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Theodore Allen Burton; D. P. Dwiggins

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RESOLVENTS, INTEGRAL EQUATIONS, LIMIT SETS

T. A. Burton, Port Angeles and Memphis, D. P. Dwiggins, Memphis

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Abstract. In this paper we study a linear integral equation $x(t)=a(t)-\int_0^t C(t,s)x(s)\,\mathrm{d}s$, its resolvent equation $R(t,s)=C(t,s)-\int_s^t C(t,u)R(u,s)\,\mathrm{d}u$, the variation of parameters formula $x(t)=a(t)-\int_0^t R(t,s)a(s)\,\mathrm{d}s$, and a perturbed equation. The kernel, $C(t,s)$, satisfies classical smoothness and sign conditions assumed in many real-world problems. We study the effects of perturbations of C and also the limit sets of the resolvent. These results lead us to the study of nonlinear perturbations.

Keywords: integral equation, resolvent

MSC 2010: 34D20

1. Introduction

In 1928 Volterra [9] noted that many kernels of integral equations had certain definite characteristics, that those characteristics were good examples of heredity found in many physical processes, and that there is a good chance of constructing Liapunov functionals for integrodifferential equations with such kernels. Volterra was right on all counts, up to a point. The typical kernel was $C(t,s)=\mathrm{e}^{-(t-s)}$ which had the general properties

$$C(t,s)\geqslant 0,\quad C_s(t,s)\geqslant 0,\quad C_{st}(t,s)\leqslant 0,\quad C_t(t,0)\leqslant 0. \tag{1}$$

But the idea was suspect because no reasonable investigator could seriously advance the notion that we can measure any real-world situation or process with the degree of accuracy dictated by (1). We clutch at straws and one can find today a great many papers boldly demanding (1) in such areas as circulating fuel nuclear reactors [3], population biology [9], viscoelasticity [6], and one of the authors has even dared postulate such behavior in neural networks [2]. The presence of (1) is ubiquitous in population biology. It was picked up by Ergen [3] for nuclear reactors, and greatly advanced by Levin [5] and others from 1963 onward when Levin actually completed Volterra's conjecture and constructed a Liapunov functional centering on (1). Levin continued the study for many years.

In this paper we will note that in some problems conditions in (1) are actually stable under significant perturbations, thereby lending integrity to the practice of asking (1). In the process we construct Liapunov functionals which also establish some fundamental properties of the resolvent for integral and integrodifferential equations. There are two properties which are new and of considerable interest. First, the resolvent will act to produce a copy of the forcing function both in L^p and pointwise. The L^p copy was known, but the pointwise copy is new. Second, under additional conditions we will show that the resolvent, $R(t,s)$, satisfies $R(t,s) \to 0$ as $t \to \infty$ for fixed $s \geqslant 0$. This will enable us to show $\int_0^T |R(t,s)|\,\mathrm{d}s \to 0$ as $t \to \infty$ and that is of great importance in perturbation problems. The same behavior is observed in both the fading memory kernels and the growing memory kernels.

2. Limit sets and Liapunov functionals

Principal players in scalar integral equations are introduced as

$$x(t) = a(t) - \int_0^t C(t,s)x(s)\,\mathrm{d}s \tag{2}$$

with a and C known, while x is the unknown solution, $a\colon [0,\infty) \to \mathbb{R}$ and $C\colon [0,\infty) \times [0,\infty) \to \mathbb{R}$ are both continuous. There is then the resolvent equation written two ways as

$$\begin{aligned} R(t,s) &= C(t,s) - \int_s^t C(t,u)R(u,s)\,\mathrm{d}u \\ &= C(t,s) - \int_s^t R(t,u)C(u,s)\,\mathrm{d}u, \end{aligned} \tag{3}$$

and, finally, the variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)\,\mathrm{d}s. \tag{4}$$

Thus, if R can be found (and it usually can not) then we can solve (2) for any $a(t)$.

