

Mingjin Wang

On another extension of q -Pfaff-Saalschütz formula

Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 4, 1131–1137

Persistent URL: <http://dml.cz/dmlcz/140811>

Terms of use:

© Institute of Mathematics AS CR, 2010

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

ON ANOTHER EXTENSION OF q -PFAFF-SAALSCHÜTZ FORMULA

MINGJIN WANG, Changzhou

(Received August 9, 2009)

Abstract. In this paper we give an extension of q -Pfaff-Saalschütz formula by means of Andrews-Askey integral. Applications of the extension are also given, which include an extension of q -Chu-Vandermonde convolution formula and some other q -identities.

Keywords: Andrews-Askey integral, ${}_r+1\varphi_r$ basic hypergeometric series, q -Pfaff-Saalschütz formula, q -Chu-Vandermonde convolution formula

MSC 2010: 05A30, 33D15, 33D05

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

The following is Andrews-Askey integral [1] which can be derived from Ramanujan's ${}_1\psi_1$ summation:

$$(1.1) \quad \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}$$

provided that no zero factors occur in the denominators of the integral.

Andrews-Askey integral is an important formula in basic hypergeometric series. In [4], the author gives a more general q -integral: If $|q| < 1$ and no zero factors occur in the denominators of the integral, then

$$(1.2) \quad \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q) P_m(\omega, d/b; q)}{(a\omega, b\omega; q)_\infty} d_q \omega \\ = \frac{t(1-q)(c; q)_n (d; q)_m (q, tq/s, s/t, abst; q)_\infty}{a^n b^m (as, at, bs, bt; q)_\infty} \\ \times \sum_{k=0}^n \frac{(q^{-n}, as, at; q)_k q^k}{(q, c, abst; q)_k} {}_3\varphi_2 \left(\begin{matrix} bs, bt, q^{-m} \\ d, abstq^k \end{matrix}; q, q \right),$$

The author was supported by STF of Jiangsu Polytechnic University.

where

$$P_0(a, b; q) = 1, \quad P_n(a, b; q) = (a - b)(a - bq) \dots (a - bq^{n-1}), \quad n \geq 1.$$

It is obvious that the case $m = n = 0$ of (1.2) results in (1.1). In this paper we use (1.2) to derive an extension of the q -Pfaff-Saalschütz formula. The following theorem is the main result of this paper.

Theorem 1.1. *If $|q| < 1$ and no zero factors occur in the denominators, then*

$$(1.3) \quad \sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} {}_3\varphi_2 \left(\begin{matrix} a, b, q^{-n} \\ cq^k, ab/cq^{n-1} \end{matrix}; q, q \right) \\ = \frac{(a, b; q)_m (c/a, c/b; q)_n}{(c; q)_{m+n} (ab/c; q)_m (c/ab; q)_n}.$$

Note that there are some important special cases of (1.3). For example, the case $m = 0$ of (1.3) results in the q -Pfaff-Saalschütz formula:

$$(1.4) \quad {}_3\varphi_2 \left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{1-n} \end{matrix}; q, q \right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}.$$

2. NOTATION AND KNOWN RESULTS

We first recall some definitions, notation and known results from [2] which will be used for the proof of Theorem 1.1. Throughout this paper, it is supposed that $0 < |q| < 1$. The q -shifted factorials are defined as

$$(2.1) \quad (a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

We also adopt the following compact notation for multiple q -shifted factorials:

$$(2.2) \quad (a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n,$$

where n is an integer or ∞ . In 1846, Heine introduced the ${}_{r+1}\varphi_r$ basic hypergeometric series, which is defined by

$$(2.3) \quad {}_{r+1}\varphi_r \left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \dots, b_r; q)_n}.$$

The q -Chu-Vandermonde sums are

$$(2.4) \quad {}_2\varphi_1 \left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q \right) = \frac{a^n (c/a; q)_n}{(c; q)_n}$$

and, reversing the order of summation, we have

$$(2.5) \quad {}_2\varphi_1 \left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, cq^n/a \right) = \frac{(c/a; q)_n}{(c; q)_n}.$$

F. H. Jackson defined the q -integral by [3]

$$(2.6) \quad \int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n$$

and

$$(2.7) \quad \int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t.$$

3. THE PROOF OF THEOREM 1.1

In this section we use the generalized Andrews-Askey integral (1.2) to prove Theorem 1.1.

