoint.*
(2) (i) *The element $[A]$ of the Calkin algebra is quasi-adjoint, and*
(ii) *for $T \in C_1(H)$, $ATA = T$ implies $A^*TA^* = T$.*

Proof. (1) $\Longrightarrow$ (2). Suppose that A is quasi-adjoint. (i) Let $\psi \in R(\Delta_{[A]})^\circ$. We define a bounded linear functional Φ on $L(H)$ by $\Phi(X) = \psi([X])$. It is clear that $\Phi \in R(\Delta_A)^\circ$ if and only if $\psi \in R(\Delta_{[A]})^\circ$. Since A is quasi-adjoint, it follows from the above Remark that $\Phi^* \in R(\Delta_A)^\circ$ and consequently $\psi^* \in R(\Delta_{[A]})^\circ$. Then $[A]$ is quasi-adjoint.

(ii) If $ATA = T$ and $T \in C_1(H)$, then Theorem 2.2 implies that $\Phi_T \in R(\Delta_A)^\circ$. Since A is quasi-adjoint, it follows that $(\Phi_T)^* = \Phi_{T^*} \in R(\Delta_A)^\circ$, from which we get $A^*TA^* = T$.

(2) $\Longrightarrow$ (1) Let $\Phi \in R(\Delta_A)^\circ$. We can write $\Phi = \Phi_\circ + \Phi_T$, where $\Phi_\circ \in R(\Delta_A)^\circ \cap K(H)^\circ$ and $T \in \ker(\Delta_A) \cap C_1(H)$. By using (ii) one obtains $A^*TA^* = T$, that is $\Phi_{T^*} \in R(\Delta_A)^\circ$. It remains to show that $\Phi_\circ^* \in R(\Delta_A)^\circ$. Let φ be the linear functional on the Calkin algebra defined by $\varphi([X]) = \Phi_\circ(X)$. Since $\Phi_\circ$ vanishes on $K(H)$, it follows that φ is well defined. From (i), $[A]$ is quasi-adjoint, hence $\varphi \in R(\Delta_{[A]})^\circ$ implies that $\varphi^* \in R(\Delta_{[A]})^\circ$, that is $\Phi_\circ^* \in R(\Delta_A)^\circ$. Thus we have shown that $\Phi^* = \Phi_\circ^* + \Phi_{T^*} \in R(\Delta_A)^\circ$, consequently A is quasi-adjoint. □

Definition 3.4. An operator $A \in L(H)$ is called generalized quasi-adjoint if $ATA = T$ and $T \in C_1(H)$ implies $AT^*A = T^*$. The set of generalized quasi-adjoint operators is denoted by $Q_\circ(H)$.

Theorem 3.5. *Let $A \in L(H)$. Then*
(i) *A is generalized quasi-adjoint if and only if $\overline{R(\Delta_A)}^{w^*}$ is self-adjoint;*
(ii) *$Q_\circ(H)$, the set of generalized quasi-adjoint operators, is self-adjoint.*

Proof. (i) $\overline{R(\Delta_A)}^{w^*}$ is self-adjoint if and only if $R(\Delta_A)^\circ \cap L'(H)^{w^*}$ is self-adjoint. It follows from Theorem 2.2 that

$$R(\Delta_A)^\circ \simeq R(\Delta_A)^\circ \cap K(H)^\circ \oplus \ker(\Delta_A) \cap C_1(H).$$

Consequently, we get

$$R(\Delta_A)^\circ \cap \overline{L'(H)}^{w^*} \cong \ker(\Delta_A) \cap C_1(H).$$

(ii) It follows immediately from the definition. $\square$

Example 3.6.

(i) If V is an isometry, in particular if $\|V^{-1}\|\|V\| = 1$, then V is a generalized quasi-adjoint operator.

(ii) Every normal operator is generalized quasi-adjoint.

(iii) Every cyclic subnormal operator is generalized quasi-adjoint.