We are going to investigate properties by means of Liapunov's direct method and it is obvious that (3) is the highest level equation here, involving only one known, $C(t,s)$. Hence, it needs to be our main focus. While we will be concerned with R, it is true that we need to show that $C(t,s)$ can be substantially perturbed without disturbing the very fundamental properties of R. Thus, we consider the perturbed equation

$$x(t) = a(t) - \int_0^t [C(t,s) + D(t,s)]x(s)\,\mathrm{d}s \tag{5}$$

and the resulting resolvent equation

$$H(t,s) = C(t,s) + D(t,s) - \int_s^t [C(t,u) + D(t,u)]H(u,s)\,\mathrm{d}u \tag{6}$$

where $D\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ is continuous, as is $\int_{t-s}^{\infty} |D(u+s,s)|\,\mathrm{d}u$.

Remark. In Theorem 2.3 we see that $H(t,s)\to B(t,s)$ as $t\to\infty$ for fixed s. Thus, if $B(t,s)\to 0$ as $t\to\infty$, so does $H(t,s)$. This means that the conditions of Theorem 2.3 could reduce the boundedness conditions (9) in the next result.

Theorem 2.1. *Let C satisfy* (1) *and suppose that there are positive numbers α and β with $\alpha+\beta<2$ and*

$$\int_0^t |D(t,s)|\,\mathrm{d}s \leqslant \alpha, \quad \int_0^\infty |D(u+t,t)|\,\mathrm{d}u \leqslant \beta. \tag{7}$$

Then:

(i) *There is a $K>0$ with*

$$\int_s^t H^2(u,s)\,\mathrm{d}u \leqslant K\int_s^t [C^2(u,s) + D^2(u,s)]\,\mathrm{d}u := L(t,s). \tag{8}$$

(ii) *If, in addition,*

$$\begin{gathered} L(t,s), \quad C(t,s)+D(t,s), \quad C_t(t,s)+D_t(t,s), \\ \int_s^t [C^2(t,u)+D^2(t,u)]\,\mathrm{d}u, \quad \int_s^t [C_t^2(t,u)+D_t^2(t,u)]\,\mathrm{d}u \end{gathered} \tag{9}$$

are all bounded, then $\sup_{0\leqslant s\leqslant t<\infty} [|H(t,s)| + |H_t(t,s)|] < \infty$ *and for fixed* $s \geqslant 0$

$$\lim_{t\to\infty} H(t,s) = 0. \tag{10}$$

(iii) *If, in addition,*

$$|B_s(t,s)| + |B(s,s)| + \int_s^t |B_s(u,s)|\,\mathrm{d}u \tag{11}$$

is also bounded, then for each $T > 0$

$$\lim_{t\to\infty}\int_0^T |H(t,s)|\,\mathrm{d}s = 0. \tag{12}$$

N o t e. The conditions on C and D are independent in the sense that we can set $C \equiv 0$ and have the result concerning the resolvent for D alone, or we can set $D \equiv 0$ and have the conditions for the resolvent for C alone.

P r o o f. We begin by defining a Liapunov functional for (6) in the form

$$\begin{aligned}V(t) = &\int_s^t \int_{t-v}^\infty |D(u+v,v)|\,\mathrm{d}u H^2(v,s)\,\mathrm{d}v + C(t,s)\Big(\int_s^t H(u,s)\,\mathrm{d}u\Big)^2\\ &+ \int_s^t C_v(t,v)\Big(\int_v^t H(u,s)\,\mathrm{d}u\Big)^2\,\mathrm{d}v.\end{aligned}$$

Taking into account that $C_{vt}(t,v) \leqslant 0$ and $C_t(t,s) \leqslant 0$ we have

$$\begin{aligned}V'(t) \leqslant &\int_0^\infty |D(u+t,t)|\,\mathrm{d}u H^2(t,s) - \int_s^t |D(t,v)|H^2(v,s)\,\mathrm{d}v\\ &+ 2H(t,s)C(t,s)\int_s^t H(u,s)\,\mathrm{d}u + 2H(t,s)\int_s^t C_v(t,v)\int_v^t H(u,s)\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

If we integrate the last term by parts we have

$$\begin{aligned}&2H(t,s)\Big[C(t,v)\int_v^t H(u,s)\,\mathrm{d}u\Big|_s^t + \int_s^t C(t,v)H(v,s)\,\mathrm{d}v\Big]\\ &\quad = 2H(t,s)\Big[-C(t,s)\int_s^t H(u,s)\,\mathrm{d}u + \int_s^t C(t,v)H(v,s)\,\mathrm{d}v\Big].\end{aligned}$$

