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## TRIANGULAR REPUNIT—THERE IS BUT 1

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*Abstract.* In this paper, we demonstrate that 1 is the only integer that is both triangular and a repunit.

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A *repunit* is any integer that can be written in decimal notation as a string of 1's. Examples include 1, 11, 111, 1111, 11111, ... Furthermore, if we denote  $r_n$  to be the  $n$ th repunit then it follows that

$$(1) \quad r_n = \frac{10^n - 1}{9}.$$

*Triangular numbers* are those integers that can be represented by the number of dots evenly arranged in an equilateral triangle. More specifically, they are the numbers 1, 3, 6, 10, 15, ... It follows that the  $n$ th triangular number,  $t_n$  is the sum of the first  $n$  consecutive integers beginning with 1. In particular, for  $n \geq 1$

$$(2) \quad t_n = \frac{n(n+1)}{2}.$$

Now, the number 1 is both triangular and a repunit. It is the objective of this note to illustrate that 1 is the only such number. To this end, we state and prove the following lemma. It incorporates a result that [1] asserts had been known by the early Pythagoreans. Appearing in *Platonic Questions*, Plutarch states that eight times any triangular number plus one is a square. The result is actually necessary and sufficient. We demonstrate it as such here.

**Lemma.** *The integer  $n$  is triangular if and only if  $8n + 1$  is a square.*

**Proof.** If  $n = t_n$ , then by (2),

$$8 \left[ \frac{n(n+1)}{2} \right] + 1 = 4n^2 + 4n + 1 = (2n+1)^2.$$

On the other hand, if  $8n + 1$  is a square, then  $8n + 1 = x^2$ , for some odd positive integer  $x$ . Hence,  $x^2 - 1$  is even, and so

$$8n = x^2 - 1 = (x+1)(x-1) = (2k+2)(2k),$$

for some positive integer  $k$ , from which it follows that  $n = \frac{1}{2}k(k+1)$ . □

The previous result may also be observed geometrically by letting an  $n \times (n+1)$  rectangle represent twice  $t_n$ . Thus, four such rectangles plus a unit square comprise a square with sides  $2n+1$ .

**Theorem.** *The only triangular repunit is 1.*

**Proof.** In light of (1) and the previous lemma, it suffices to show that  $8 \times (10^n - 1)/9 + 1$  is square only for  $n = 1$ .

To this end, we shall determine all  $n$  for which

$$8 \left[ \frac{10^n - 1}{9} \right] + 1 = (2k+1)^2,$$

for some positive integer  $k$ . Thus,

$$10^n = (2 \cdot 5)^n = \frac{(3k+2)(3k+1)}{2};$$

whence,

$$(3) \quad 5^n = \frac{(3k+2)(3k+1)}{2^{n+1}}.$$

Now,  $5 \mid 3k+2$  or  $5 \mid 3k+1$  but not both. Similarly,  $2 \mid 3k+2$  or  $2 \mid 3k+1$  but not both.

**Case 1.** Suppose  $5 \mid 3k+2$ . If  $k$  is odd, then  $3k+2$  is odd. As  $3k+1$  is even, in light of (3) we have  $3k+2 = 5^n$  and  $3k+1 = 2^{n+1}$ . So,  $2^{n+1} + 1 = 5^n$ . But this can only occur when  $n = 1$  (i.e., for  $t_n = 1$ .) This is seen upon noting that for  $n > 2$ ,  $5^n > 2^{n+1} + 1$ . On the other hand, if  $k$  is even then  $3k+1$  is odd, implying that 5 divides both  $3k+1$  and  $3k+2$ , which is impossible.

**Case 2.** Suppose  $5 \mid 3k + 1$ . If  $k$  is odd, then  $3k + 1$  is even, which implies that both 5 and 2 divide  $3k + 1$ . Hence, there must exist an integer different from 5 and 2 that is a factor of  $3k + 1$ , which is impossible. Hence,  $k$  must be even. It follows that  $3k + 1$  is odd, and so  $3k + 1 = 5^n$  and  $3k + 2 = 2^{n+1}$ . A triangular repunit then results when  $5^n + 1 = 2^{n+1}$ . But the only solution is  $n = 0$ , as  $n > 0$  implies  $5^n + 1 > 2^{n+1}$ . However,  $t_0 = 0$  is neither triangular nor a repunit.  $\square$

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#### *References*

- [1] *J. J. Tattersall: Elementary Number Theory in Nine Chapters*, 2nd ed. Cambridge University Press, Cambridge, 2005.

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