Proposition 3.7. *Let $A \in L(H)$ be a contraction. Then A is generalized quasi-adjoint.*

Proof. The result of [7] guarantees that for every $T \in C_1(H)$ we get that $\overline{R(T)}$ reduces A, and $(\ker T)^\perp$ reduces A and the restrictions $A\big|\overline{R(T)}$ and $A\big|(\ker T)^\perp$ are unitarily equivalent to unitary operators. Put $H_1 = H = \overline{R(T)} \oplus \overline{R(T)}^\perp$ and $H_2 = H = (\ker T)^\perp \oplus \ker T$. Then for $A\colon H_1 \longrightarrow H_2$ and $T\colon H_2 \longrightarrow H_1$, we get the decompositions

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, A = \begin{pmatrix} A_1' & 0 \\ 0 & A_2' \end{pmatrix}, \text{ and } T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The condition $ATA = T$ implies that $A_1T_1A_1' = T_1$. Since A_1 and A_1' are unitary operators, it follows that $A_1^*T_1{A_1'}^* = T_1$, or equivalently $A^*TA^* = T$. This completes the proof. $\square$

Proposition 3.8. *Let $A \in Q_\circ(H)$. If $H_\circ$ reduces A, then $A|H_\circ$ is a generalized quasi-adjoint operator.*

Proof. By virtue of the decomposition $H = H_\circ \oplus H_\circ^\perp$, we have $A = A_\circ \oplus A_1$. Suppose that $A_\circ T_\circ A_\circ = T_\circ$ and $T_\circ \in C_1(H_\circ)$. Define an operator T on $H = H_\circ \oplus H_\circ^\perp$ by $T = \left(\begin{smallmatrix} T_\circ & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then $ATA = T$ and $T \in C_1(H)$. Since A is generalized quasi-adjoint, it follows that $AT^*A = T^*$. Hence one obtains $A_\circ T_\circ^* A_\circ = T_\circ^*$. $\square$

Lemma 3.9. *Let $A \in L(H)$. Then the following statements are equivalent:*

(1) *A is generalized quasi-adjoint.*

(2) *If $ATA = T$ and $T \in C_1(H)$, then $\overline{R(T)}$ and $(\ker T)^\perp$ reduce A and $A|\overline{R(T)}$ and $A|(\ker T)^\perp$ are normal operators.*

Proof. We omit the proof which may be based entirely on the proof of the well known Lemma [8]. $\square$

Theorem 3.10. *Let $A \in L(H)$. If $T \in C_1(H)$ is such that $T = U|T|$ is the polar decomposition of T, then the operator A is generalized quasi-adjoint if and only if $A|T| = |T|A$, $A|T^*| = |T^*|A$ and $\Delta_A(U) = 0$.*

Proof. Assume A is generalized quasi-adjoint. Let $T \in C_1(H)$ have the polar decomposition $T = U|T|$. If $ATA = T$, it follows that $AT^*A = T^*$.

Then we have

$$A|T|^2 = AT^*T = AT^*ATA = T^*TA = |T|^2A.$$

Analogously,

$$A|T^*|^2 = ATT^* = ATAT^*A = TT^*A = |T^*|^2A,$$

and by the functional calculus both operators $|T|$ and $|T^*|$ commute with A. Hence, we get $A|T| = |T|A$ and $A|T^*| = |T^*|A$.

Moreover, $ATA = T$ implies that $(AUA - U)|T| = 0$. Consequently, $(AUA - U)|\overline{R(T)} = 0$, that is $\Delta_A(U)|\overline{R(T)} = 0$. Since $A\colon \ker T \longrightarrow \ker T$, we obtain that $\Delta_A(U) = 0$.

Conversely, the conditions $A|T| = |T|A$ and $\Delta_A(U) = 0$ imply that $ATA = T$. Since A commutes with $|T|$ and $|T^*|$, it follows from the Fuglede-Putnam Theorem that $\overline{R(T)}$ and $(\ker T)^{\perp}$ reduce A, and the restrictions $A_1 = A|\overline{R(T)}$ and $A_1' = A|(\ker T)^{\perp}$ are normal operators. Take the following two decompositions of H:

$$H_1 = H = \overline{R(T)} \oplus \overline{R(T)}^{\perp}, \text{ and } H_2 = H = \ker T^{\perp} \oplus \ker T.$$