Proof. Using the Andrews-Askey integral (1.1) we arrive at

$$(3.1) \quad \int_c^d \frac{(qt/c, qt/d; q)_{\infty}}{(atq^n, btq^m; q)_{\infty}} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcdq^{m+n}; q)_{\infty}}{(acq^n, adq^n, bcq^m, bdq^m; q)_{\infty}}.$$

On the other hand, if we employ the formulas

$$(3.2) \quad (at; q)_n = (-1)^n a^n q^{\binom{n}{2}} P_n(t, 1/aq^{n-1}; q),$$

$$(3.3) \quad (bt; q)_m = (-1)^m b^m q^{\binom{m}{2}} P_m(t, 1/bq^{m-1}; q)$$

and use the generalized Andrews-Askey integral (1.2), we obtain

$$\begin{aligned}
 (3.4) \quad & \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(atq^n, btq^m; q)_\infty} d_q t = \int_c^d \frac{(qt/c, qt/d; q)_\infty (at; q)_n (bt; q)_m}{(at, bt; q)_\infty} d_q t \\
 & = (-1)^{n+m} a^n b^m q^{\binom{n}{2} + \binom{m}{2}} \\
 & \quad \times \int_c^d \frac{(qt/c, qt/d; q)_\infty P_m(t, a/abq^{m-1}; q) P_n(t, b/baq^{n-1}; q)}{(at, bt; q)_\infty} d_q t \\
 & = (-1)^{n+m} a^n b^m q^{\binom{n}{2} + \binom{m}{2}} \\
 & \quad \times \frac{d(1-q)(a/bq^{m-1}; q)_m (b/aq^{n-1}; q)_n (q, dq/c, c/d, abcd; q)_\infty}{a^m b^n (ac, ad, bc, bd; q)_\infty} \\
 & \quad \times \sum_{k=0}^m \frac{(q^{-m}, ac, ad; q)_k q^k}{(q, a/bq^{m-1}, abcd; q)_k} {}_3\varphi_2 \left(\begin{matrix} bc, bd, q^{-n} \\ b/aq^{n-1}, abcdq^k \end{matrix}; q, q \right).
 \end{aligned}$$

Substituting the relations

$$(-1)^m (b/a)^m q^{\binom{m}{2}} (a/bq^{m-1}; q)_m = (b/a; q)_m,$$

and

$$(-1)^n (a/b)^n q^{\binom{n}{2}} (b/aq^{n-1}; q)_n = (a/b; q)_n$$

into (3.4) we obtain

$$\begin{aligned}
 (3.5) \quad & \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(atq^n, btq^m; q)_\infty} d_q t \\
 & = \frac{d(1-q)(b/a; q)_m (a/b; q)_n (q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty} \\
 & \quad \times \sum_{k=0}^m \frac{(q^{-m}, ac, ad; q)_k q^k}{(q, a/bq^{m-1}, abcd; q)_k} {}_3\varphi_2 \left(\begin{matrix} bc, bd, q^{-n} \\ b/aq^{n-1}, abcdq^k \end{matrix}; q, q \right).
 \end{aligned}$$

Combining (3.1) and (3.5) yields

$$\begin{aligned}
 (3.6) \quad & \frac{d(1-q)(b/a; q)_m (a/b; q)_n (q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty} \\
 & \quad \times \sum_{k=0}^m \frac{(q^{-m}, ac, ad; q)_k q^k}{(q, a/bq^{m-1}, abcd; q)_k} {}_3\varphi_2 \left(\begin{matrix} bc, bd, q^{-n} \\ b/aq^{n-1}, abcdq^k \end{matrix}; q, q \right) \\
 & = \frac{d(1-q)(q, dq/c, c/d, abcdq^{m+n}; q)_\infty}{(acq^n, adq^n, bcq^m, bdq^m; q)_\infty}.
 \end{aligned}$$

Replacing bc, bd and $abcd$ by a, b and c , respectively, and making simple rearrangements, we have (1.3). \square

Letting $a \rightarrow \infty$, $a \rightarrow 0$ in (1.3), respectively, we obtain the following extensions of the q -Chu-Vandermonde convolution formula.

