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C-GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

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Abstract. By analogy with the projective, injective and flat modules, in this paper we study some properties of C-Gorenstein projective, injective and flat modules and discuss some connections between C-Gorenstein injective and C-Gorenstein flat modules. We also investigate some connections between C-Gorenstein projective, injective and flat modules of change of rings.

Keywords: C-Gorenstein projective module, C-Gorenstein injective module, C-Gorenstein flat module

MSC 2010: 13D07, 16E65

1. Introduction

Unless stated otherwise, throughout this paper R is a commutative and noetherian ring with unit and C is a semi-dualizing R-module. By $\mathcal{P}(R)$ and $\mathcal{I}(R)$ we denote the class of all projective and injective R-modules, respectively. For any R-module M, $\mathrm{pd}_R M$, $\mathrm{id}_R M$ and $\mathrm{fd}_R M$ denote the projective, injective and flat dimension, respectively. The character module $\mathrm{Hom}_Z(M, Q/Z)$ is denoted by M^+.

For any semi-dualizing module (in fact, complex) C over R and any complex Z with bounded and finitely generated homology, Christensen introduced the dimension $\text{G-dim}_C Z$ and developed a satisfactory theory for this new invariant. If C is a semi-dualizing R-module and M is any R-complex, then Holm and Jørgensen suggested in [5] the viewpoint that one should change rings from R to $R \ltimes C$ (the trivial extension of R by C) and then consider the three changed "ring" Gorenstein dimensions: $\mathrm{Gid}_{R\ltimes C} M$, $\mathrm{Gpd}_{R\ltimes C} M$, $\mathrm{Gfd}_{R\ltimes C} M$. The usefulness of this viewpoint

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was demonstrated as it enabled them to introduce three new Cohen-Macaulay dimensions, which characterize Cohen-Macaulay rings in a way one could hope for. For every semi-dualizing R-module C Holm and Jørgensen in [6] defined, three new Gorenstein dimensions: $C\text{-Gid}_R M$, $C\text{-Gpd}_R M$, $C\text{-Gfd}_R M$, which are called the C-Gorenstein injective, C-Gorenstein projective and C-Gorenstein flat dimension respectively, and proved how they are related to the "changed ring" Gorenstein dimensions over $R \propto C$. They compared $C\text{-Gpd}_R(-)$ with $\text{G-dim}_C(-)$ and interpreted the C-Gorenstein dimensions in terms of Auslander and Bass categories.

In Section 2, we study some properties of C-Gorenstein projective and injective modules. We prove that the union of a continuous chain of C-Gorenstein projective modules is C-Gorenstein projective and the well-ordered continuous inverse system of C-Gorenstein injective R-modules is C-Gorenstein injective. In Section 3, we discuss some connections between C-Gorenstein injective and C-Gorenstein flat modules. We prove that if R is artinian, then M is C-Gorenstein injective if and only if M^+ is C-Gorenstein flat. In Section 4, we show that some studies of homological properties of change of rings can be generalized to C-Gorenstein homological properties. The two structural operations addressed later are the information of m-adic completion and polynomial rings.

We first recall some concepts. Let $\mathcal{X}$ be a class of R-modules. We call $\mathcal{X}$ projectively resolving if $\mathcal{P}(R) \subseteq \mathcal{X}$ and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent. Injectively resolving is defined dually. A semi-dualizing module C is finitely generated so that $\text{Hom}_R(C,C)$ is canonically isomorphic to R and $\text{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$.

An R-module M is said to be C-Gorenstein injective if

(I1) $\text{Ext}^i_R(\text{Hom}_R(C,I), M) = 0$ for all injective R-modules I and all $i \geqslant 1$;

(I2) there exist injective R-modules $I_0, I_1, \ldots$ together with an exact sequence

$$\ldots \longrightarrow \text{Hom}_R(C, I_1) \longrightarrow \text{Hom}_R(C, I_0) \longrightarrow M \longrightarrow 0,$$

and also, this sequence stays exact when we apply to it the functor

$$\text{Hom}_R(\text{Hom}_R(C, J), -)$$

for any injective R-module J.

An R-module M is said to be C-Gorenstein projective if

(P1) $\text{Ext}^i_R(M, C \otimes_R P) = 0$ for all projective R-modules P and all $i \geqslant 1$;

(P2) there exist projective R-modules $P^0, P^1, \ldots$ together with an exact sequence

$$0 \longrightarrow M \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \ldots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\text{Hom}_R(-, C \otimes_R Q)$ for any projective R-module Q.

An R-module M is said to be C-Gorenstein flat if
(F1) $\operatorname{Tor}_i^R(\operatorname{Hom}_R(C,I),M)=0$ for all injective R-modules I and all $i\geqslant 1$;
(F2) there exist flat R-modules $F^0,F^1,\ldots$ together with an exact sequence

$$0\longrightarrow M\longrightarrow C\otimes_R F^0\longrightarrow C\otimes_R F^1\longrightarrow\ldots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\operatorname{Hom}_R(C,I)\otimes_R -$ for any injective R-module I.

Remark. (a) If I is an injective R-module, then $\operatorname{Hom}_R(C,I)$ and I are C-Gorenstein injective. If P is a projective R-module, then $C\otimes_R P$ and P are C-Gorenstein projective. If F is a flat R-module, then $C\otimes_R F$ and F are C-Gorenstein flat.
(b) Note that when $C=R$ in the above definition, we recover the categories of ordinary Gorenstein injective, Gorenstein projective and Gorenstein flat R-modules.

If C is any R-module, then the direct sum $R\oplus C$ can be equipped with the product $(r,c)(r',c')=(rr',rc'+r'c)$. This turns $R\oplus C$ into a ring, which is called the trivial extension of R by C and denoted $R\propto C$. There are canonical ring homomorphisms $R\rightleftarrows R\propto C$, which enables us to view R-modules as $R\propto C$-modules, and vice versa.

2. C-Gorenstein projective and injective modules

In this section we study some properties of C-Gorenstein projective modules and C-Gorenstein injective modules.

Proposition 2.1. *The class C-$\mathcal{GP}(R)$ of all C-Gorenstein projective R-modules is projectively resolving. Furthermore, C-$\mathcal{GP}(R)$ is closed under arbitrary direct sums and arbitrary direct summands.*

Proof. By [4, Theorem 2.5] and [6, Proposition 2.13]. □

Proposition 2.2. *The class C-$\mathcal{GI}(R)$ of all C-Gorenstein injective R-modules is injectively resolving. Furthermore, C-$\mathcal{GI}(R)$ is closed under arbitrary direct products and arbitrary direct summands.*

Proof. By [4, Theorem 2.6] and [6, Proposition 2.13]. □

Given an ordinal number λ and a family $(M_\alpha)_{\alpha<\lambda}$ of submodules of a module M, we say that the family is a continuous (well ordered) chain of submodules if $M_\alpha\subseteq M_\beta$ whenever $\alpha\leqslant\beta<\lambda$ and if $M_\beta=\bigcup\limits_{\alpha<\beta}M_\alpha$ whenever $\beta<\lambda$ is a limit ordinal. A family $(M_\alpha)_{\alpha\leqslant\lambda}$ is called a continuous chain if $(M_\alpha)_{\alpha<\lambda+1}$ is such (see [2, Definition 7.3.3]). A continuous chain of projective R-modules is projective by [2, p. 162, Exercise 2].

Theorem 2.3. *Let L be an R-module and suppose L is the union of a continuous chain of submodules $(L_\alpha)_{\alpha\leqslant\lambda}$. If L_0 and $L_{\alpha+1}/L_\alpha$ are C-Gorenstein projective R-modules whenever $\alpha+1\leqslant\lambda$, then L is C-Gorenstein projective.*

Proof. Let $\alpha+1\leqslant\lambda$. If α is not a limit ordinal, then L_α and $L_{\alpha+1}/L_\alpha$ are C-Gorenstein projective, and so there exist projective R-modules $P^0_\alpha, P^1_\alpha,\ldots$ and $Q^0, Q^1,\ldots$ together with exact sequences

$$\begin{gathered} 0\longrightarrow L_\alpha\longrightarrow C\otimes_R P^0_\alpha\to C\otimes_R P^1_\alpha\longrightarrow\ldots,\\ 0\longrightarrow L_{\alpha+1}/L_\alpha\longrightarrow C\otimes_R Q^0\longrightarrow C\otimes_R Q^1\longrightarrow\ldots, \end{gathered}$$

such that those sequences stay exact when we apply the functor $\operatorname{Hom}_R(-,C\otimes_R Q)$ to them for any projective R-module Q. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_\alpha & \longrightarrow & L_{\alpha+1} & \longrightarrow & L_{\alpha+1}/L_\alpha & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C\otimes_R P^0_\alpha & \longrightarrow & C\otimes_R (P^0_\alpha\oplus Q^0) & \longrightarrow & C\otimes_R Q^0 & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C\otimes_R P^1_\alpha & \longrightarrow & C\otimes_R (P^1_\alpha\oplus Q^1) & \longrightarrow & C\otimes_R Q^1 & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Then $0\to L_{\alpha+1}\to C\otimes_R(P^0_\alpha\oplus Q^0)\to C\otimes_R(P^1_\alpha\oplus Q^1)\to\ldots$ is exact such that this sequence stays exact when we apply to it the functor $\operatorname{Hom}_R(-,C\otimes_R Q)$ for any projective R-module Q. If α is a limit ordinal, set $P^i_\alpha=\bigcup\limits_{\beta<\alpha}P^i_\beta$ for $i=0,1,\ldots$. Then $0\to L_\alpha\to C\otimes_R P^0_\alpha\to C\otimes_R P^1_\alpha\to\ldots$ is exact. So $(P^i_\alpha)_{\alpha\leqslant\lambda}$ is a continuous chain for all $i=0,1,\ldots$. Set $P^0=\bigcup\limits_{\alpha\leqslant\lambda}P^0_\alpha$, $P^1=\bigcup\limits_{\alpha\leqslant\lambda}P^1_\alpha,\ldots$. Then

$$\mathbb{W}\colon\ 0\longrightarrow L\longrightarrow C\otimes_R P^0\longrightarrow C\otimes_R P^1\longrightarrow\ldots$$

is exact and each P^i is projective. Let Q be any projective R-module. Then

$$\operatorname{Ext}^i_R(L_0,C\otimes_R Q)=0=\operatorname{Ext}^i_R(L_{\alpha+1}/L_\alpha,C\otimes_R Q)\quad\forall i\geqslant 1,\ \text{whenever } \alpha+1\leqslant\lambda.$$