In terms of these decompositions of H, for $A\colon H_2 \longrightarrow H_1$ we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \; A^* = \begin{pmatrix} {A_1'}^* & 0 \\ 0 & {A_2'}^* \end{pmatrix} \text{ and } T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From $ATA = T$ it follows that $A_1T_1A_1' = T_1$. Since A_1 and A_1' are normal operators, we get $A_1T_1^*A_1' = T_1^*$, or equivalently $AT^*A = T^*$. This completes the proof. □

Proposition 3.11. *Let A and B be generalized quasi-adjoint operators. If $1 \notin \sigma(A)\sigma(B)$, then $A \oplus B$ is a generalized quasi-adjoint operator.*

Proof. Let $T = \left(\begin{smallmatrix} T_\circ & T_1 \\ T_2 & T_3 \end{smallmatrix}\right)$ be a trace class operator on $H \oplus H$. It is easily seen that $(A \oplus B)T(A \oplus B) = T$ implies that

$$AT_\circ A = T_\circ, \; AT_1B = T_1, \; BT_2A = T_2 \text{ and } BT_3B = T_3.$$

Since $1 \notin \sigma(A)\sigma(B)$, it follows from Rosenblum's Theorem [9] that the operators $\Delta_{A,B}$ and $\Delta_{B,A}$ are invertible. Consequently, we get $T_1 = T_2 = 0$.

Moreover, A and B are generalized quasi-adjoint operators, hence $AT_\circ A = T_\circ$ implies $AT_\circ^*A = T_\circ^*$ and $BT_3B = T_3$ implies $BT_3^*B = T_3^*$. Thus $(A \oplus B)T^*(A \oplus B) = T^*$. The proof is complete. □

Proposition 3.12. *Let $A \in L(H)$. If there exist $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta = 1$ and nonzero vectors $f, g \in H$ such that*
(i) *$Af = \alpha f$ and $\|A^*f\| \neq \|\alpha f\|$,*
(ii) *$A^*g = \overline{\beta}g$.*
Then A is not a generalized quasi-adjoint operator.

Proof. A is generalized quasi-adjoint if and only if $\overline{R(\Delta_A)}^{w^*}$ is self adjoint. Under the preceding hypothesis, we will show that $\overline{R(\Delta_A)}^{w^*} \neq \overline{R(\Delta_{A^*})}^{w^*}$. Suppose first that $A^*f \neq 0$. We consider the operator $T = g \otimes A^*f$. It is easily seen that

$$\langle (AYA - Y)f, g\rangle = 0$$

for all $Y \in L(H)$. On the other hand, one obtains that

$$\langle (A^*TA^* - T)f, g\rangle = \overline{\beta}(\|A^*f\|^2 - \|\alpha f\|^2)\|g\|^2.$$

If $A^*TA^* - T \in \overline{R(\Delta_A)}^{w^*}$, then there exists a generalized sequence $(X_\alpha)_\alpha$ in $L(H)$ such that

$$AX_\alpha A - X_\alpha \longrightarrow A^*TA^* - T.$$

This implies that

$$0 = \langle (AX_\alpha A - X_\alpha)f, g\rangle \longrightarrow \langle (A^*TA^* - T)f, g\rangle = \overline{\beta}(\|A^*f\|^2 - \|\alpha f\|^2)\|g\|^2.$$

It follows that $\overline{\beta}(\|A^*f\|^2 - \|\alpha f\|^2)\|g\|^2 = 0$ which is absurd. If $A^*f = 0$ we consider the operator $T = g \otimes f$. By repeating the same argument we get the result. $\square$

Some open problems

(1) Let $(e_n)_{n=-\infty}^{n=+\infty}$ be an orthonormal basis for H and let S be the bilateral weighted shift $Se_n = \omega_n e_{n+1}$ for all $n \in \mathbb{Z}$, with nonzero weights ω_n. We ask if there exist necessary and sufficient conditions on the weights of S in order that S be a quasi-adjoint operator.

(2) Which weighted shifts are generalized quasi-adjoint operators?

(3) Is the set $Q_\circ(H)$ of generalized quasi-adjoint operators norm closed?

(4) What characterizes compact generalized quasi-adjoint operators?

Acknowledgement. It is our great pleasure to thank the referee for his careful reading of the paper and for several helpful suggestions.

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