Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 60 (2010), No. 4, 1171–1180

Persistent URL: `http://dml.cz/dmlcz/140815`

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Czechoslovak Mathematical Journal, 60 (135) (2010), 1171–1180

A HYBRID MEAN VALUE RELATED TO DEDEKIND SUMS

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(Received November 27, 2009)

Abstract. The main purpose of this paper is to study a hybrid mean value problem related to the Dedekind sums by using estimates of character sums and analytic methods.

Keywords: the Dedekind sum, hybrid mean value, asymptotic formula, identity

MSC 2010: 11F20

1. Introduction

For a positive integer k and an arbitrary integer h, the classical Dedekind sum $S(h,k)$ is defined by

$$S(h,k)=\sum_{a=1}^{k}\Big(\Big(\frac{a}{k}\Big)\Big)\Big(\Big(\frac{ah}{k}\Big)\Big),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2} & \text{if } x \text{ is not an integer;}\\ 0 & \text{if } x \text{ is an integer.}\end{cases}$$

The various properties of $S(h,k)$ were investigated by many authors, see [2] and [3]. For example, L. Carlitz [3] obtained a reciprocity theorem of $S(h,k)$. J. B. Conrey et al. [4] studied the mean value distribution of $S(h,k)$, and proved the following important and interesting asymptotic formula:

$$(1)\qquad {\sum_{h=1}^{k}}'|S(h,k)|^{2m}=f_m(k)\Big(\frac{k}{12}\Big)^{2m}+O\big(\big(k^{9/5}+k^{2m-1+(1/m+1)}\log^3 k\big),$$

This work has been supported by the N.S.F. (10671155) of P. R. China.

where $\sum_h'$ denotes summation over all h such that $(k,h)=1$ and

$$\sum_{n=1}^{\infty}\frac{f_m(n)}{n^s}=2\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta-function.

Jia Chaohua [5] improved the error term in (1) to $O(k^{2m-1})$ provided $m\geqslant 2$. The second author [6] improved the error term of (1) for $m=1$. That is, he proved the asymptotic formula

$$\sum_{h=1}^{k}{}'|S(h,k)|^2=\frac{5}{144}k\varphi(k)\cdot\frac{\prod_{p^\alpha\|k}\left(\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha+1}}\right)}{\prod_{p|k}\left(1+\frac{1}{p}+\frac{1}{p^2}\right)}+O\Big(k\exp\Big(\frac{4\ln k}{\ln\ln k}\Big)\Big),$$

where $p^\alpha\|k$ denotes that $p^\alpha|k$ and $p^{\alpha+1}\dagger k$.

In this paper, we consider the following hybrid mean value problem related to the Dedekind sums:

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\frac{1}{ab}S(a\overline{b},q), \tag{2}$$

where $\overline{b}$ denotes the solution of the congruent equation $x\cdot b\equiv 1 \pmod q$.

It is clear that this sum is a finite sum, so it will be an interesting problem to study its arithmetical properties. About the asymptotic properties of (2), it seems that no one had studied it, at least we have not seen any related result before. In this paper, we use the properties of character sums and analytic methods to study the asymptotic properties of (2), and obtain an interesting hybrid mean value formula and identity. That is, we shall prove the following two theorems:

Theorem 1. *Let $q\geqslant 3$ be an integer. Then we have the asymptotic formula*

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\frac{1}{ab}S(a\overline{b},q)=\frac{5\pi^2}{144}q\cdot\prod_{p|q}\frac{(p^2-1)^2}{p^2(p^2+1)}+O\Big(\exp\Big(\frac{7\ln q}{\ln\ln q}\Big)\Big),$$

where $\sum_{a=1}^{q}{}'$ denotes the summation over all integers $1\leqslant a\leqslant q$ with $(a,q)=1$, $\prod_{p|q}$ denotes the product over all different prime divisors of q, and $\exp(y)=\mathrm{e}^y$.