Hence $\operatorname{Ext}_R^i(L, C \otimes_R Q) = 0$ by [2, Theorem 7.3.4] for all $i \geqslant 1$, and so $\operatorname{Hom}_R(\mathbb{W}, C \otimes_R Q)$ is exact by analogy with the proof of [10, Theorem 2.1]. Thus L is C-Gorenstein projective. □

Let μ be an ordinal and $\mathcal{A} = (A_\alpha\colon \alpha \leqslant \mu)$ a sequence of modules. Let $(f_{\beta\alpha}\colon \alpha \leqslant \beta \leqslant \mu)$ be a sequence of monomorphisms (with $f_{\beta\alpha} \in \operatorname{Hom}_R(A_\alpha, A_\beta)$) such that $\mathcal{I} = \{(A_\alpha, f_{\beta\alpha})\colon \alpha \leqslant \beta \leqslant \mu\}$ is an direct system of modules. $\mathcal{I}$ is called continuous provided that $A_0 = 0$ and $A_\alpha = \varinjlim_{\beta<\alpha} A_\beta$ for all limit ordinals. Let $(g_{\alpha\beta}\colon \alpha \leqslant \beta \leqslant \mu)$ be a sequence of epimorphisms (with $g_{\alpha\beta} \in \operatorname{Hom}_R(A_\beta, A_\alpha)$) such that $\mathcal{I} = \{(A_\alpha, g_{\alpha\beta})\colon \alpha \leqslant \beta \leqslant \mu\}$ is an inverse system of modules. $\mathcal{I}$ is called continuous provided that $A_0 = 0$ and $A_\alpha = \varprojlim_{\beta<\alpha} A_\beta$ for all limit ordinals (see [15, Definition 2.1]). It is well known that the class $\mathcal{L}$ of Gorenstein projective (injective) objects in a Grothendieck category $\mathcal{A}$ is closed under direct (inverse) transfinite extensions by [3, Theorem 3.2].

Corollary 2.4. *Let $\mathcal{I} = \{(L_\alpha, f_{\beta\alpha})\colon \alpha \leqslant \beta \leqslant \mu\}$ be a well-ordered continuous direct system of modules. If $C_\alpha = \mathrm{Coker}(L_\alpha \to L_{\alpha+1})$ is a C-Gorenstein projective R-module whenever $\alpha + 1 \leqslant \mu$, then $L = \varinjlim_{\alpha\leqslant\mu} L_\alpha$ is a C-Gorenstein projective R-module.*

Theorem 2.5. *Let $L_0 \leftarrow L_1 \leftarrow L_2 \leftarrow \ldots$ be a continuous inverse system of modules. If $K_n = \operatorname{Ker}(L_{n+1} \to L_n)$ is a C-Gorenstein injective R-module for each n, then $L = \varprojlim L_n$ is a C-Gorenstein injective R-module.*

Proof. For each n there exist injective R-modules $I_n^0, I_n^1, \ldots$ together with an exact sequence

$$\ldots \longrightarrow \operatorname{Hom}_R(C, I_n^1) \longrightarrow \operatorname{Hom}_R(C, I_n^0) \longrightarrow L_n \longrightarrow 0,$$

such that the sequence stays exact when we apply the functor $\operatorname{Hom}_R(\operatorname{Hom}_R(C, J), -)$ to it for all injective R-modules J. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \operatorname{Hom}_R(C, I_2^1) & \xrightarrow{f_{1,2}^1} & \operatorname{Hom}_R(C, I_1^1) & \xrightarrow{f_{0,1}^1} & \operatorname{Hom}_R(C, I_0^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \operatorname{Hom}_R(C, I_2^0) & \xrightarrow{f_{1,2}^0} & \operatorname{Hom}_R(C, I_1^0) & \xrightarrow{f_{0,1}^0} & \operatorname{Hom}_R(C, I_0^0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_2 & \xrightarrow{f_{1,2}} & L_1 & \xrightarrow{f_{0,1}} & L_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then $g^i_{n,n+1} = C \otimes_R f^i_{n,n+1}\colon I^i_{n+1} \to I^i_n$ is an epimorphism. So $(\mathrm{Hom}_R(C, I^i_n))$ and (I^i_n) are continuous inverse systems for all $i = 0, 1, \ldots$ Set $I^0 = \varprojlim I^0_n$, $I^1 = \varprojlim I^1_n, \ldots$ Then

$$\mathbb{V}\colon \ \ldots \longrightarrow \mathrm{Hom}_R(C, I^1) \longrightarrow \mathrm{Hom}_R(C, I^0) \longrightarrow L \longrightarrow 0$$

is exact by [2, Theorem 1.5.14] and [15, Lemma 2.2] and I^0, $I^1, \ldots$ are injective R-modules by [15, Lemma 2.3]. Let I be any injective R-module. Then

$$\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, I), L_0) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, I), K_n) \quad \forall i \geqslant 1 \text{ and each } n,$$

and so $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, I), L) = 0$ by [15, Lemma 2.3] for all $i \geqslant 1$, which gives that $\mathrm{Hom}_R(\mathrm{Hom}_R(C, I), \mathbb{V})$ is exact by analogy with the proof of [10, Theorem 2.1]. Thus L is C-Gorenstein injective. □

Proposition 2.6. *Let Q be a projective R-module. If M is a C-Gorenstein projective R-module, then $M \otimes_R Q$ is a C-Gorenstein projective R-module.*

Proof. There exist projective R-modules $P^0, P^1, \ldots$ together with an exact sequence

$$\mathbb{W}\colon \ 0 \longrightarrow M \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \ldots.$$

Then $\mathbb{W} \otimes_R Q\colon \ 0 \to M \otimes_R Q \to C \otimes_R (P^0 \otimes_R Q) \to C \otimes_R (P^1 \otimes_R Q) \to \ldots$ is exact and each $P^i \otimes_R Q$ is projective. Let P be any projective R-module. By [13, p. 258, 9.20],

$$\mathrm{Ext}^i_R(M \otimes_R Q, C \otimes_R P) \cong \mathrm{Hom}_R(Q, \mathrm{Ext}^i_R(M, C \otimes_R P)) = 0 \quad \forall i \geqslant 1,$$
$$\mathrm{Hom}_R(\mathbb{W} \otimes_R Q, C \otimes_R P) \cong \mathrm{Hom}_R(Q, \mathrm{Hom}_R(\mathbb{W}, C \otimes_R P))$$

is exact. So $M \otimes_R Q$ is a C-Gorenstein projective R-module. □

Proposition 2.7. *Let P be a finitely generated projective R-module. If M is a C-Gorenstein projective R-module, then $\mathrm{Hom}_R(P, M)$ is a C-Gorenstein projective R-module.*

Proof. Let Q be a projective R-module and let $B \to C \to 0$ be exact. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_R(\mathrm{Hom}_R(P, Q), B) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(P, Q), C) & & \\
\cong\downarrow & & \cong\downarrow & & \\
P \otimes_R \mathrm{Hom}_R(Q, B) & \longrightarrow & P \otimes_R \mathrm{Hom}_R(Q, C) & \longrightarrow & 0
\end{array}$$

with the lower row exact. Then

$$\mathrm{Hom}_R(\mathrm{Hom}_R(P,Q),B) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(P,Q),C) \longrightarrow 0$$

is exact, and hence $\mathrm{Hom}_R(P,Q)$ is projective. Since M is a C-Gorenstein projective R-module, there exist projective R-modules $P^0, P^1, \ldots$ together with an exact sequence

$$\mathbb{W}\colon\ 0 \longrightarrow M \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \ldots.$$

