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THE GROTHENDIECK PROPERTY FOR INJECTIVE TENSOR PRODUCTS OF BANACH SPACES

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Abstract. Let X be a Banach space with the Grothendieck property, Y a reflexive Banach space, and let $X\check{\otimes}_{\varepsilon}Y$ be the injective tensor product of X and Y.

(a) If either X^{**} or Y has the approximation property and each continuous linear operator from X^* to Y is compact, then $X\check{\otimes}_{\varepsilon}Y$ has the Grothendieck property.

(b) In addition, if Y has an unconditional finite dimensional decomposition, then $X\check{\otimes}_{\varepsilon}Y$ has the Grothendieck property if and only if each continuous linear operator from X^* to Y is compact.

Keywords: Banach space, Grothendieck property, tensor product

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González and Gutiérrez in [5] showed that if X is a Banach space with the Grothendieck property, Y is a reflexive Banach space, and each continuous linear operator from X to Y^* is compact, then $X\hat{\otimes}_{\pi}Y$, the projective tensor product of X and Y, has the Grothendieck property; in addition, if Y^* has the bounded compact approximation property, they also showed that $X\hat{\otimes}_{\pi}Y$ has the Grothendieck property if and only if each continuous linear operator from X to Y^* is compact. Bu and Emmanuele in [1] showed that the injective tensor product $L_p[0,1]\check{\otimes}_{\varepsilon}X$ $(1 < p < \infty)$ has the Grothendieck property if and only if X has the Grothendieck property and each continuous linear operator from X^* to $L_p[0,1]$ is compact. In this paper, we will give sufficient conditions for $X\check{\otimes}_{\varepsilon}Y$, the injective tensor product of X and Y, to have the Grothendieck property, and then we will show that these conditions are also necessary under special circumstances.

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For a Banach space X, B_X will denote its closed unit ball and X^* will denote its topological dual space. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ will denote the space of continuous linear operators and the space of compact operators from X to Y, respectively, and $\mathcal{N}(X,Y)$ will denote the space of nuclear operators from X to Y with its nuclear operator norm $\|\cdot\|_{\mathrm{nuc}}$. $X\hat{\otimes}_\pi Y$ and $X\check{\otimes}_\varepsilon Y$ will denote the *projective tensor product* of X and Y with its projective tensor norm $\pi(\cdot)$ and the *injective tensor product* of X and Y with its injective tensor norm $\varepsilon(\cdot)$, respectively (see [10]). For $T \in \mathcal{L}(X,Y)$, T^* will denote its adjoint operator.

A *Schauder decomposition* of a Banach space X is a sequence $\{P_n\}_1^\infty$ of continuous projections on X such that $P_i \circ P_j = 0$ whenever $i \neq j$, and $x = \sum\limits_{k=1}^{\infty} P_n x$ for each x in X (see [7] or [9, § 1.g]). A Schauder decomposition $\{P_n\}_1^\infty$ of X is called *unconditional* if for each $x \in X$, the series $\sum\limits_n P_n x$ converges to x unconditionally. Let K denote the unconditional constant of the unconditional Schauder decomposition $\{P_n\}_1^\infty$ of X. Then for each $x \in X$ and each sequence $\{\theta_n\}_1^\infty$ of signs,

$$\left\|\sum_{n=1}^{\infty} \theta_n P_n x\right\| \leqslant K \cdot \left\|\sum_{n=1}^{\infty} P_n x\right\| = K \cdot \|x\|. \tag{1}$$

A Banach space X is said to have *a finite dimensional decomposition* (FDD for short) if X has a Schauder decomposition $\{P_n\}_1^\infty$ such that $P_n[X]$ is finite dimensional for each $n \in \mathbb{N}$. In addition, if $\{P_n\}_1^\infty$ is also unconditional then X is said to have an *unconditional FDD*. Each Banach space with an unconditional basis has an unconditional FDD.

For convenience, throughout this paper we will write the Radon-Nikodym property simply as RNP and the approximation property simply as AP.

Recall that $(X\hat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$ under the dual operation

$$\langle T, u\rangle = \sum_{k=1}^{n} \langle T(x_k), y_k\rangle$$

for each $T \in \mathcal{L}(X, Y^*)$ and each $u \in X \otimes Y$ with a representation $u = \sum\limits_{k=1}^{n} x_k \otimes y_k$. Also recall that if X^* or Y has AP then $\mathcal{N}(X,Y) = X^*\hat{\otimes}_\pi Y$ and $\mathcal{K}(X,Y) = X^*\check{\otimes}_\varepsilon Y$ (see [6, Ch. I, § 5.1] or [10, § 4.1]). Note that if a dual Banach space has AP then its predual also has AP. Hence if X^{**} or Y^* has AP and if $\mathcal{L}(X^*, Y^*) = \mathcal{K}(X^*, Y^*)$ then