Canceling terms and taking (6) into account we have

$$\begin{aligned}V' \leqslant &\ \beta H^2(t,s) - \int_s^t |D(t,v)|H^2(v,s)\,\mathrm{d}v\\ &+ 2H(t,s)\Big[C(t,s) + D(t,s) - H(t,s) - \int_s^t D(t,u)H(u,s)\,\mathrm{d}u\Big]\\ \leqslant &\ \beta H^2(t,s) - \int_s^t |D(t,v)|H^2(v,s)\,\mathrm{d}v\\ &+ 2H(t,s)[C(t,s) + D(t,s) - H(t,s)]\\ &+ \int_s^t |D(t,u)|\big(H^2(u,s) + H^2(t,s)\big)\,\mathrm{d}u\\ \leqslant &\ (\alpha+\beta)H^2(t,s) + 2H(t,s)[C(t,s) + D(t,s) - H(t,s)]\\ \leqslant &\ (\alpha+\beta)H^2(t,s) + M\big(C^2(t,s) + D^2(t,s)\big) - \gamma H^2(t,s)\end{aligned}$$

where γ can be chosen so that $\alpha+\beta<\gamma<2$, and then for $\eta=\gamma-(\alpha+\beta)$ we have

$$V'(t)\leqslant -\eta H^2(t,s)+M\bigl(C^2(t,s)+D^2(t,s)\bigr)$$

and (8) holds.

Notice that if (9) holds then H is bounded, as may be seen using the Schwarz inequality on the last term of H. Having H bounded, we have H_t bounded, as may be seen using the Schwarz inequality on the last term of H_t. As $L(t,s)$ is bounded, it follows that $\lim\limits_{t\to\infty} H(t,s)=0$ for fixed s.

Next, rewrite the resolvent as $H(t,s)=B(t,s)-\int_s^t H(t,u)B(u,s)\,\mathrm{d}u$ and use (11) to show that H_s is bounded. We are now prepared to prove the following proposition which will complete the proof. □

Proposition 2.1. *Let $H(t,s)$ be continuous for $0\leqslant s\leqslant t<\infty$, and assume $\lim\limits_{t\to\infty} H(t,s)=0$ for each fixed $s\geqslant 0$. Assume also that $H(t,s)$ is globally Lipschitz in s, i.e., there is an $M>0$ such that $0\leqslant s_1\leqslant s_2\leqslant t<\infty$ implies $|H(t,s_1)-H(t,s_2)|\leqslant M|s_1-s_2|$. Then for each $T>0$ we have $\lim\limits_{t\to\infty}\int_0^T |H(t,s)|\,\mathrm{d}s=0$.*

Proof. Assume by way of contradiction there is a $T>0$, there is an $\varepsilon>0$, and a sequence $\{t_n\}\uparrow\infty$ such that $\int_0^T |H(t_n,s)|\,\mathrm{d}s>\varepsilon$ for each n. Now, if $|H(t_n,s)|\leqslant\varepsilon/T$ for each $s\in[0,T]$ then $\int_0^T |H(t_n,s)|\,\mathrm{d}s\leqslant\varepsilon$, so it follows that for each n there is a point $s_n\in[0,T]$ such that $|H(t_n,s_n)|>\varepsilon/T$. The sequence $\{s_n\}$ is contained in the compact interval $[0,T]$, so there must be some convergent subsequence $s_{n_k}\to s^*$. For the given $\varepsilon>0$, find $N>0$ such that $n_k>N$ implies that $|s_{n_k}-s^*|<\varepsilon/(2MT)$. Thus we have

$$\begin{aligned}\frac{\varepsilon}{T}<|H(t_{n_k},s_{n_k})|&\leqslant|H(t_{n_k},s_{n_k})-H(t_{n_k},s^*)|+|H(t_{n_k},s^*)|\\&\leqslant M|s_{n_k}-s^*|+|H(t_{n_k},s^*)|,\end{aligned}$$

so $n_k>N\Longrightarrow|H(t_{n_k},s^*)|>\varepsilon/(2T)$, contradicting $H(t,s^*)\to 0$. □

Remark. There is a list of conclusions which can be drawn from the theorem.