Then

$$\mathrm{Hom}_R(P,\mathbb{W})\colon\ 0 \to \mathrm{Hom}_R(P,M) \to C\otimes_R \mathrm{Hom}_R(P,P^0) \to C\otimes_R \mathrm{Hom}_R(P,P^1) \to \ldots$$

is exact and each $\mathrm{Hom}_R(P,P^i)$ is a projective R-module. Let Q be any projective R-module and let $E_\bullet$ be an injective resolution of $C \otimes_R Q$. Then

$$\begin{aligned}
\mathrm{Ext}^i_R(\mathrm{Hom}_R(P,M), C\otimes_R Q) &= \mathrm{H}^i(\mathrm{Hom}_R(\mathrm{Hom}_R(P,M),E_\bullet))\\
&\cong \mathrm{H}^i(P\otimes_R \mathrm{Hom}_R(M,E_\bullet))\\
&\cong P\otimes_R \mathrm{Ext}^i_R(M, C\otimes_R Q) = 0, \qquad \forall i \geqslant 1,\\
\mathrm{Hom}_R(\mathrm{Hom}_R(P,\mathbb{W}), C\otimes_R Q) &\cong P\otimes_R \mathrm{Hom}_R(\mathbb{W}, C\otimes_R Q)
\end{aligned}$$

is exact. So $\mathrm{Hom}_R(P,M)$ is C-Gorenstein projective. □

Let M be an R-module of finite Gorenstein projective dimension. Then there exists a short exact sequence of R-modules $0 \to M \to H \to A \to 0$, where A is Gorenstein projective and $\mathrm{pd}_R H = \mathrm{Gpd}_R M$ by [1, Lemma 2.17]. □

Theorem 2.8. *Let M be an R-module of finite C-Gorenstein projective dimension. Then there exists an exact sequence of R-modules $0 \to M \to H \to A \to 0$ such that there is an exact sequence $0 \to C\otimes_R P_n \to \ldots \to C\otimes_R P_0 \to H \to 0$, where A is C-Gorenstein projective, $n = C\text{-}\mathrm{Gpd}_R M$ and each P_i is projective.*

Proof. If M is C-Gorenstein projective, we take $0 \to M \to H \to A \to 0$ to be the first short exact sequence. We may now assume that $C\text{-}\mathrm{Gpd}_R M = n > 0$. Then there exists an exact sequence $0 \to K \to A' \to M \to 0$, where A' is Gorenstein projective over $R \ltimes C$ and $\mathrm{pd}_{R\ltimes C} K = n-1$ by [6, Proposition 2.13] and [4, Theorem 2.10]. Let $0 \to Q_{n-1} \to \ldots \to Q_0 \to K \to 0$ be a projective resolution of K over $R \ltimes C$. We successively pick projective $R \ltimes C$-modules $Q'_0, \ldots, Q'_{n-1}$ such that

$$Q_0 \oplus Q'_0 \cong (R\ltimes C)\otimes_R P_0,\ \ Q_i \oplus Q'_{i-1}\oplus Q'_i \cong (R\ltimes C)\otimes_R P_i \ \text{ for } i = 1,\ldots,n-1$$

by [6, Lemma 1.5]. Then $0 \to Q_{n-1} \oplus Q'_{n-2} \to (R \propto C) \otimes_R P_{n-2} \to \ldots \to (R \propto C) \otimes_R P_0 \to K \to 0$ is exact. By adding $0 \to (Q'_{n-1} \oplus Q_{n-1} \oplus Q'_{n-2})^{(\mathbb{N})} \to (Q'_{n-1} \oplus Q_{n-1} \oplus Q'_{n-2})^{(\mathbb{N})} \to 0$ to the above sequence in degree $n-1$ and $n-2$, we have that

$$\begin{aligned}0 \longrightarrow (R \propto C) \otimes_R P_{n-1}^{(\mathbb{N})} &\longrightarrow (R \propto C) \otimes_R (P_{n-1}^{(\mathbb{N})} \oplus P_{n-2}) \\ &\longrightarrow \ldots \longrightarrow (R \propto C) \otimes_R P_0 \longrightarrow K \longrightarrow 0\end{aligned}$$

is exact. Since $\mathrm{Ext}^i_R(R, C \otimes_R P) = 0$, hence $\mathrm{Ext}^i_{R\propto C}(R, (R \propto C) \otimes_R P) = 0$ by [6, Corollary 2.3] and [6, Lemma 1.5] for any projective R-module P. So $0 \to \mathrm{Hom}_{R\propto C}(R, (R \propto C) \otimes_R P_{n-1}^{(\mathbb{N})}) \to \mathrm{Hom}_{R\propto C}(R, (R \propto C) \otimes_R (P_{n-1}^{(\mathbb{N})} \oplus P_{n-2})) \to \ldots \to \mathrm{Hom}_{R\propto C}(R, (R \propto C) \otimes_R P_0) \to \mathrm{Hom}_{R\propto C}(R, K) \to 0$ is exact, and hence

$$0 \longrightarrow C \otimes_R P_{n-1}^{(\mathbb{N})} \longrightarrow C \otimes_R (P_{n-1}^{(\mathbb{N})} \oplus P_{n-2}) \longrightarrow \ldots \longrightarrow C \otimes_R P_0 \longrightarrow K \to 0$$

is exact by [6, Lemma 2.2]. Since A' is a Gorenstein projective $R \propto C$-module, hence A' is a C-Gorenstein projective R-module by [6, Proposition 2.13]. So there is an exact sequence $0 \to A' \to C \otimes_R Q \to A \to 0$, where A is C-Gorenstein projective. Consider the pushout of $A' \to M$ and $A' \to C \otimes_R Q$:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & A' & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & C \otimes_R Q & \longrightarrow & H & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & A & = & A & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

If $H \cong C \otimes_R Q'$ for some projective R-module Q', then M is C-Gorenstein projective by Proposition 2.1, which is a contradiction. So $0 \to M \to H \to A \to 0$ is the desired sequence such that $0 \to C \otimes_R P_{n-1}^{(\mathbb{N})} \to C \otimes_R (P_{n-1}^{(\mathbb{N})} \oplus P_{n-2}) \to \ldots \to C \otimes_R P_0 \to C \otimes_R Q \to H \to 0$ is exact. $\square$

By analogy with the proof of Theorem 2.8, we have the following result.

Theorem 2.9. *Let M be an R-module of finite C-Gorenstein injective dimension. Then there exists an exact sequence of R-modules $0 \to B \to H \to M \to 0$ such that there is an exact sequence $0 \to H \to \mathrm{Hom}_R(C, E^0) \to \ldots \to \mathrm{Hom}_R(C, E^n) \to 0$, where B is C-Gorenstein injective, $n = C\text{-}\mathrm{Gid}_R M$ and each E^i is injective.*

It is well known that R is a noetherian ring if and only if any direct limit of injective R-modules is injective by [2, Theorem 3.1.17]. Let R be a local Cohen-Macaulay ring with residue field k and Ω a dualizing module (see [2, Definition 9.5.14]). If $\dim R = 0$, then $\Omega = E(k)$ is a semi-dualizing module of R and R is an artinian ring.

Theorem 2.10. *Let R be artinian. If $M_0 \to M_1 \to M_2 \to \ldots$ is a sequence of C-Gorenstein injective R-modules, then the direct limit $\varinjlim M_n$ is again C-Gorenstein injective.*

Proof. For each n there exist injective R-modules $I_n^0, I_n^1, \ldots$ together with an exact sequence

$$\mathbb{V}_n\colon\ \ldots \longrightarrow \mathrm{Hom}_R(C, I_n^1) \longrightarrow \mathrm{Hom}_R(C, I_n^0) \longrightarrow M_n \longrightarrow 0$$

such that the sequence stays exact when we apply the functor $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), -)$ to it for all injective R-modules J. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{Hom}_R(C, I_0^1) & \longrightarrow & \mathrm{Hom}_R(C, I_0^0) & \longrightarrow & M_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_{10}^1} & & \downarrow{\scriptstyle \varphi_{10}^0} & & \downarrow{\scriptstyle \varphi_{10}} & & \\
\cdots & \longrightarrow & \mathrm{Hom}_R(C, I_1^1) & \longrightarrow & \mathrm{Hom}_R(C, I_1^0) & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Then $\varphi_{n+1,n}^k = \mathrm{Hom}_R(C, \psi_{n+1,n}^k)$ for some homomorphism; namely $\psi_{n+1,n}^k = C \otimes_R \varphi_{n+1,n}^k$ since $C \otimes_R \mathrm{Hom}_R(C, I_n^k) \cong I_n^k$ by [2, Theorem 3.2.11]. So (I_n^k) is a direct system for $k = 0, 1, \ldots$, which gives that

$$\varinjlim \mathbb{V}_n\colon\ \ldots \longrightarrow \mathrm{Hom}_R(C, \varinjlim I_n^1) \longrightarrow \mathrm{Hom}_R(C, \varinjlim I_n^0) \longrightarrow \varinjlim M_n \longrightarrow 0$$

is exact and each $\varinjlim I_n^k$ is an injective R-module. Let J be any injective R-module. Then $J = \bigoplus\limits_{\Lambda} J_\alpha$, where J_α is an injective envelope of some simple R-module for any $\alpha \in \Lambda$ by [8, Theorem 6.6.4]. So

$$\mathrm{Ext}_R^i(\mathrm{Hom}_R(C, J), \varinjlim M_n) \cong \varinjlim \prod_{\alpha \in \Lambda} \mathrm{Ext}_R^i(\mathrm{Hom}_R(C, J_\alpha), M_n) = 0 \quad \forall i \geqslant 1,$$

$$\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), \varinjlim \mathbb{V}_n) \cong \varinjlim \prod_{\alpha \in \Lambda} \mathrm{Hom}_R(\mathrm{Hom}_R(C, J_\alpha), \mathbb{V}_n)$$

is exact since C and $\operatorname{Hom}_R(C, J_\alpha)$ are finitely generated by [9, Theorem 3.64]. Therefore $\varinjlim M_n$ is C-Gorenstein injective. $\square$

3. C-Gorenstein flat modules

In this section we discuss some connections between C-Gorenstein flat modules and C-Gorenstein injective modules. Holm in [4, Theorem 3.6] proved that if R is right coherent, then M is a Gorenstein flat left R-module if and only if M^+ is a Gorenstein injective right R-module.