Theorem 2. *Let p be a prime with $p \geqslant 3$. Then we have the identity*

$$\sum_{a=1}^{\infty}{}' \sum_{b=1}^{\infty}{}' \frac{1}{ab} S(a\overline{b}, p) = \frac{\pi^2}{p} \cdot \sum_{a=1}^{p-1} |S(a,p)|^2.$$

Note that $S(-a,p) = -S(a,p)$, so the sums on the left-hand side of the displayed formula in Theorem 2 is conditionally convergent, its limit being a finite sum. From this result and [6] we can also deduce the following

Corollary. *For any prime $p \geqslant 3$, we have the asymptotic formula*

$$\sum_{a=1}^{\infty}{}' \sum_{b=1}^{\infty}{}' \frac{1}{ab} S(a\overline{b}, p) = \frac{5\pi^2}{144} \cdot p + O\Big(\exp\Big(\frac{4 \ln p}{\ln \ln p}\Big)\Big).$$

2. Some lemmas

In this section we shall give some lemmas which are necessary in the proof of our theorems. First we have

Lemma 1. *Let $q > 2$ be an integer. Then for any integer a with $(a,q) = 1$ we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d|q} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) |L(1,\chi)|^2,$$

where $\varphi(n)$ is the Euler function, $\sum\limits_{\substack{\chi \bmod d \\ \chi(-1)=-1}}$ denotes the summation over all odd character modulo d, and $L(s,\chi)$ is the Dirichlet L-function corresponding to χ modulo d.

Proof. See Lemma 2 of [7]. □

Lemma 2. *Let $q > 2$ be an integer, $d > 1$ a divisor of q, let χ_q^0 denote the principal character modulo q. Then we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |L(1,\chi\chi_q^0)|^2 \cdot |L(1,\chi)|^2$$
$$= \frac{5\pi^4}{144} \varphi(d) \prod_{p|q} \frac{(p^2-1)^2}{p^2(p^2+1)} \prod_{p|d} \Big(1 - \frac{1}{p^2}\Big) + O\Big(\exp\Big(\frac{6 \ln d}{\ln \ln d}\Big)\Big),$$

where $\prod\limits_{p|d}$ denotes the product over all different prime divisors of d, $\exp(y) = \mathrm{e}^y$.

P r o o f. Let χ be any odd character modulo d with $d \mid q$. It is clear that

$$L(1,\chi\chi_q^0)L(1,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)U(n)}{n},$$

where $U(n)=\sum_{r\mid n}\chi_q^0(r)$. Let $A(y,\chi)=\sum_{d<n\leqslant y}\chi(n)U(n)$, then for any real number $N\geqslant d$, applying Abel's identity (see [1]) we have

$$L(1,\chi\chi_q^0)L(1,\chi)=\sum_{1\leqslant n\leqslant N}\frac{\chi(n)U(n)}{n}+\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,\mathrm{d}y.$$

Therefore,

$$\begin{aligned}(3)\qquad &\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|L(1,\chi\chi_q^0)|^2\cdot|L(1,\chi)|^2\\
&=\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\sum_{1\leqslant n\leqslant N}\frac{\chi(n)U(n)}{n}+\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,\mathrm{d}y\right|^2\\
&=\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\sum_{1\leqslant n\leqslant N}\frac{\chi(n)U(n)}{n}\right|^2\\
&\quad+\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left(\sum_{1\leqslant n\leqslant N}\frac{\chi(n)U(n)}{n}\right)\left(\int_N^{\infty}\frac{\overline{A(y,\chi)}}{y^2}\,\mathrm{d}y\right)\\
&\quad+\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left(\sum_{1\leqslant n\leqslant N}\frac{\overline{\chi(n)}U(n)}{n}\right)\left(\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,\mathrm{d}y\right)\\
&\quad+\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,\mathrm{d}y\right|^2\equiv M_1+M_2+M_3+M_4.\end{aligned}$$

Now we estimate M_1, M_2, M_3, and M_4. From the orthogonal relations for character sums modulo d we have

$$\begin{aligned}(4)\quad M_1&=\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\sum_{1\leqslant n\leqslant N}\frac{\chi(n)U(n)}{n}\right|^2\\
&=\sum_{1\leqslant m\leqslant N}\sum_{1\leqslant n\leqslant N}\frac{U(m)U(n)}{mn}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(m)\overline{\chi(n)}\end{aligned}$$

$$= \frac{\varphi(d)}{2} \sideset{}{'}\sum_{\substack{1\leqslant m\leqslant N \\ m\equiv n \bmod d}} \sideset{}{'}\sum_{1\leqslant m\leqslant N} \frac{U(m)U(n)}{mn} - \frac{\varphi(d)}{2} \sideset{}{'}\sum_{\substack{1\leqslant m\leqslant N \\ m\equiv -n \bmod d}} \sideset{}{'}\sum_{1\leqslant m\leqslant N} \frac{U(m)U(n)}{mn}$$

$$= \frac{\varphi(d)}{2} \sideset{}{'}\sum_{1\leqslant m\leqslant N} \frac{U^2(m)}{m^2} + \varphi(d) \sideset{}{'}\sum_{1\leqslant m< N} \sum_{1\leqslant k\leqslant (N-m)/d} \frac{U(m)U(kd+m)}{m(kd+m)}$$

$$+ O\biggl(\frac{\varphi(d)}{2} \sideset{}{'}\sum_{1\leqslant m< N} \sum_{m/d<k\leqslant 2N/d} \frac{U(m)U(kd-m)}{m(kd-m)}\biggr).$$