(i) The same conclusions follow from the derivative conditions (1), or from the integral conditions (7), or certain linear combinations of the two.

(ii) The integral $\int_s^t[C(t,u)+D(t,u)]H(u,s)\,\mathrm{d}u$ constructs both an L^2 and a pointwise copy of $C(t,s)+D(t,s)$ in t for fixed s.

(iii) A parallel Liapunov functional can be constructed for (5) under the conditions of (1) and (7) which will yield $x\in L^2$ for $a\in L^2$ and show that $x(t)$ converges to zero, under the conditions which send $H(t,s)$ to zero. The integral $\int_s^t H(t,u)a(u)\,\mathrm{d}u$ constructs both an L^2 and an asymptotic pointwise copy of every $a\in L^2$. That property will be used shortly in a perturbation problem.

Now there is a fundamental result which can be obtained from the fact that $H(t,s) \to 0$ as $t \to \infty$. It was proved by Strauss [8] that if there is an $\alpha < 1$ with $\int_0^t |B(t,s)|\,\mathrm{d}s \leqslant \alpha$ then $\int_0^t |H(t,s)|\,\mathrm{d}s \leqslant \alpha/(1-\alpha)$, and if $\lim\limits_{t\to\infty} \int_0^T |B(t,s)|\,\mathrm{d}s \to 0$ for each $T > 0$ then the corresponding result also holds for $H(t,s)$.

This result proved fundamental in perturbation problems and Strauss remarked that $\int_0^t |B(t,s)|\,\mathrm{d}s \leqslant \alpha < 1$ could not be improved. One of our main theses here is that the result can, indeed, hold with far less than Strauss's condition, and it hinges on our new conclusion that $H(t,s) \to 0$ as $t \to \infty$ for fixed s, along with Proposition 2.1.

Proposition 2.2. *If $\varphi \in L^1[0,\infty)$, if φ is continuous or in $L^2[0,\infty)$, if $H(t,s)$ is bounded and continuous, and if for each $T > 0$ it follows that $\int_0^T |H(t,s)|\,\mathrm{d}s \to 0$ as $t \to \infty$, then $\int_0^t |H(t,s)|\,\varphi(s)\,\mathrm{d}s \to 0$ as $t \to \infty$.*

Proof. Let $|H(t,s)| \leqslant M$ and let $\varepsilon > 0$ be given. Find $T > 0$ so that $\int_T^\infty |\varphi(s)|\,\mathrm{d}s < \varepsilon/2M$. Then for φ continuous we have

$$\int_0^t |H(t,s)\varphi(s)|\,\mathrm{d}s \leqslant \|\varphi\|^{[0,T]} \int_0^T |H(t,s)|\,\mathrm{d}s + M \int_T^\infty |\varphi(s)|\,\mathrm{d}s < \varepsilon$$

for large t. On the other hand, if $\varphi \in L^2[0,\infty)$ then we have

$$\begin{aligned}\int_0^t |H(t,s)\varphi(s)|\,\mathrm{d}s &\leqslant \int_0^T |H(t,s)||\varphi(s)|\,\mathrm{d}s + M\int_T^\infty |\varphi(s)|\,\mathrm{d}s \\ &\leqslant \sqrt{\int_0^T H^2(t,s)\,\mathrm{d}s \int_0^\infty \varphi^2(s)\,\mathrm{d}s} + M\int_T^\infty |\varphi(s)|\,\mathrm{d}s.\end{aligned}$$

As $\int_0^T H^2(t,s)\,\mathrm{d}s \leqslant \|H\| \int_0^T |H(t,s)|\,\mathrm{d}s \to 0$, the result follows as before. □