Theorem 3.1. *M is a C-Gorenstein flat R-module if and only if M^+ is a C-Gorenstein injective R-module.*

Proof. "⇒" There exist flat R-modules $F^0, F^1, \ldots$ together with an exact sequence

$$\mathbb{X}\colon\ 0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \ldots.$$

Then $\mathbb{X}^+\colon \ldots \to \operatorname{Hom}_R(C, F^{1+}) \to \operatorname{Hom}_R(C, F^{0+}) \to M^+ \to 0$ is exact and each F^{i+} is an injective R-module. Let J be any injective R-module. Then

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(C, J), M^+) \cong \operatorname{Tor}^R_i(\operatorname{Hom}_R(C, J), M)^+ = 0 \ \ \forall i \geqslant 1,$$

$$\operatorname{Hom}_R(\operatorname{Hom}_R(C, J), \mathbb{X}^+) \cong (\operatorname{Hom}_R(C, J) \otimes_R \mathbb{X})^+$$

is exact. Hence M^+ is a C-Gorenstein injective R-module.

"⇐" There are injective R-modules $I_0, I_1, \ldots$ together with an exact sequence

$$\mathbb{V}\colon\ \ldots \longrightarrow \operatorname{Hom}_R(C, I_1) \longrightarrow \operatorname{Hom}_R(C, I_0) \longrightarrow M^+ \longrightarrow 0.$$

We successively pick injective R-modules $I'_0, I'_1, \ldots$ such that

$$I_0 \oplus I'_0 \cong I_0^{++},\ \ I'_i \oplus I_{i+1} \oplus I'_{i+1} \cong (I'_i \oplus I_{i+1})^{++} \ \text{ for } i = 0, 1, \ldots.$$

By adding $0 \to \operatorname{Hom}_R(C, I'_i) \to \operatorname{Hom}_R(C, I'_i) \to 0$ to the sequence $\mathbb{V}$ in degree $i+2$ and $i+1$ for all $i = 0, 1, \ldots$, we obtain an exact sequence

$$\mathbb{V}'\colon\ \ldots \longrightarrow \operatorname{Hom}_R(C, (I'_0 \oplus I_1)^{++}) \longrightarrow \operatorname{Hom}_R(C, I_0^{++}) \longrightarrow M^+ \longrightarrow 0,$$

and so $\mathbb{X}\colon\ 0 \to M \to C \otimes_R I_0^+ \to C \otimes_R (I'_0 \oplus I_1)^+ \to \ldots$ is exact. Let I be any injective R-module. Then

$$\operatorname{Tor}^R_i(\operatorname{Hom}_R(C, I), M)^+ \cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, I), M^+) = 0 \ \ \forall i \geqslant 1,$$
$$(\operatorname{Hom}_R(C, I) \otimes_R \mathbb{X})^+ \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, I), \mathbb{V}')$$

is exact. Thus M is a C-Gorenstein flat R-module. $\square$

Corollary 3.2. *The following conditions are equivalent for an R-module M:*

(1) *M is C-Gorenstein flat;*

(2) $\mathrm{Hom}_R(M,E)$ *is C-Gorenstein injective for all injective R-modules E;*

(3) $\mathrm{Hom}_R(M,E)$ *is C-Gorenstein injective for any injective cogenerator E for R-Mod.*

Proof. (1) $\Rightarrow$ (2) Let E be any injective R-module. Then E is isomorphic to a summand of R^{+X} for some set X. Thus $\mathrm{Hom}_R(M,E)$ is isomorphic to a summand of $\mathrm{Hom}_R(M,R^{+X}) \cong M^{+X}$; it follows that $\mathrm{Hom}_R(M,E)$ is C-Gorenstein injective by Theorem 3.1 and Proposition 2.2.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Since R^+ is an injective cogenerator, we see that $M^+ \cong \mathrm{Hom}_R(M,R^+)$ is C-Gorenstein injective, and so M is C-Gorenstein flat by Theorem 3.1. □

Proposition 3.3. *The class C-$\mathcal{GF}(R)$ of all C-Gorenstein flat R-modules is projectively resolving. Furthermore, C-$\mathcal{GF}(R)$ is closed under arbitrary direct sums and arbitrary direct summands.*

Proof. Using Proposition 2.2 and Theorem 3.1. □

Theorem 3.4. *Let R be artinian. Then M is a C-Gorenstein injective R-module if and only if M^+ is a C-Gorenstein flat R-module.*

Proof. "$\Rightarrow$" There exist injective R-modules $I_0, I_1, \ldots$ together with an exact sequence

$$\mathbb{V}\colon\ \ldots \longrightarrow \mathrm{Hom}_R(C,I_1) \longrightarrow \mathrm{Hom}_R(C,I_0) \longrightarrow M \longrightarrow 0.$$

Then $\mathbb{V}^+\colon\ 0 \to M^+ \to C \otimes_R I_0^+ \to C \otimes_R I_1^+ \to \ldots$ is exact by [2, Theorem 3.2.11] and I_i^+ is flat for all $i = 0,1,\ldots$. Let J be any injective R-module. Then $J = \bigoplus\limits_{\Lambda} J_\alpha$, where J_α is an injective envelope of some simple R-module for any $\alpha \in \Lambda$ by [8, Theorem 6.6.4]. Since C and $\mathrm{Hom}_R(C,J_\alpha)$ are finitely generated by [9, Theorem 3.64], we have that

$$\begin{aligned}\mathrm{Tor}_i^R(\mathrm{Hom}_R(C,J),M^+) &\cong \bigoplus_{\alpha\in\Lambda} \mathrm{Tor}_i^R(\mathrm{Hom}_R(C,J_\alpha),M^+)\\ &\cong \bigoplus_{\alpha\in\Lambda} \mathrm{Ext}_R^i(\mathrm{Hom}_R(C,J_\alpha),M)^+ = 0 \quad \forall i \geqslant 1,\\ \mathrm{Hom}_R(C,J) \otimes_R \mathbb{V}^+ &\cong \bigoplus_{\alpha\in\Lambda} \mathrm{Hom}_R(C,J_\alpha) \otimes_R \mathbb{V}^+ \cong \bigoplus_{\alpha\in\Lambda} \mathrm{Hom}_R(\mathrm{Hom}_R(C,J_\alpha),\mathbb{V})^+\end{aligned}$$

is exact by [2, Theorem 3.2.11] and [2, Theorem 3.2.13]. So M^+ is C-Gorenstein flat.

"⇐" There exist flat R-modules $F^0, F^1, \ldots$ together with an exact sequence

$$\mathbb{X}\colon\ 0 \longrightarrow M^+ \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \ldots.$$

Then $\mathbb{X}^+\colon\ \ldots \to \mathrm{Hom}_R(C, F^{1+}) \to \mathrm{Hom}_R(C, F^{0+}) \to M^{++} \to 0$ is exact. We successively pick injective R-modules $E^0, E^1, \ldots$ such that

$$F^{0+} \oplus E^0 \cong F^{0+++},\ \ F^{i+} \oplus E^{i-1} \oplus E^i \cong (F^{i+} \oplus E^{i-1})^{++} \ \text{ for } i = 1, 2, \ldots.$$

By adding $0 \to \mathrm{Hom}_R(C, E^i) \to \mathrm{Hom}_R(C, E^i) \to 0$ to the sequence $\mathbb{X}^+$ in degree $i+2$ and $i+1$ for all $i = 0, 1, \ldots$, we obtain an exact sequence

$$\ldots \longrightarrow \mathrm{Hom}_R(C, (F^{1+} \oplus E^0)^{++}) \longrightarrow \mathrm{Hom}_R(C, F^{0+++}) \longrightarrow M^{++} \longrightarrow 0.$$