$$\mathcal{N}(X,Y)^* = (X^*\hat{\otimes}_\pi Y)^* = \mathcal{L}(X^*, Y^*) = \mathcal{K}(X^*, Y^*) = X^{**}\check{\otimes}_\varepsilon Y^*$$

under the dual operation

$$\langle T, u\rangle = \sum_{k=1}^{n} \langle x_k^{**}, T^*(y_k^*)\rangle$$

for each $T \in \mathcal{N}(X, Y)$ and each $u \in X^{**} \otimes Y^*$ with a representation $u = \sum_{k=1}^{n} x_k^{**} \otimes y_k^*$.

Proposition 1. *Suppose that either X^{**} or Y^* has AP and $\mathcal{L}(X^*, Y^*) = \mathcal{K}(X^*, Y^*)$. Let $\{T_n\}_1^\infty$ be a bounded sequence in $\mathcal{N}(X, Y)$ and $T \in \mathcal{N}(X, Y)$. Then $\lim_n T_n = T$ weakly in $\mathcal{N}(X, Y)$ if and only if $\lim_n T_n^{**}(x^{**}) = T^{**}(x^{**})$ weakly in Y for each $x^{**} \in X^{**}$.*

P r o o f. Recall that each nuclear operator is weakly compact. Thus for each $x^{**} \in X^{**}$ and each $n \in \mathbb{N}$, $T_n^{**}(x^{**}) \in Y$ and $T^{**}(x^{**}) \in Y$. First assume that $\lim_n T_n = T$ weakly in $\mathcal{N}(X, Y)$. For each $x^{**} \in X^{**}$ and each $y^* \in Y^*$, since $x^{**} \otimes y^* \in X^{**} \check{\otimes}_\varepsilon Y^* = \mathcal{N}(X, Y)^*$, we have

$$\lim_n \langle T_n^{**}(x^{**}) - T^{**}(x^{**}), y^*\rangle = \lim_n \langle T_n - T, x^{**} \otimes y^*\rangle = 0.$$

Thus $\lim_n T_n^{**}(x^{**}) = T^{**}(x^{**})$ weakly in Y.

Now assume that $\lim_n T_n^{**}(x^{**}) = T^{**}(x^{**})$ weakly in Y for each $x^{**} \in X^{**}$. Then for each $v \in X^{**} \otimes Y^*$ with a representation $v = \sum_{k=1}^{m} x_k^{**} \otimes y_k^*$,

$$\lim_n \langle T_n - T, v\rangle = \lim_n \sum_{k=1}^{m} \langle T_n^{**}(x_k^{**}) - T^{**}(x_k^{**}), y_k^*\rangle = 0.$$

Note that $X^{**} \otimes Y^*$ is dense in $X^{**} \check{\otimes}_\varepsilon Y^* = \mathcal{N}(X, Y)^*$. Therefore, $\lim_n T_n = T$ weakly in $\mathcal{N}(X, Y)$. □

Recall that if either X^* or Y^* has RNP then $(X \check{\otimes}_\varepsilon Y)^* = \mathcal{N}(X, Y^*)$ (see [6, Ch. I, § 4.1] or [2, p. 524]). Similarly to the proof of Proposition 1, we have a characterization of weak* convergent sequences in $\mathcal{N}(X, Y^*)$ as a dual space of $X \check{\otimes}_\varepsilon Y$.

Proposition 2. *Suppose that either X^* or Y^* has RNP. Let $\{T_n\}_1^\infty$ be a bounded sequence in $\mathcal{N}(X, Y^*)$ and $T \in \mathcal{N}(X, Y^*)$. Then $\lim_n T_n = T$ weak* in $\mathcal{N}(X, Y^*)$ if and only if $\lim_n T_n(x) = T(x)$ weak* in Y^* for each $x \in X$.*

Recall that a Banach space X is said to have the *Grothendieck property* (or said to be a *Grothendieck space*) if each weak* convergent sequence in X^* is weakly convergent.