Rather than deal with C we wish to deal with

$$B(t,s) := C(t,s) + D(t,s) \tag{13}$$

where C satisfies (1) and D satisfies (7) so that the conclusions of Theorem 2.1 hold. It is well-known (see Miller [6; p. 190–192], Burton [1; p. 162], and applications in Islam and Neugebauer [4]) that the perturbed equation

$$x(t) = a(t) - \int_0^t B(t,s)[x(s) + G(s,x(s))]\,\mathrm{d}s \tag{14}$$

can be decomposed into

$$y(t) = a(t) - \int_0^t B(t,s)y(s)\,\mathrm{d}s \tag{15}$$

and

$$x(t) = y(t) - \int_0^t H(t,s)G(s,x(s))\,\mathrm{d}s. \tag{16}$$

Thus, we can work with (15), perhaps differentiating it, and establish the properties of the resolvent $H(t,s)$ and then apply those properties to (16) without disturbing $G(t,x)$. Under the conditions of Theorem 2.1 we can already say that $a \in L^2$ and bounded implies $y \in L^2$ and y is bounded. We can directly obtain a very nice result for (14). We remark that there are many conditions known to ensure $y \in L^2$ and bounded. See, for example, Burton [1], especially the material around p. 78. These properties of y can frequently be proved without assigning the corresponding property to a.

Theorem 2.2. *Let H satisfy the conditions of Proposition* 2.1, *let $y \in L^2[0,\infty)$ be bounded, let*

$$|G(t,x)| \leqslant \varphi(t)|x|, \ \ \varphi \in L^1[0,\infty),$$

and let φ be continuous or in $L^2[0,\infty)$. If $x(t)$ is any solution of (14) *then x is bounded and $x(t) - y(t) \to 0$ as $t \to \infty$.*

Proof. If $x(t)$ is not bounded, then there is a sequence $\{t_n\} \uparrow \infty$ such that $|x(t)| \leqslant |x(t_n)|$ for $0 \leqslant t \leqslant t_n$ and $|x(t_n)| \uparrow \infty$. Thus,

$$|x(t_n)| \leqslant \|y\| + \int_0^{t_n} |H(t_n,s)|\varphi(s)\,\mathrm{d}s|x(t_n)|.$$

For large n we have $\int_0^{t_n} |H(t_n,s)|\varphi(s)\,\mathrm{d}s < 1/2$ by Proposition 2.2 and this contradicts $|x(t_n)| \uparrow \infty$. It then follows that $\varphi(t)|x(t)| \in L^1[0,\infty)$ and so

$$\int_0^t H(t,s)\varphi(s)|x(s)|\,\mathrm{d}s \to 0$$

and the result is proved. □

Example 2.1. Note in the listed assumptions on $B(t,s) = C(t,s) + D(t,s)$ we have $\int_0^t |D(t,s)|\,\mathrm{d}s \leqslant \alpha < 2$, but we require no similar integral bound on either C or B. Thus, our result shows the Strauss condition $\sup\limits_{t>0} \int_0^t |C(t,s)|\,\mathrm{d}s < 1$ is unnecessary in order to conclude that $\lim\limits_{t\to\infty} \int_0^T |R(t,s)|\,\mathrm{d}s = 0$ for each $T > 0$. Indeed, the kernel $B(t,s) = C(t,s) = \mathrm{e}^{-k(t-s)}$ (with $D = 0$, so $H = R$) satisfies the conditions of Theorem 2.1 and Proposition 2.1 for every $k > 0$, but also has $\sup\limits_{t>0} | \int_0^t |C(t,s)|\,\mathrm{d}s = 1/k$, and this integral bound on C may be as large as desired by taking $0 < k < 1$ as small as needed.

Here, the conditions of Strauss's theorem are not met, but $\sup\limits_{t>0} \int_0^t |R(t,s)|\,\mathrm{d}s < \infty$ (this supremum is equal to $1/(k+1)$ in this example) and $\lim\limits_{t\to\infty} \int_0^T |R(t,s)|\,\mathrm{d}s = 0$ for every $T > 0$. These results are explicitly obtained by showing $R(t,s) = \mathrm{e}^{-(k+1)(t-s)}$ is the resolvent corresponding to the kernel $C(t,s) = \mathrm{e}^{-k(t-s)}$.

The theory of the resolvent is filled with instances of similarity between H and B. There are many ways to obtain relations such as (8) when assumptions different from (1) and (7) are made. The following two results show general relations between H and B whenever relations like (8) and (9) hold. The latter is a mild extension of Theorem 2.6.1.7 in Burton [1; p. 118]. Here, we note the close relation between them when s is near t.