Hence $\mathbb{V}\colon\ \ldots \to \mathrm{Hom}_R(C, F^{1+} \oplus E^0) \to \mathrm{Hom}_R(C, F^{0+}) \to M \to 0$ is exact and F^{0+}, $F^{i+} \oplus E^{i-1}$ are injective for $i = 1, 2, \ldots$. Let J be any injective R-module. Then $J = \bigoplus_{\Lambda} J_\alpha$, where J_α is an injective envelope of some simple R-module for any $\alpha \in \Lambda$ by [8, Theorem 6.6.4]. Thus $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J_\alpha), \mathbb{X}^+) \cong (\mathrm{Hom}_R(C, J_\alpha) \otimes_R \mathbb{X})^+$ is exact, which implies that

$$\mathrm{Hom}_R(\mathrm{Hom}_R(C, J_\alpha), \mathbb{V})^{++} \cong (\mathrm{Hom}_R(C, J_\alpha) \otimes_R \mathbb{V}^+)^+ \cong \mathrm{Hom}_R(\mathrm{Hom}_R(C, J_\alpha), \mathbb{V}^{++})$$

is exact by [2, Theorem 3.2.11] since $\mathrm{Hom}_R(C, J_\alpha)$ is finitely generated for any $\alpha \in \Lambda$. So

$$\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), M) \cong \prod_{\alpha \in \Lambda} \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J_\alpha), M) = 0 \ \ \forall i \geqslant 1,$$
$$\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), \mathbb{V}) \cong \prod_{\alpha \in \Lambda} \mathrm{Hom}_R(\mathrm{Hom}_R(C, J_\alpha), \mathbb{V})$$

is exact since C is finitely generated. Thus M^+ is C-Gorenstein flat. □

Corollary 3.5. *Let R be artinian. The following conditions are equivalent for an R-module M:*

(1) *M is C-Gorenstein injective;*
(2) $\mathrm{Hom}_R(M, E)$ *is C-Gorenstein flat for all injective R-modules E;*
(3) $\mathrm{Hom}_R(M, E)$ *is C-Gorenstein flat for any injective cogenerator E for R-Mod;*
(4) *$M \otimes_R F$ is C-Gorenstein injective for all flat R-modules F;*
(5) *$M \otimes_R F$ is C-Gorenstein injective for any faithfully flat R-module F.*

P r o o f. (1) $\Rightarrow$ (2) Let I be any injective R-module. Then $I = \bigoplus_{\Lambda} I_\alpha$, where I_α is an injective envelope of some simple R-module for any $\alpha \in \Lambda$ by [8, Theorem 6.6.4], and so

$$\begin{aligned}\operatorname{Tor}_i^R(&\operatorname{Hom}_R(C, I), \operatorname{Hom}_R(M, E))\\ &\cong \bigoplus_{\alpha\in\Lambda} \operatorname{Hom}_R(\operatorname{Ext}_R^i(\operatorname{Hom}_R(C, I_\alpha), M), E) = 0 \quad \forall i \geqslant 1\end{aligned}$$

by [2, Theorem 3.2.13] for any injective R-module E since $\operatorname{Hom}_R(C, I_\alpha)$ is finitely generated. Since M is C-Gorenstein injective, there exist injective R-modules I_0, $I_1, \ldots$ together with an exact sequence

$$\mathbb{V}\colon \ \ldots \longrightarrow \operatorname{Hom}_R(C, I_1) \longrightarrow \operatorname{Hom}_R(C, I_0) \longrightarrow M \longrightarrow 0.$$

Then

$$\operatorname{Hom}_R(\mathbb{V}, E)\colon\ 0 \to \operatorname{Hom}_R(M, E) \to C \otimes_R \operatorname{Hom}_R(I_0, E) \to C \otimes_R \operatorname{Hom}_R(I_1, E) \to \ldots$$

is exact by [2, Theorem 3.2.11] and each $\operatorname{Hom}_R(I_i, E)$ is flat. By [2, Theorem 3.2.11], $\forall i, \alpha$

$$\begin{aligned}\operatorname{Hom}_R(C, I_\alpha) \otimes_R \operatorname{Hom}_R(M, E) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I_\alpha), M), E),\\ \operatorname{Hom}_R(C, I_\alpha) \otimes_R C \otimes_R \operatorname{Hom}_R(I_i, E) &\cong C \otimes_R \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I_\alpha), I_i), E)\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, \operatorname{Hom}_R(\operatorname{Hom}_R(C, I_\alpha), I_i)), E)\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I_\alpha), \operatorname{Hom}_R(C, I_i)), E).\end{aligned}$$

Denoting $H = \operatorname{Hom}_R(C, I_\alpha)$, consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(\operatorname{Hom}_R(H, M), E) & \longrightarrow & \operatorname{Hom}_R(\operatorname{Hom}_R(H, \operatorname{Hom}_R(C, I_0)), E) & \longrightarrow & \cdots\\ & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & H \otimes_R \operatorname{Hom}_R(M, E) & \longrightarrow & H \otimes_R C \otimes_R \operatorname{Hom}_R(I_0, E) & \longrightarrow & \cdots\end{array}$$

with the upper row exact. Then $\operatorname{Hom}_R(C, I) \otimes_R \operatorname{Hom}_R(\mathbb{V}, E) \cong \bigoplus_{\alpha\in\Lambda} (\operatorname{Hom}_R(C, I_\alpha) \otimes_R \operatorname{Hom}_R(\mathbb{V}, E))$ is exact, and so $\operatorname{Hom}_R(M, E)$ is C-Gorenstein flat.

(3) $\Rightarrow$ (1) Since $M^+ \cong \operatorname{Hom}_R(M, R^+)$ is C-Gorenstein flat, we have that M is C-Gorenstein injective by Theorem 3.4.

(2) $\Rightarrow$ (4) Let F be any flat R-module. Then $(M \otimes_R F)^+ \cong \operatorname{Hom}_R(M, F^+)$ is C-Gorenstein flat, and so $M \otimes_R F$ is C-Gorenstein injective by Theorem 3.4.

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) are obvious. □

If T is a Gorenstein flat R module, then $\operatorname{Ext}_R^i(T, K) = 0$ for all $i \geqslant 1$ and all cotorsion R-modules K with finite flat dimension by [4, Proposition 3.22].

Proposition 3.6. *If M is a C-Gorenstein flat R-module, then $\operatorname{Ext}_R^i(M, C \otimes_R K) = 0$ for all $i \geqslant 1$ and all cotorsion R-modules K with finite flat dimension.*

Proof. We use induction on the finite number $\operatorname{fd}_R K = n$. Assume $n = 0$. Then K is flat, and hence K is a summand of an R-module $\operatorname{Hom}_R(E, E')$, where E, E' are injective by [1, Lemma 2.3] and $\operatorname{Hom}_R(C, C \otimes_R K) \cong K$. By [2, Theorem 3.2.11] and [2, Theorem 3.2.1],

$$\begin{aligned}\operatorname{Ext}_R^i(M, C \otimes_R \operatorname{Hom}_R(E, E')) &\cong \operatorname{Ext}_R^i(M, \operatorname{Hom}_R(\operatorname{Hom}_R(C, E), E')) \\ &\cong \operatorname{Hom}_R(\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, E), M), E') = 0 \quad \forall i \geqslant 1.\end{aligned}$$

So $\operatorname{Ext}_R^i(M, C \otimes_R K) = 0$ for all $i \geqslant 1$. Now assume that $\operatorname{fd}_R K = n > 0$. Let $F \to K$ be a flat cover of K with kernel L. Then L is cotorsion and $\operatorname{fd}_R L = n - 1$. Consider the commutative diagram

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & L & \longrightarrow & F \\ & & \downarrow & & \downarrow{\scriptstyle \mu_L} & & \downarrow{\scriptstyle \mu_F} \\ 0 & \longrightarrow & \operatorname{Hom}_R(C, \operatorname{Tor}_1^R(C, K)) & \longrightarrow & \operatorname{Hom}_R(C, C \otimes_R L) & \longrightarrow & \operatorname{Hom}_R(C, C \otimes_R F)\end{array}$$

Then μ_L is an isomorphism by the induction hypothesis, and so we get $\operatorname{Hom}_R(C, \operatorname{Tor}_1^R(C, K)) = 0$, which means that $\operatorname{Tor}_1^R(C, K) = 0$ since C is faithfully semidualizing by [7, Proposition 3.6]. Thus $0 \to C \otimes_R L \to C \otimes_R F \to C \otimes_R K \to 0$ is exact. Applying the induction hypothesis and the long exact sequence

$$0 = \operatorname{Ext}_R^i(M, C \otimes_R F) \longrightarrow \operatorname{Ext}_R^i(M, C \otimes_R K) \longrightarrow \operatorname{Ext}_R^{i+1}(M, C \otimes_R L) = 0,$$

we have the desired conclusion. □

Proposition 3.7. *Let Q be a flat R-module. If M is a C-Gorenstein flat R-module, then $M \otimes_R Q$ is a C-Gorenstein flat R-module.*

Proof. There exist flat R-modules $F^0, F^1, \ldots$ together with an exact sequence

$$\mathbb{X}\colon\ 0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \ldots.$$