Proposition 3. *Suppose that X and Y are Banach spaces with the Grothendieck property such that either X^* or Y^* has RNP, either X^{**} or Y^{**} has AP, and $\mathcal{L}(X^*, Y^{**}) = \mathcal{K}(X^*, Y^{**})$. Then $X\check{\otimes}_\varepsilon Y$ has the Grothendieck property.*

Proof. Recall that $(X\check{\otimes}_\varepsilon Y)^* = \mathcal{N}(X, Y^*)$. Take a sequence $\{T_n\}_1^\infty$ in $\mathcal{N}(X, Y^*)$ such that $\lim_n T_n = 0$ weak* in $\mathcal{N}(X, Y^*)$. Then $\{T_n\}_1^\infty$ is a bounded sequence in $\mathcal{N}(X, Y^*)$. By Proposition 2, for each $x \in X$, $\lim_n T_n(x) = 0$ weak* in Y^*. Since Y has the Grothendieck property, $\lim_n T_n(x) = 0$ weakly in Y^*. That is, for each $y^{**} \in Y^{**}$, $\lim_n \langle T_n(x), y^{**}\rangle = 0$. Thus $\lim_n T_n^*(y^{**}) = 0$ weak* in X^*. Since X has the Grothendieck property, $\lim_n T_n^*(y^{**}) = 0$ weakly in X^*. That is, for each $x^{**} \in X^{**}$, $\lim_n \langle x^{**}, T_n^*(y^{**})\rangle = 0$. So $\lim_n T_n^{**}(x^{**}) = 0$ weakly in Y^*. It follows from Proposition 1 that $\lim_n T_n = 0$ weakly in $\mathcal{N}(X, Y^*)$ and hence, $X\check{\otimes}_\varepsilon Y$ has the Grothendieck property. □

Note that if Y has the Grothendieck property and Y^* has RNP then Y is reflexive (see [3, p. 215]). Thus Proposition 3 is reformulated to the following theorem.

Theorem 4. *Suppose that X is a Banach space with the Grothendieck property and Y is a reflexive Banach space such that either X^{**} or Y has AP and each continuous linear operator from X^* to Y is compact. Then $X\check{\otimes}_\varepsilon Y$ has the Grothendieck property.*

González and Gutiérrez in [5] showed that if $X\hat{\otimes}_\pi Y$ has the Grothendieck property, then either X or Y is reflexive. However, we do not know if the assumption that $X\check{\otimes}_\varepsilon Y$ has the Grothendieck property implies that either X or Y is reflexive.

The following lemma is a special case of Lemma 4 on page 259 of Dunford and Schwartz's book [4].

Lemma 5. *Let $\{P_n\}_1^\infty$ be an unconditional Schauder decomposition of a Banach space X and let B be a subset of X. Then B is relatively compact if and only if $P_n(B)$ is relatively compact for each $n \in \mathbb{N}$ and*

$$\lim_n \sup\left\{\left\|\sum_{i=n}^{\infty} P_i(x)\right\| : x \in B\right\} = 0.$$

Lemma 6. *Suppose that X is a reflexive Banach space with an unconditional FDD. If each bounded sequence in $\mathcal{N}(X, Y)$ which converges in the weak operator topology is weakly convergent in $\mathcal{N}(X, Y)$, then each continuous linear operator from X^* to Y^* is compact.*

Proof. Let $\{P_n\}_1^\infty$ be an unconditional Schauder decomposition of X such that each P_n is of a finite rank. Take any $S \in \mathcal{L}(X^*, Y^*)$. Let $R = S^*|_Y$. Then $R^* = S$. To show that S is compact, we need only to show that R is compact. Suppose that R is not compact, that is, $R[B_Y]$ is not a relatively compact subset of X. By Lemma 5,

$$\lim_n \sup\biggl\{\biggl\|\sum_{i=n}^{\infty} P_i(R(y))\biggr\|_X : y \in B_Y\biggr\} \neq 0.$$

Then there are $\varepsilon_0 > 0$, $y_k \in B_Y$ for $k \in \mathbb{N}$, and a subsequence $n_1 < n_2 < \ldots$ such that

$$\biggl\|\sum_{i=n_k}^{\infty} P_i(R(y_k))\biggr\|_X > \varepsilon_0, \quad k = 1, 2, \ldots.$$

Choose $x_k^* \in B_{X^*}$ such that

$$\biggl|\sum_{i=n_k}^{\infty} \langle P_i(R(y_k)), x_k^*\rangle\biggr| > \varepsilon_0, \quad k = 1, 2, \ldots. \tag{2}$$

Define a linear functional z_k^* on X by

$$z_k^*(x) = \biggl\langle \sum_{i=n_k}^{\infty} P_i(x), x_k^* \biggr\rangle, \quad \forall\, x \in X. \tag{3}$$

Then $z_k^* \in X^*$ and by (1), $\|z_k^*\| \leqslant K$. Let $T_k = z_k^* \otimes y_k$. Then $T_k \in \mathcal{N}(X, Y)$ and $\|T_k\|_{\mathrm{nuc}} \leqslant K$. Moreover, for each $x \in X$ and each $y^* \in Y^*$,

$$\begin{aligned} |\langle T_k(x), y^*\rangle| &= \biggl|\biggl\langle \sum_{i=n_k}^{\infty} P_i(x), x_k^* \biggr\rangle \cdot y^*(y_k)\biggr| \\ &\leqslant \|y^*\| \cdot \|y_k\| \cdot \|x_k^*\| \cdot \biggl\|\sum_{i=n_k}^{\infty} P_i(x)\biggr\| \\ &\leqslant \|y^*\| \cdot \biggl\|\sum_{i=n_k}^{\infty} P_i(x)\biggr\|. \end{aligned}$$

Thus $\{T_k\}_1^\infty$ converges to 0 in the weak operator topology in $\mathcal{N}(X, Y)$ and by hypothesis, it converges to 0 weakly in $\mathcal{N}(X, Y)$.