Theorem 2.3. *Suppose there are positive constants M and K such that $\int_0^t B^2(t,u)\,\mathrm{d}u \leqslant M$ and for fixed $s \geqslant 0$ we have*

$$\int_s^t H^2(u,s)\,\mathrm{d}u \leqslant K \int_s^t B^2(u,s)\,\mathrm{d}u \leqslant M$$

for $s \leqslant t$. If, in addition, for all large fixed $T > s$ we have $\int_s^T B^2(t,u)\,\mathrm{d}u \to 0$ as $t \to \infty$, then

$$H(t,s) - B(t,s) \to 0$$

as $t \to \infty$.

Proof. Notice that with s fixed we have $H^2(t,s) \in L^1[s,\infty)$. From (3) we have for $s < T < t$ that

$$\begin{aligned}
&\frac{1}{2}\big(H(t,s) - B(t,s)\big)^2 \\
&\quad = \frac{1}{2}\bigg(\int_s^t B(t,u)H(u,s)\,\mathrm{d}u\bigg)^2 \\
&\quad \leqslant \bigg(\int_s^T B(t,u)H(u,s)\,\mathrm{d}u\bigg)^2 + \bigg(\int_T^t B(t,u)H(u,s)\,\mathrm{d}u\bigg)^2 \\
&\quad \leqslant \int_s^T H^2(u,s)\,\mathrm{d}u \int_s^T B^2(t,u)\,\mathrm{d}u + \int_T^t B^2(t,u)\,\mathrm{d}u \int_T^t H^2(u,s)\,\mathrm{d}u \\
&\quad \leqslant M \int_s^T B^2(t,u)\,\mathrm{d}u + M \int_T^t H^2(u,s)\,\mathrm{d}u.
\end{aligned}$$

For a given $\varepsilon > 0$, take T so large that $\int_T^\infty H^2(u,s)\,\mathrm{d}u < \varepsilon/2M$. Then take t so large that $M\int_s^T B^2(t,u)\,\mathrm{d}u < \varepsilon/2$. This will complete the proof. □

We see that the totally unknown function, $H(t,s)$, converges pointwise to the clearly visible $B(t,s)$.

The result can be very useful in conjunction with Proposition 2.1 since if $B(t,s) \to 0$ as $t \to \infty$ for fixed s, then the same is true for $H(t,s)$ and the consequence of Proposition 2.1 is very important for our work with (14) in Theorem 2.2.

Our work here is most emphatically not for the convolution case, but for purpose of illustration note that in the convolution case $H(t,s) = H(t-s)$ and for $s = 0$ we see that $H(t) \to B(t)$. In the next result we would have $\int_0^\infty \big(H(t) - B(t)\big)^2 \, \mathrm{d}t < \infty$. Under the conditions of both theorems H converges to B both pointwise and in $L^2[0,\infty)$.

One by one, we transfer the asymptotic properties of B to H. One of the ultimate goals is to relate H and B so closely with integral inequalities that the unknown function H can, in effect, be replaced by the known function B in establishing long-term qualitative properties of y in (15) using its variation of parameters formula $y(t) = a(t) - \int_0^t H(t,s)a(s)\,\mathrm{d}s$.

Theorem 2.4. *Suppose that there are positive constants K, M_1, M_2, and M_3 with*

$$\int_s^t H^2(u,s)\,\mathrm{d}u \leqslant K \int_s^t B^2(u,s)\,\mathrm{d}u,$$

$$\int_s^t |B(v,s)|\,\mathrm{d}v \leqslant M_2,$$

and

$$\int_s^t |B(t,u)|\,\mathrm{d}u \leqslant M_3$$

all for $0 \leqslant s \leqslant t < \infty$. Then

$$\int_s^t \big(H(u,s) - B(u,s)\big)^2\,\mathrm{d}u \leqslant M_2 M_3 K \int_s^t B^2(u,s)\,\mathrm{d}u.$$