Then $\mathbb{X} \otimes_R Q\colon\ 0 \to M \otimes_R Q \to C \otimes_R (F^0 \otimes_R Q) \to C \otimes_R (F^1 \otimes_R Q) \to \ldots$ is exact and each $F^i \otimes_R Q$ is flat by [2, p. 43, Exercise 9]. Let I be any injective R-module and let $F_\bullet$ be a flat resolution of M. Since $I \otimes_R Q$ is an injective R-module, we have

$$\begin{aligned}\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, I), M \otimes_R Q) &= \operatorname{H}_i(\operatorname{Hom}_R(C, I) \otimes_R F_\bullet \otimes_R Q) \\ &\cong \operatorname{H}_i(\operatorname{Hom}_R(C, I \otimes_R Q) \otimes_R F_\bullet) \\ &= \operatorname{Tor}_i^R(\operatorname{Hom}_R(C, I \otimes_R Q), M) = 0 \quad \forall i \geqslant 1, \\ \operatorname{Hom}_R(C, I) \otimes_R (\mathbb{X} \otimes_R Q) &\cong \operatorname{Hom}_R(C, I \otimes_R Q) \otimes_R \mathbb{X}\end{aligned}$$

is exact. Hence $M \otimes_R Q$ is C-Gorenstein flat. □

Proposition 3.8. *Let P be a finitely generated projective R-module. If M is a C-Gorenstein flat R-module, then $\operatorname{Hom}_R(P,M)$ is a C-Gorenstein flat R-module.*

P r o o f. Let Q be any flat R-module. Then $\operatorname{Hom}_R(P,Q)$ is flat by analogy with the proof of Proposition 2.7. Since M is C-Gorenstein flat, there exist flat R-modules $F^0, F^1, \ldots$ together with an exact sequence

$$\mathbb{X}\colon\ 0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \ldots.$$

Then

$$\operatorname{Hom}_R(P,\mathbb{X})\colon\ 0 \to \operatorname{Hom}_R(P,M) \to C \otimes_R \operatorname{Hom}_R(P,F^0) \to C \otimes_R \operatorname{Hom}_R(P,F^1) \to \ldots$$

is exact and each $\operatorname{Hom}_R(P,F^i)$ is flat. Let I be an injective R-module and $F_\bullet$ a flat resolution of $\operatorname{Hom}_R(C,I)$. Since

$$\begin{aligned}\operatorname{Tor}_i^R(\operatorname{Hom}_R(P,M),\operatorname{Hom}_R(C,I)) &= \operatorname{H}_i(\operatorname{Hom}_R(P,M) \otimes_R F_\bullet)\\ &\cong \operatorname{H}_i(\operatorname{Hom}_R(P, M \otimes_R F_\bullet))\\ &\cong \operatorname{Hom}_R(P, \operatorname{Tor}_i^R(M,\operatorname{Hom}_R(C,I))) = 0 \quad \forall i \geqslant 1,\end{aligned}$$

$$\operatorname{Hom}_R(P,\mathbb{X}) \otimes_R \operatorname{Hom}_R(C,I) \cong \operatorname{Hom}_R(P, \mathbb{X} \otimes_R \operatorname{Hom}_R(C,I))$$

is exact, hence $\operatorname{Hom}_R(P,M)$ is C-Gorenstein flat. □

4. C-Gorenstein modules and change of rings

In this section we investigate some connections between C-Gorenstein projective, injective and flat modules of change of rings. We shall now be concerned with what happens when certain modifications are made to a ring. The two structural operations addressed later are the information of m-adic completion and polynomial rings.

Let (R,m) be a commutative local noetherian ring with residue field k and let $E(k)$ be the injective envelope of k. $\hat{R}$, $\hat{M}$ will denote the m-adic completion of a ring R and an R-module M and M^v will denote the Matlis dual $\operatorname{Hom}_R(M,E(k))$.

Lemma 4.1. *Let (R,m) be a local ring. Then $\hat{C}$ is a semi-dualizing module of $\hat{R}$.*

P r o o f. Since $\operatorname{Hom}_{\hat{R}}(\hat{C},\hat{C}) \cong \operatorname{Hom}_R(C,C) \otimes_R \hat{R} \cong \hat{R}$, hence $\hat{C}$ is a semi-dualizing module of $\hat{R}$. □

Proposition 4.2. *Let (R, m) be a local ring and M an R-module. If $\hat{R}$ is a projective R-module and M is a C-Gorenstein projective R-module, then $M \otimes_R \hat{R}$ is a $\hat{C}$-Gorenstein projective $\hat{R}$-module.*

Proof. There exist projective R-modules $P^0, P^1, \ldots$ together with an exact sequence

$$\mathbb{W}\colon\ 0 \longrightarrow M \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \ldots.$$

Then $\mathbb{W} \otimes_R \hat{R}\colon\ 0 \to M \otimes_R \hat{R} \to \hat{C} \otimes_{\hat{R}} (P^0 \otimes_R \hat{R}) \to \hat{C} \otimes_{\hat{R}} (P^1 \otimes_R \hat{R}) \to \ldots$ is exact and each $P^i \otimes_R \hat{R}$ is a projective $\hat{R}$-module since $\operatorname{Ext}^1_{\hat{R}}(P^i \otimes_R \hat{R}, -) \cong \operatorname{Ext}^1_R(P^i, -) = 0$ by [13, p. 258, 9.21]. Let $\overline{P}$ be any projective $\hat{R}$-module. Then $\overline{P}$ is a projective R-module, and so

$$\operatorname{Ext}^i_{\hat{R}}(M \otimes_R \hat{R}, \hat{C} \otimes_{\hat{R}} \overline{P}) \cong \operatorname{Ext}^i_R(M, C \otimes_R \overline{P}) = 0 \quad \forall i \geqslant 1,$$
$$\operatorname{Hom}_{\hat{R}}(\mathbb{W} \otimes_R \hat{R}, \hat{C} \otimes_{\hat{R}} \overline{P}) \cong \operatorname{Hom}_R(\mathbb{W}, C \otimes_R \overline{P})$$

is exact, which gives that $M \otimes_R \hat{R}$ is a $\hat{C}$-Gorenstein projective $\hat{R}$-module. □

Proposition 4.3. *Let (R, m) be a local ring and M an R-module. If $\hat{R}$ is a projective R-module, then*

(1) *if M is a C-Gorenstein injective R-module, then $\operatorname{Hom}_R(\hat{R}, M)$ is a $\hat{C}$-Gorenstein injective $\hat{R}$-module;*

(2) *$\operatorname{Hom}_R(\hat{R}, M)$ is a $\hat{C}$-Gorenstein injective $\hat{R}$-module if and only if $\operatorname{Hom}_R(\hat{R}, M)$ is a C-Gorenstein injective R-module.*

Proof. (1) There exist injective R-modules $I_0, I_1, \ldots$ together with an exact sequence

$$\mathbb{V}\colon\ \ldots \longrightarrow \operatorname{Hom}_R(C, I_1) \longrightarrow \operatorname{Hom}_R(C, I_0) \longrightarrow M \longrightarrow 0.$$

Then $\operatorname{Hom}_R(\hat{R}, \mathbb{V})\colon\ \ldots \to \operatorname{Hom}_{\hat{R}}(\hat{C}, \operatorname{Hom}_R(\hat{R}, I_1)) \to \operatorname{Hom}_{\hat{R}}(\hat{C}, \operatorname{Hom}_R(\hat{R}, I_0)) \to \operatorname{Hom}_R(\hat{R}, M) \to 0$ is exact and every $\operatorname{Hom}_R(\hat{R}, I_i)$ is an injective $\hat{R}$-module since $\operatorname{Hom}_R(\hat{R}, \operatorname{Hom}_R(C, I_i)) \cong \operatorname{Hom}_{\hat{R}}(\hat{C}, \operatorname{Hom}_R(\hat{R}, I_i))$. Let $\overline{I}$ be any injective $\hat{R}$-module. Then $\overline{I}$ is an injective R-module. By [13, p. 258, 9.21], we have

$$\operatorname{Ext}^i_{\hat{R}}(\operatorname{Hom}_{\hat{R}}(\hat{C}, \overline{I}), \operatorname{Hom}_R(\hat{R}, M)) \cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, \overline{I}), M) = 0 \quad \forall i \geqslant 1,$$
$$\operatorname{Hom}_{\hat{R}}(\operatorname{Hom}_{\hat{R}}(\hat{C}, \overline{I}), \operatorname{Hom}_R(\hat{R}, \mathbb{V})) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, \overline{I}), \mathbb{V})$$

is exact. Hence $\operatorname{Hom}_R(\hat{R}, M)$ is a $\hat{C}$-Gorenstein injective $\hat{R}$-module.

(2) "⇒" There exist injective $\hat{R}$-modules $\overline{I}_0, \overline{I}_1, \ldots$ together with an exact sequence

$$\overline{\mathbb{V}}\colon\ \ldots \longrightarrow \operatorname{Hom}_{\hat{R}}(\hat{C}, \overline{I}_1) \longrightarrow \operatorname{Hom}_{\hat{R}}(\hat{C}, \overline{I}_0) \longrightarrow \operatorname{Hom}_R(\hat{R}, M) \longrightarrow 0.$$

Then $\overline{\mathbb{V}}'\colon \ldots \to \operatorname{Hom}_R(C,\overline{I}_1) \to \operatorname{Hom}_R(C,\overline{I}_0) \to \operatorname{Hom}_R(\hat{R},M) \to 0$ is exact and each $\overline{I}_i$ is an injective R-module. Let I be any injective R-module. Then I is isomorphic to a summand of $E(k)^X$ for some set X, and so $I \otimes_R \hat{R}$ is isomorphic to a summand of $E(k)^X \otimes_R \hat{R} \cong E_{\hat{R}}(\hat{R}/\hat{m})^X \otimes_R \hat{R}$ by [2, Theorem 3.4.1]. Thus $I \otimes_R \hat{R}$ is an injective $\hat{R}$-module by [2, Theorem 3.2.16]. Now by [13, p. 258, 9.21] and [2, Theorem 3.2.4], we see that $\forall i \geqslant 1$

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(C,I),\operatorname{Hom}_R(\hat{R},M)) \cong \operatorname{Ext}^i_{\hat{R}}(\operatorname{Hom}_{\hat{R}}(\hat{C},I\otimes_R\hat{R}),\operatorname{Hom}_R(\hat{R},M)) = 0,$$
$$\operatorname{Hom}_R(\operatorname{Hom}_R(C,I),\overline{\mathbb{V}}') \cong \operatorname{Hom}_{\hat{R}}(\operatorname{Hom}_{\hat{R}}(\hat{C},I\otimes_R\hat{R}),\overline{\mathbb{V}})$$

is exact, which implies that $\operatorname{Hom}_R(\hat{R},M)$ is a C-Gorenstein injective R-module.