Note that if $(m,q)=1$, then $U(m)=d(m)$, $\zeta(2)=\frac{1}{6}\pi^2$, $\zeta(4)=\frac{1}{90}\pi^4$, where $d(n)$ is the Dirichlet divisor function and $\zeta(s)$ is the Riemann zeta-function. So from the Euler product formula we have

$$\sideset{}{'}\sum_{1\leqslant m\leqslant N} \frac{U^2(m)}{m^2} = \frac{\zeta^4(2)}{\zeta(4)} \prod_{p\mid q} \frac{(p^2-1)^2}{p^2(p^2+1)} \prod_{p\mid d}\Bigl(1-\frac{1}{p^2}\Bigr) + O\Bigl(\frac{1}{d}\Bigr) \tag{5}$$

$$= \frac{5\pi^4}{72} \prod_{p\mid q} \frac{(p^2-1)^2}{p^2(p^2+1)} \prod_{p\mid d}\Bigl(1-\frac{1}{p^2}\Bigr) + O\Bigl(\frac{1}{d}\Bigr),$$

$$\sideset{}{'}\sum_{1\leqslant m< N} \sum_{1\leqslant k\leqslant (N-m)/d} \frac{U(m)U(kd+m)}{m(kd+m)} \ll \frac{1}{d}\exp\Bigl(\frac{2\ln N}{\ln\ln N}\Bigr), \tag{6}$$

$$\sideset{}{'}\sum_{1\leqslant m< N} \sum_{m/d<k\leqslant 2N/d} \frac{U(m)U(kd-m)}{m(kd-m)} \ll \frac{1}{d}\exp\Bigl(\frac{2\ln N}{\ln\ln N}\Bigr). \tag{7}$$

From (4), (5), (6), and (7) we have

$$M_1 = \frac{5\pi^4}{144}\varphi(d) \prod_{p\mid q} \frac{(p^2-1)^2}{p^2(p^2+1)} \prod_{p\mid d}\Bigl(1-\frac{1}{p^2}\Bigr) + O\Bigl(\exp\Bigl(\frac{2\ln N}{\ln\ln N}\Bigr)\Bigr). \tag{8}$$

Similarly, by the orthogonal relations for the character sums modulo d we can also deduce that

$$M_2 = \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \biggl(\sum_{1\leqslant n\leqslant N} \frac{\chi(n)U(n)}{n}\biggr)\biggl(\int_N^\infty \frac{\overline{A(y,\chi)}}{y^2}\,\mathrm{d}y\biggr) \ll \frac{d^{3/2}}{\sqrt{N}}, \tag{9}$$

$$M_3 = \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \biggl(\sum_{1\leqslant n\leqslant N} \frac{\overline{\chi(n)}U(n)}{n}\biggr)\biggl(\int_N^\infty \frac{A(y,\chi)}{y^2}\,\mathrm{d}y\biggr) \ll \frac{d^{3/2}}{\sqrt{N}}, \tag{10}$$

$$M_4 = \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \biggl|\int_N^\infty \frac{A(y,\chi)}{y^2}\,\mathrm{d}y\biggr|^2 \ll \frac{\varphi^2(d)}{N}. \tag{11}$$

Taking $N = d^3$, combining (3), (8), (9), (10), and (11) we immediately get

$$\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} |L(1,\chi\chi_q^0)|^2 \cdot |L(1,\chi)|^2$$
$$= \frac{5\pi^4}{144}\varphi(d)\prod_{p|q}\frac{(p^2-1)^2}{p^2(p^2+1)}\prod_{p|d}\Big(1-\frac{1}{p^2}\Big)+O\Big(\exp\Big(\frac{6\ln d}{\ln\ln d}\Big)\Big).$$

This proves Lemma 2. □

Lemma 3. *Let p be a prime with $p > 2$. Then we have the identity*

$$\sideset{}{'}\sum_{a=1}^{\infty}\sideset{}{'}\sum_{b=1}^{\infty}\frac{1}{ab}S(a\overline{b},p) = \frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}|L(1,\chi)|^4.$$

P r o o f. Let p be a prime with $p > 2$. Then for any integers a and b with $(ab,p) = 1$, Lemma 1 implies

$$S(a\overline{b},p) = \frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}\chi(a)\overline{\chi}(b)|L(1,\chi)|^2.$$