Corollary 3.1. *We have*

$$(3.7) \quad \sum_{k=0}^m \frac{(q^{-m}, c/b; q)_k q^k}{(q, c; q)_k} {}_2\varphi_1 \left(\begin{matrix} b, q^{-n} \\ cq^k \end{matrix}; q, \frac{cq^n}{b} \right) = \left(\frac{c}{b} \right)^m \frac{(b; q)_m (c/b; q)_n}{(c; q)_{m+n}}$$

and

$$(3.8) \quad \sum_{k=0}^m \frac{(q^{-m}, c/b; q)_k}{(q, c; q)_k} (bq^m)^k {}_2\varphi_1 \left(\begin{matrix} b, q^{-n} \\ cq^k \end{matrix}; q, q \right) = b^n \frac{(b; q)_m (c/b; q)_n}{(c; q)_{m+n}}.$$

It is easy to see that the case $m = 0$ or $n = 0$ in (3.7) or (3.8) results in the q -Chu-Vandermonde convolution formula.

4. SOME APPLICATIONS

In this section we give some q -identities as applications of (1.3). First we give the following q -identity.

Theorem 4.1. *For any integer $n \geq 1$ we have*

$$(4.1) \quad {}_3\varphi_2 \left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{2-n} \end{matrix}; q, q \right) = \left\{ 1 - \frac{(1-a)(1-b)}{(1-cq^{n-1})(1-abq/c)} \right\} \frac{(c/a, c/b; q)_{n-1}}{(c, c/ab; q)_{n-1}}.$$

Proof. Let $m = 1$ in (1.3) to get

$$(4.2) \quad {}_3\varphi_2 \left(\begin{matrix} a, b, q^{-n} \\ c, ab/cq^{n-1} \end{matrix}; q, q \right) + \frac{q(1-q^{-1})(1-c/a)(1-c/b)}{(1-q)(1-c)(1-c/ab)} {}_3\varphi_2 \left(\begin{matrix} a, b, q^{-n} \\ cq, ab/cq^{n-1} \end{matrix}; q, q \right) \\ = \frac{(1-a)(1-b)(c/a, c/b; q)_n}{(1-ab/c)(c; q)_{n+1}(c/ab; q)_n}.$$

Substituting the q -Pfaff-Saalschütz formula (1.4) on the left-hand side of (4.2) and making some simple rearrangements, we have

$$(4.3) \quad {}_3\varphi_2 \left(\begin{matrix} a, b, q^{-n} \\ cq, abc^{-1}q^{1-n} \end{matrix}; q, q \right) = \left\{ 1 - \frac{(1-a)(1-b)}{(1-cq^n)(1-ab/c)} \right\} \frac{(cq/a, cq/b; q)_{n-1}}{(cq, cq/ab; q)_{n-1}}.$$

After letting $cq = c$ in (4.3), we get (4.1). □

Corollary 4.2. For any integer $n \geq 1$ we have

$$(4.4) \quad {}_2\varphi_1 \left(\begin{matrix} b, q^{-n} \\ c \end{matrix}; q, \frac{cq^{n-1}}{b} \right) = \left(1 - \frac{c-bc}{bq-bcq^n} \right) \frac{(c/b; q)_{n-1}}{(c; q)_{n-1}}.$$

Proof. Letting $a \rightarrow \infty$ in (4.1), we obtain (4.4). \square

Similarly, if we let $m = 2, 3, \dots$, in (1.3), we can get some more identities like (4.1). Then we give another kind of a q -identity.