"$\Leftarrow$" Since $\operatorname{Hom}_R(\hat{R}\otimes_R\hat{R},E(k)) \cong \operatorname{Hom}_R(\hat{R},\operatorname{Hom}_R(\hat{R},E(k))) \cong \operatorname{Hom}_R(\hat{R},E(k))$ by the proof of [14, Corollary 2.5], hence $\hat{R}\otimes_R\hat{R} \cong \hat{R}$, and so $\operatorname{Hom}_R(\hat{R},M)$ is a $\hat{C}$-Gorenstein injective $\hat{R}$-module by (1). □

Proposition 4.4. *Let (R,m) be a local ring. Then the following conditions are equivalent for a finitely generated R-module M:*

(1) *M is a C-Gorenstein flat R-module;*
(2) *$\hat{M}$ is a $\hat{C}$-Gorenstein flat $\hat{R}$-module;*
(3) *$\hat{M}$ is a C-Gorenstein flat R-module.*

Proof. Since $\operatorname{Tor}^{\hat{R}}_i(\operatorname{Hom}_{\hat{R}}(\hat{C},E(k)),\hat{M}) \cong \operatorname{Tor}^{\hat{R}}_i(\operatorname{Hom}_R(C,E(k))\otimes_R\hat{R},\hat{M}) \cong \operatorname{Tor}^R_i(\operatorname{Hom}_R(C,E(k)),M)\otimes_R\hat{R}$ by [2, Theorem 2.1.11], hence $\operatorname{Tor}^{\hat{R}}_i(\operatorname{Hom}_{\hat{R}}(\hat{C},E(k)),\hat{M}) = 0$ if and only if $\operatorname{Tor}^R_i(\operatorname{Hom}_R(C,E(k)),M) = 0$ for all $i \geqslant 1$.

(1) $\Rightarrow$ (2) There exist flat R-modules $F^0, F^1, \ldots$ together with an exact sequence

$$\mathbb{X}\colon\ 0 \longrightarrow M \longrightarrow C\otimes_R F^0 \longrightarrow C\otimes_R F^1 \longrightarrow \ldots.$$

Then $\mathbb{X}\otimes_R\hat{R}\colon\ 0 \to \hat{M} \to \hat{C}\otimes_{\hat{R}}(F^0\otimes_R\hat{R}) \to \hat{C}\otimes_{\hat{R}}(F^1\otimes_R\hat{R}) \to \ldots$ is exact and every $F^i\otimes_R\hat{R}$ is a flat $\hat{R}$-module by [2, p. 43, Exercise 9]. Let $\overline{I}$ be any injective $\hat{R}$-module. Then $\overline{I}$ is an injective R-module, and so $\operatorname{Hom}_{\hat{R}}(\hat{C},\overline{I})\otimes_{\hat{R}}\hat{R}\otimes_R\mathbb{X} \cong \operatorname{Hom}_R(C,\overline{I})\otimes_R\mathbb{X}$ is exact. Since $\overline{I}$ is isomorphic to a summand of $E(k)^X$ for some set X and $\operatorname{Tor}^{\hat{R}}_i(\operatorname{Hom}_{\hat{R}}(\hat{C},E(k)^X),\hat{M}) \cong \operatorname{Tor}^{\hat{R}}_i(\operatorname{Hom}_{\hat{R}}(\hat{C},E(k)),\hat{M})^X = 0$ by [2, Theorem 3.2.26] we have $\operatorname{Tor}^{\hat{R}}_i(\operatorname{Hom}_{\hat{R}}(\hat{C},\overline{I}),\hat{M}) = 0$ for all $i \geqslant 1$. Therefore $\hat{M}$ is a $\hat{C}$-Gorenstein flat $\hat{R}$-module.

(2) $\Rightarrow$ (1) There exist flat $\hat{R}$-modules $\overline{F}^0, \overline{F}^1, \ldots$ together with an exact sequence

$$\overline{\mathbb{X}}\colon\ 0 \longrightarrow \hat{M} \longrightarrow \hat{C}\otimes_{\hat{R}}\overline{F}^0 \longrightarrow \hat{C}\otimes_{\hat{R}}\overline{F}^1 \longrightarrow \ldots.$$

Then $\mathbb{X}\colon\ 0 \to M \to C\otimes_R\overline{F}^0 \to C\otimes_R\overline{F}^1 \to \ldots$ is exact since $\hat{R}$ is a faithfully flat R-module and each $\overline{F}^i \cong \overline{F}^i\otimes_{\hat{R}}\hat{R} \cong \overline{F}^i\otimes_{\hat{R}}(\hat{R}\otimes_R\hat{R}) \cong \overline{F}^i\otimes_R\hat{R}$ is a flat R-module.

Let J be any injective R-module. Then $J \otimes_R \hat{R}$ is an injective $\hat{R}$-module. Thus $\operatorname{Hom}_R(C, J) \otimes_R \mathbb{X} \otimes_R \hat{R} \cong \operatorname{Hom}_{\hat{R}}(\hat{C}, J \otimes_R \hat{R}) \otimes_{\hat{R}} \overline{\mathbb{X}}$ is exact by [2, Theorem 3.2.4], and hence $\operatorname{Hom}_R(C, J) \otimes_R \mathbb{X}$ is exact. Since J is isomorphic to a summand of $E(k)^X$ for some set X and $\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, E(k)^X), M) \cong \operatorname{Tor}_i^R(\operatorname{Hom}_R(C, E(k)), M)^X = 0$ by [2, Theorem 3.2.26] we have $\operatorname{Tor}_i^R(\operatorname{Hom}_R(C, J), M) = 0$ for all $i \geqslant 1$. Thus M is a C-Gorenstein flat R-module.

(2) $\Leftrightarrow$ (3) By $\hat{R} \otimes_R \hat{R} \cong \hat{R}$.

If R is a ring, then $R[x]$ is the polynomial ring. If M is an R-module, write $M[x] = R[x] \otimes_R M$. Since $R[x]$ is a free R-module and since the tensor product commutes with sums, we may regard the elements of $M[x]$ as 'vectors' $(x^i \otimes_R m_i)$, $i \geqslant 0$, $m_i \in M$ with almost all $m_i = 0$. $M[[x^{-1}]]$ is the $R[x]$-module such that $x(m_0 + m_1 x^{-1} + \ldots) = m_1 + m_2 x^{-1} + \ldots$ and $r(m_0 + m_1 x^{-1} + \ldots) = rm_0 + rm_1 x^{-1} + \ldots$, where $r \in R$. □

Lemma 4.5. *$C[x]$ is a semi-dualizing module of $R[x]$.*

Proof. By analogy with the proof of Lemma 4.1 □

Proposition 4.6. *M is a C-Gorenstein projective R-module if and only if $M[x]$ is a $C[x]$-Gorenstein projective $R[x]$-module.*

Proof. "⇒" There exist projective R-modules $P^0, P^1, \ldots$ together with an exact sequence

$$\mathbb{W}\colon\ 0 \longrightarrow M \longrightarrow C \otimes_R P^0 \longrightarrow C \otimes_R P^1 \longrightarrow \ldots.$$

Then $\mathbb{W} \otimes_R R[x]\colon\ 0 \to M[x] \to C[x] \otimes_{R[x]} P^0[x] \to C[x] \otimes_{R[x]} P^1[x] \to \ldots$ is exact and each $P^i[x]$ is a projective $R[x]$-module by [11, Proposition 5.11]. Let $\bar{Q}$ be any projective $R[x]$-module. Then $\bar{Q}$ is a projective R-module, and so

$$\operatorname{Ext}^i_{R[x]}(M[x], C[x] \otimes_{R[x]} \bar{Q}) \cong \operatorname{Ext}^i_R(M, C \otimes_R \bar{Q}) = 0 \quad \forall i \geqslant 1,$$
$$\operatorname{Hom}_{R[x]}(\mathbb{W} \otimes_R R[x], C[x] \otimes_{R[x]} \bar{Q}) \cong \operatorname{Hom}_R(\mathbb{W}, C \otimes_R \bar{Q})$$

is exact. Therefore $M[x]$ is a $C[x]$-Gorenstein projective $R[x]$-module.