Summation for all positive integers a and b with $(ab,p) = 1$ in the above formula immediately yields

$$\sideset{}{'}\sum_{a=1}^{\infty}\sideset{}{'}\sum_{b=1}^{\infty}\frac{1}{ab}S(a\overline{b},p) = \frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}\sideset{}{'}\sum_{a=1}^{\infty}\sideset{}{'}\sum_{b=1}^{\infty}\frac{\chi(a)\overline{\chi}(b)}{ab}|L(1,\chi)|^2$$
$$= \frac{p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}|L(1,\chi)|^4.$$

This proves Lemma 3. □

3. Proofs of Theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1. From Lemma 1 we have

$$
\begin{aligned}
(12)\quad & \sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\frac{1}{ab}S(a\overline{b},q)\\
&= \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\frac{\chi(a)\overline{\chi}(b)}{ab}|L(1,\chi)|^2\\
&= \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\sum_{a=1}^{q}\frac{\chi(a)\chi_q^0(a)}{a}\right|^2\cdot|L(1,\chi)|^2\\
&= \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\\
&\quad\times\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|L(1,\chi\chi_q^0)-\int_q^\infty\frac{1}{y^2}\Big(\sum_{q<n\leqslant y}\chi(n)\chi_q^0(n)\Big)\,\mathrm{d}y\right|^2\cdot|L(1,\chi)|^2\\
&= \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}|L(1,\chi\chi_q^0)^2\cdot|L(1,\chi)|^2\\
&\quad-\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}L(1,\chi\chi_q^0)\Big(\int_q^\infty\frac{1}{y^2}\Big(\sum_{q<n\leqslant y}\overline{\chi}(n)\chi_q^0(n)\Big)\,\mathrm{d}y\Big)\\
&\quad\times|L(1,\chi)|^2\\
&\quad-\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}(1,\overline{\chi}\chi_q^0)\Big(\int_q^\infty\frac{1}{y^2}\Big(\sum_{q<n\leqslant y}\chi(n)\chi_q^0(n)\Big)\,\mathrm{d}y\Big)\\
&\quad\times|L(1,\chi)|^2\\
&\quad+\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|\int_q^\infty\frac{1}{y^2}\Big(\sum_{q<n\leqslant y}\chi(n)\chi_q^0(n)\Big)\,\mathrm{d}y\right|^2\cdot|L(1,\chi)|^2\\
&\equiv N_1-N_2-N_3+N_4.
\end{aligned}
$$

Note that from the identity

$$\sum_{d|q}d^2\prod_{p|d}\Big(1-\frac{1}{p^2}\Big)=q^2,$$

applying Lemma 2 we get

$$(13)\quad N_1 = \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\varphi(d)}\Big[\frac{5\pi^4}{144}\varphi(d)\prod_{p|q}\frac{(p^2-1)^2}{p^2(p^2+1)}\prod_{p|d}\Big(1-\frac{1}{p^2}\Big)+O\Big(\exp\Big(\frac{6\ln d}{\ln\ln d}\Big)\Big)\Big]$$
$$= \frac{5\pi^2}{144}q\cdot\prod_{p|q}\frac{(p^2-1)^2}{p^2(p^2+1)}+O\Big(\exp\Big(\frac{7\ln q}{\ln\ln q}\Big)\Big).$$

For any non-principal character χ modulo d, from the Pólya-Vinogradov's inequality we have the estimate

$$\sum_{n=N+1}^{N+H}\chi(n)\ll d^{1/2}\ln d.$$

Using this estimate and Abel's identity we deduce that

$$(14)\quad L(1,\chi) = \sum_{1\leqslant n\leqslant d}\frac{\chi(n)}{n}+\int_d^\infty\frac{1}{y^2}\Big(\sum_{d<n\leqslant y}\chi(n)\Big)\,\mathrm{d}y = \sum_{1\leqslant n\leqslant d}\frac{\chi(n)}{n}+O\Big(\frac{\ln d}{\sqrt{d}}\Big),$$

$$(15)\quad L(1,\chi\chi_q^0) = \sum_{1\leqslant n\leqslant d}\frac{\chi(n)\chi_q^0(n)}{n}+\int_d^\infty\frac{1}{y^2}\Big(\sum_{d<n\leqslant y}\chi(n)\chi_q^0(n)\Big)\,\mathrm{d}y = \sum_{1\leqslant n\leqslant d}\frac{\chi(n)\chi_q^0(n)}{n}+O\Big(\frac{\ln d}{\sqrt{d}}\cdot 2^{\omega(q)}\Big),$$

where $\omega(q)$ denotes the number of all different prime divisors of q.