Theorem 4.3. For any integer $m \geq 1$, we have

$$(4.5) \quad \sum_{k=0}^m \frac{(q^{-m}, a, b; q)_k}{(q, c, ab/cq^{m-1}; q)_k} \cdot \frac{q^k}{1-cq^k} \\ = \left\{ 1 - \frac{(1-a)(1-b)}{(1-cq^m)(1-ab/c)} \right\} \frac{(c/a, c/b; q)_{m-1}}{(c; q)_m (c/ab; q)_{m-1}}.$$

Proof. Let $n = 1$ in (1.3) to get

$$\sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} {}_3\varphi_2 \left(\begin{matrix} a, b, q^{-1} \\ cq^k, ab/c \end{matrix}; q, q \right) \\ = \sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} \left\{ 1 + \frac{q(1-q^{-1})(1-a)(1-b)}{(1-q)(1-ab/c)(1-cq^k)} \right\} \\ = \sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} - \frac{(1-a)(1-b)}{(1-ab/c)} \sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} \cdot \frac{q^k}{1-cq^k} \\ = \frac{(1-c/a)(1-c/b)(a, b; q)_m}{(1-c/ab)(c; q)_{m+1}(ab/c; q)_m}.$$

Hence, we have

$$(4.6) \quad \frac{(1-a)(1-b)}{(1-ab/c)} \sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} \cdot \frac{q^k}{1-cq^k} \\ = \sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k q^k}{(q, c, c/abq^{m-1}; q)_k} - \frac{(1-c/a)(1-c/b)(a, b; q)_m}{(1-c/ab)(c; q)_{m+1}(ab/c; q)_m}.$$

We use the q -Pfaff-Saalschütz formula (1.4) in (4.6) with $n = m$, $a = c/a$ and $b = c/b$. After simple rearrangements, we have

$$(4.7) \quad \sum_{k=0}^m \frac{(q^{-m}, c/a, c/b; q)_k}{(q, c, c/abq^{m-1}; q)_k} \cdot \frac{q^k}{1-cq^k} \\ = \left\{ 1 - \frac{(1-c/a)(1-c/b)}{(1-cq^m)(1-c/ab)} \right\} \frac{(a, b; q)_{m-1}}{(c; q)_m (ab/c; q)_{m-1}},$$

which is equivalent to (4.5). \square

Corollary 4.4. For any integer $m \geq 1$ we have

$$(4.8) \quad \sum_{k=0}^m \frac{(q^{-m}, b; q)_k}{(q; q)_k (c; q)_{k+1}} \left(\frac{cq^m}{b} \right)^k = \left(1 - \frac{c - bc}{b - bcq^m} \right) \frac{(c/b; q)_{m-1}}{(c; q)_m},$$

and

$$(4.9) \quad \sum_{k=0}^m \frac{(q^{-m}, b; q)_k}{(q, c; q)_k} \cdot \frac{q^k}{1 - cq^k} = \left(1 - \frac{1 - b}{1 - cq^m} \right) \frac{(c/b; q)_{m-1}}{(c; q)_m}.$$

Proof. Letting $a \rightarrow \infty$, or $a \rightarrow 0$ in (4.5), we obtain, respectively, (4.8) and (4.9). □

References

- [1] *G. E. Andrews and R. Askey:* Another q -extension of the beta function. Proc. Amer. Math. Soc. *81* (1981), 97–100.
- [2] *G. E. Andrews:* q -Series: Their Development and Applications in Analysis, Number Theory, Combinatorics, Physics and Computer Algebra. CBMS Regional Conference Lecture Series, vol. 66, Amer. Math, Providences, RI, 1986.
- [3] *F. H. Jackson:* On q -definite integrals. Quart. J. Pure and Appl. Math. *41* (1910), 193–203.
- [4] *M. Wang:* A remark on Andrews-Askey integral. J. Math. Anal. Appl. *341/2* (2008), 14870–1494.

Author's address: Mingjin Wang, Department of Applied Mathematics, Changzhou University, Changzhou, Jiangsu, 213164, P. R. China, e-mail: wang197913@126.com, wmj@jpu.edu.cn.