"⇐" There exist projective $R[x]$-modules $\overline{P}^0, \overline{P}^1, \ldots$ together with an exact sequence

$$\overline{\mathbb{W}}\colon\ 0 \longrightarrow M[x] \longrightarrow C[x] \otimes_{R[x]} \overline{P}^0 \longrightarrow C[x] \otimes_{R[x]} \overline{P}^1 \longrightarrow \ldots.$$

Then $\overline{\mathbb{W}}'\colon 0 \to M[x] \to C \otimes_R \overline{P}^0 \to C \otimes_R \overline{P}^1 \to \ldots$ is exact and every $\overline{P}^i$ is a projective R-module. Let Q be any projective R-module. Then

$$0 = \operatorname{Ext}^i_{R[x]}(M[x], C[x] \otimes_{R[x]} Q[x]) \cong \operatorname{Ext}^i_R(M[x], C \otimes_R Q[x]) \quad \forall i \geqslant 1,$$
$$\operatorname{Hom}_R(\overline{\mathbb{W}}', C \otimes_R Q[x]) \cong \operatorname{Hom}_R(\overline{\mathbb{W}}, \operatorname{Hom}_{R[x]}(R[x], C \otimes_R Q[x]))$$
$$\cong \operatorname{Hom}_{R[x]}(\overline{\mathbb{W}}, C[x] \otimes_{R[x]} Q[x])$$

is exact, and hence $\operatorname{Hom}_R(\overline{\mathbb{W}}', C \otimes_R Q)$ is exact and $\operatorname{Ext}^i_R(M[x], C \otimes_R Q) = 0$ for all $i \geqslant 1$ since Q is isomorphic to a summand of $Q[x]$. Thus $M[x]$ is a C-Gorenstein projective R-module, and it follows that M is a C-Gorenstein projective R-module by Proposition 2.1. □

Proposition 4.7. *M is a C-Gorenstein injective R-module if and only if $M[[x^{-1}]]$ is a $C[x]$-Gorenstein injective $R[x]$-module.*

Proof. "⇒" There exist injective R-modules $I_0, I_1, \ldots$ together with an exact sequence

$$\mathbb{V}\colon \ldots \longrightarrow \operatorname{Hom}_R(C, I_1) \longrightarrow \operatorname{Hom}_R(C, I_0) \longrightarrow M \longrightarrow 0.$$

Then $\operatorname{Hom}_R(R[x], \mathbb{V})\colon \ldots \to \operatorname{Hom}_{R[x]}(C[x], \operatorname{Hom}_R(R[x], I_0)) \to \operatorname{Hom}_R(R[x], M) \to 0$ is exact and each $\operatorname{Hom}_R(R[x], I_i)$ is an injective $R[x]$-module. Let $\overline{E}$ be any injective $R[x]$-module. Then $\overline{E}$ is an injective R-module. By [13, p. 258, 9.21] we have

$$\operatorname{Ext}^i_{R[x]}(\operatorname{Hom}_{R[x]}(C[x], \overline{E}), \operatorname{Hom}_R(R[x], M)) \cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, \overline{E}), M) = 0 \quad \forall i \geqslant 1,$$
$$\operatorname{Hom}_{R[x]}(\operatorname{Hom}_{R[x]}(C[x], \overline{E}), \operatorname{Hom}_R(R[x], \mathbb{V})) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, \overline{E}), \mathbb{V})$$

is exact, and so $M[[x^{-1}]] \cong \operatorname{Hom}_R(R[x], M)$ is a $C[x]$-Gorenstein injective $R[x]$-module.

"⇐" There exist injective $R[x]$-modules $\overline{I}_0, \overline{I}_1, \ldots$ together with an exact sequence

$$\overline{\mathbb{V}}\colon \ldots \longrightarrow \operatorname{Hom}_{R[x]}(C[x], \overline{I}_1) \longrightarrow \operatorname{Hom}_{R[x]}(C[x], \overline{I}_0) \longrightarrow \operatorname{Hom}_R(R[x], M) \longrightarrow 0.$$

Then $\overline{\mathbb{V}}'\colon \ldots \to \operatorname{Hom}_R(C, \overline{I}_1) \to \operatorname{Hom}_R(C, \overline{I}_0) \to \operatorname{Hom}_R(R[x], M) \to 0$ is exact and every $\overline{I}_i$ is an injective R-module. Let E be any injective R-module. Then $\operatorname{Hom}_R(R[x], E)$ is an injective $R[x]$-module, and so

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(C, \operatorname{Hom}_R(R[x], E)), M[[x^{-1}]])$$
$$\cong \operatorname{Ext}^i_{R[x]}(\operatorname{Hom}_{R[x]}(C[x], \operatorname{Hom}_R(R[x], E)), M[[x^{-1}]]) = 0 \quad \forall i \geqslant 1,$$
$$\operatorname{Hom}_R(\operatorname{Hom}_R(C, \operatorname{Hom}_R(R[x], E)), \overline{\mathbb{V}}')$$
$$\cong \operatorname{Hom}_{R[x]}(\operatorname{Hom}_{R[x]}(C[x], \operatorname{Hom}_R(R[x], E)), \overline{\mathbb{V}})$$

is exact, which gives that $\operatorname{Hom}_R(\operatorname{Hom}_R(C,E),\overline{\mathbb{V}}')$ is exact and $\operatorname{Ext}^i_R(\operatorname{Hom}_R(C,E),$ $\operatorname{Hom}_R(R[x],M)) = 0$ for all $i \geqslant 1$ since E is isomorphic to a summand of $\operatorname{Hom}_R(R[x],E)$. Thus $M[[x^{-1}]]$ is a C-Gorenstein injective R-module, and hence M is a C-Gorenstein injective R-module by Proposition 2.2. □

Proposition 4.8. *M is a C-Gorenstein flat R-module if and only if $M[x]$ is a $C[x]$-Gorenstein flat $R[x]$-module.*

Proof. "⇒" There exist flat R-modules $F^0, F^1, \ldots$ together with an exact sequence

$$\mathbb{X}\colon\ 0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \ldots.$$

Then $\mathbb{X} \otimes_R R[x]\colon\ 0 \to M[x] \to C[x] \otimes_{R[x]} F^0[x] \to C[x] \otimes_{R[x]} F^1[x] \to \ldots$ is exact and every $F^i[x]$ is a flat $R[x]$-module. Let $\overline{E}$ be any injective $R[x]$-module. Then $\overline{E}$ is an injective R-module, and so

$$\begin{aligned}\operatorname{Tor}_i^{R[x]}(\operatorname{Hom}_{R[x]}(C[x],\overline{E}), M[x])^+ &\cong \operatorname{Ext}^i_{R[x]}(M[x], \operatorname{Hom}_R(C,\overline{E})^+)\\ &\cong \operatorname{Ext}^i_R(M, \operatorname{Hom}_R(C,\overline{E})^+)\\ &\cong \operatorname{Tor}_i^R(\operatorname{Hom}_R(C,\overline{E}), M)^+ = 0 \quad \forall i \geqslant 1,\\ \operatorname{Hom}_{R[x]}(C[x],\overline{E}) \otimes_{R[x]} \mathbb{X} \otimes_R R[x] &\cong \operatorname{Hom}_R(C,\overline{E}) \otimes_R \mathbb{X}\end{aligned}$$

is exact. Thus $M[x]$ is a $C[x]$-Gorenstein flat $R[x]$-module.

"⇐" There exist flat $R[x]$-modules $\overline{F}^0, \overline{F}^1, \ldots$ together with an exact sequence

$$\overline{\mathbb{X}}\colon\ 0 \longrightarrow M[x] \longrightarrow C[x] \otimes_{R[x]} \overline{F}^0 \longrightarrow C[x] \otimes_{R[x]} \overline{F}^1 \longrightarrow \ldots.$$

Then $\overline{\mathbb{X}}'\colon\ 0 \to M[x] \to C \otimes_R \overline{F}^0 \to C \otimes_R \overline{F}^1 \to \ldots$ is exact and each $\overline{F}^i$ is a flat R-module. Let E be any injective R-module. Then

$$\begin{aligned}0 &= \operatorname{Tor}_i^{R[x]}(M[x], \operatorname{Hom}_{R[x]}(C[x], \operatorname{Hom}_R(R[x],E)))^+\\ &\cong \operatorname{Ext}^i_R(M[x], \operatorname{Hom}_R(C, \operatorname{Hom}_R(R[x],E))^+)\\ &\cong \operatorname{Tor}_i^R(M[x], \operatorname{Hom}_R(C, \operatorname{Hom}_R(R[x],E)))^+ \quad \forall i \geqslant 1,\\ \overline{\mathbb{X}}' \otimes_R \operatorname{Hom}_R(C, \operatorname{Hom}_R(R[x],E)) &\cong \overline{\mathbb{X}} \otimes_{R[x]} \operatorname{Hom}_{R[x]}(C[x], \operatorname{Hom}_R(R[x],E))\end{aligned}$$

is exact, which implies that $\overline{\mathbb{X}}' \otimes_R \operatorname{Hom}_R(C,E)$ is exact and moreover $\operatorname{Ext}^i_R(M[x],$ $\operatorname{Hom}_R(C,E)) = 0$ for all $i \geqslant 1$. Thus $M[x]$ is a C-Gorenstein flat R-module, and so M is a C-Gorenstein flat R-module by Proposition 3.3. □

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