From (14), (15) and the orthogonal relations for character sums modulo d we obtain

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} L(1,\chi\chi_q^0)\Big(\int_q^\infty\frac{1}{y^2}\Big(\sum_{q<n\leqslant y}\overline{\chi}(n)\chi_q^0(n)\Big)\,\mathrm{d}y\Big)\cdot|L(1,\chi)|^2$$
$$= \int_q^\infty\frac{1}{y^2}\Big(\sum_{1\leqslant l\leqslant d}\sum_{1\leqslant m\leqslant d}\sum_{1\leqslant n\leqslant d}\sum_{q\leqslant u\leqslant y}\frac{\chi_q^0(l)\chi_q^0(u)}{lmn}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(l)\chi(m)\overline{\chi}(n)\overline{\chi}(u)\Big)\,\mathrm{d}y$$
$$+O\Big(\frac{\sqrt{d}\cdot 2^{\omega(q)}\cdot\ln^2 q}{\sqrt{q}}\Big)$$

$$\ll \int_q^\infty \frac{1}{y^2}\Bigg(\sum_{1\leqslant l\leqslant d}\ \sum_{\substack{1\leqslant m\leqslant d\\ lm\equiv un \bmod d}}\ \sum_{1\leqslant n\leqslant d}\ \sum_{q\leqslant u\leqslant y\leqslant 2q} \frac{\chi_q^0(l)\chi_q^0(u)}{lmn}\Bigg)\,\mathrm{d}y + \frac{\sqrt{d}\cdot 2^{\omega(q)}\cdot \ln^2 q}{\sqrt{q}}$$

$$\ll \frac{1}{q}\sum_{1\leqslant l\leqslant d}\ \sum_{1\leqslant m\leqslant d}\ \sum_{1\leqslant n\leqslant d} \frac{\chi_q^0(l)\chi_q^0(u)}{lmn}\,\frac{q}{d} + \frac{\sqrt{d}\cdot 2^{\omega(q)}\cdot \ln^2 q}{\sqrt{q}}$$

$$\ll \frac{\ln^3 d}{d} + \frac{\sqrt{d}\cdot 2^{\omega(q)}\cdot \ln^2 q}{\sqrt{q}}.$$

Consequently,

$$N_2 \ll \frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)}\Big[\frac{\ln^3 d}{d} + \frac{\sqrt{d}\cdot 2^{\omega(q)}\cdot \ln^2 q}{\sqrt{q}}\Big] \ll \exp\Big(\frac{4\ln q}{\ln\ln q}\Big). \tag{16}$$

Similarly, we can also get the estimates

$$N_3 \ll \frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)}\Big[\frac{\ln^3 d}{d} + \frac{\sqrt{d}\cdot 2^{\omega(q)}\cdot \ln^2 q}{\sqrt{q}}\Big] \ll \exp\Big(\frac{4\ln q}{\ln\ln q}\Big) \tag{17}$$

and

$$\begin{aligned} N_4 &\ll \frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \Big|\int_q^\infty \frac{1}{y^2}\Big(\sum_{q<n\leqslant y}\chi(n)\chi_q^0(n)\Big)\,\mathrm{d}y\Big|^2 \cdot |L(1,\chi)|^2 \\ &\ll \frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)} \sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \Big|\int_q^\infty \frac{\sqrt{q}\ln q}{y^2}\,\mathrm{d}y\Big|^2 \cdot |L(1,\chi)|^2 \\ &\ll \exp\Big(\frac{3\ln q}{\ln\ln q}\Big). \end{aligned} \tag{18}$$

Combining (12), (13), (16), (17), and (18) we immediately deduce the asymptotic formula

$$\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\frac{1}{ab}S(a\bar{b},q) = \frac{5\pi^2}{144}q\cdot\prod_{p\mid q}\frac{(p^2-1)^2}{p^2(p^2+1)} + O\Big(\exp\Big(\frac{7\ln q}{\ln\ln q}\Big)\Big).$$

This proves Theorem 1. □

From the orthogonal relations for character sums modulo p and Lemma 1 we have the identity

$$\sum_{a=1}^{p-1}|S(a,p)|^2 = \frac{p^2}{\pi^4(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^4. \tag{19}$$

Now Theorem 2 follows from Lemma 3 and formula (19).

This completes the proof of Theorem 2. □

Acknowledgment. The authors express their gratitude to the referee for his/her very helpful and detailed comments.

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