On the other hand, since X is reflexive and has an unconditional FDD, X^* has AP and so $\mathcal{N}(X, Y) = X^* \hat{\otimes}_\pi Y$. Thus $T_k \in X^* \hat{\otimes}_\pi Y$, too. Recall that $S \in \mathcal{L}(X^*, Y^*) =$

$(X^*\hat{\otimes}_\pi Y)^*$. So $\lim_k \langle T_k, S\rangle = 0$. However, by (2) and (3) one has

$$\begin{aligned} |\langle T_k, S\rangle| &= |\langle S(z_k^*), y_k\rangle| = |\langle R^*(z_k^*), y_k\rangle| \\ &= |\langle z_k^*, R(y_k)\rangle| = \left|\left\langle \sum_{i=n_k}^{\infty} P_i(R(y_k)), x_k^*\right\rangle\right| > \varepsilon_0 \end{aligned}$$

for each $k \in \mathbb{N}$. This contradiction shows that R is compact and hence, S is compact. □

Theorem 7. *Suppose that X is a Banach space and Y is a reflexive Banach space with an unconditional FDD. Then $X\check{\otimes}_\varepsilon Y$ has the Grothendieck property if and only if X has the Grothendieck property and each continuous linear operator from X^* to Y is compact.*

Proof. Suppose that $X\check{\otimes}_\varepsilon Y$ has the Grothendieck property. Since X is complemented in $X\check{\otimes}_\varepsilon Y$, X has the Grothendieck property. Let $\{T_n\}_1^\infty$ be a bounded sequence in $\mathcal{N}(Y, X^*)$ such that $\{T_n\}_1^\infty$ converges to 0 in the weak operator topology in $\mathcal{N}(Y, X^*)$. By Proposition 2, $\{T_n\}_1^\infty$ converges to 0 weak* in $\mathcal{N}(Y, X^*)$. Note that $(X\check{\otimes}_\varepsilon Y)^* = \mathcal{N}(Y, X^*)$ and $X\check{\otimes}_\varepsilon Y$ has the Grothendieck property. $\{T_n\}_1^\infty$ converges to 0 weakly in $\mathcal{N}(Y, X^*)$. It follows from Lemma 6 that each continuous linear operator from Y^* to X^{**} is compact, which is equivalent to the fact that each continuous linear operator from X^* to Y is compact since Y is reflexive. □

Note that if Y is a reflexive Banach space with AP then $\mathcal{K}(X, Y) = X^*\check{\otimes}_\varepsilon Y$, and note that each bounded linear operator from X^{**} to Y is compact if and only if each bounded linear operator from X to Y is compact. Thus we have the following consequence.

Corollary 8. *Suppose that X is a Banach space and Y is a reflexive Banach space with an unconditional FDD. Then $\mathcal{K}(X, Y)$ has the Grothendieck property if and only if X^* has the Grothendieck property and each continuous linear operator from X to Y is compact.*

If X and Y are reflexive Banach spaces, one of them has AP, and each continuous linear operator from X^* to Y is compact, then $X\check{\otimes}_\varepsilon Y$ is reflexive (see [8] or [10, p. 85, Theorem 4.21]) and hence, has the Grothendieck property. But right now we do not have an example in which $X\check{\otimes}_\varepsilon Y$ is not reflexive but has the Grothendieck property. However, Theorem 7 provides us with more examples of Banach spaces without the Grothendieck property. In fact, if X is an infinite-dimensional reflexive Banach space and K is an infinite-dimensional compact Hausdorff space, then there is a non-compact bounded linear operator from X^* to $C(K)$. Thus there is a non-compact bounded linear operator from $C(K)^*$ to X. If, in addition, X has an unconditional FDD, then Theorem 7 informs us that $C(K,X) = C(K)\check{\otimes}_\varepsilon X$ does not have the Grothendieck property even though $C(K)$ has the Grothendieck property (in this case, K is a Stonean space). As examples, $C(K,\ell_p)$, $C(K,L_p[0,1])$, $\ell_\infty\check{\otimes}_\varepsilon\ell_p$, and $\ell_\infty\check{\otimes}_\varepsilon L_p[0,1]$ $(1 < p < \infty)$ do not have the Grothendieck property.

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