If, in addition,

$$\int_s^\infty B^2(u,s)\,\mathrm{d}u \leqslant M_1$$

for any fixed $s \geqslant 0$, then

$$\int_s^\infty \big(H(u,s) - B(u,s)\big)^2\,\mathrm{d}u \leqslant M_1 M_2 M_3 K.$$

P r o o f. From (3) we have

$$\begin{aligned}\big(H(t,s) - B(t,s)\big)^2 &= \Big(\int_s^t B(t,u)H(u,s)\,\mathrm{d}u\Big)^2 \\ &\leqslant \int_s^t |B(t,u)|\,\mathrm{d}u \int_s^t |B(t,u)|H^2(u,s)\,\mathrm{d}u\end{aligned}$$

so

$$\begin{aligned}\int_s^t \big(H(u,s)-B(u,s)\big)^2\,\mathrm{d}u &\leqslant M_3\int_s^t\int_s^v |B(v,u)|H^2(u,s)\,\mathrm{d}u\,\mathrm{d}v\\ &= M_3\int_s^t\int_u^t |B(v,u)|H^2(u,s)\,\mathrm{d}v\,\mathrm{d}u\\ &\leqslant M_3\int_s^t H^2(u,s)\int_u^t |B(v,u)|\,\mathrm{d}v\,\mathrm{d}u\\ &\leqslant M_2M_3\int_s^t H^2(u,s)\,\mathrm{d}u\\ &\leqslant M_2M_3K\int_s^t B^2(u,s)\,\mathrm{d}u.\end{aligned}$$

This completes the proof. □

Application of Theorem 2.4. Obviously, we long to replace the unknown function H by the known function B. Consider (15) with variation of parameters formula

$$y(t)=a(t)-\int_0^t H(t,s)a(s)\,\mathrm{d}s$$

which we compare with

$$Y(t)=a(t)-\int_0^t B(t,s)a(s)\,\mathrm{d}s.$$

If we square $y-Y$, use the Schwarz inequality taking $\sqrt{|a(t)|}\sqrt{|a(t)|}$, integrate both sides from 0 to t, interchange order of integration, then we conclude that $a\in L^1[0,\infty)$ implies $y-Y\in L^2[0,\infty)$. Even on a finite interval we have readily measurable errors of $y-Y$. Without the squaring or the second integration, we easily show that if $a\in L^1[0,\infty)$, if $B-H$ is bounded, and if $\int_0^T |B(t,s)-H(t,s)|\,\mathrm{d}s\to 0$ as $t\to\infty$ for fixed $T>0$, then $y(t)\to Y(t)$ pointwise as $t\to\infty$.

3. The derived equation

Volterra introduced the idea of heredity in differential and integral equations with the simple observation that the solution is inheriting its past through $\int_0^t C(t,s)x(s)\,\mathrm{d}s$. But in the kernels Volterra considered is the idea that x is not inheriting its past but, rather, it is forgetting its past.

Inheritance becomes more prominent with passing time. The infant has no resemblance to the parent, but the family connection is clear by age 20, and, finally, at age 40 the child "becomes" the parent.

Consider the pair of equations

$$x(t) = a(t) - \int_0^t \ln[\mathrm{e} + t - s]x(s)\,\mathrm{d}s$$

and

$$y(t) = a(t) - \int_0^t [1 + t - s]^{-3/4}x(s)\,\mathrm{d}s.$$

The first has a growing memory, the second a fading memory. The second already satisfies (1), while the first will when we write

$$R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)\,\mathrm{d}u$$

and then

$$R_t(t,s) = C_t(t,s) - C(t,t)R(t,s) - \int_s^t C_t(t,u)R(u,s)\,\mathrm{d}u. \tag{17}$$

Replacing C with C_t we now see (1) is satisfied, and the Liapunov functional is

$$V(t) = R^2(t,s) + \int_s^t C_{tv}(t,v)\Big(\int_v^t R(u,s)\,\mathrm{d}u\Big)^2\,\mathrm{d}v + C_t(t,s)\Big(\int_s^t R(u,s)\,\mathrm{d}u\Big)^2.$$

In the proof of Theorem 3.1 we will see how to conclude that

$$V'(t) \leqslant C_t^2(t,s) - R^2(t,s).$$

We have $\int_s^t R^2(u,s)\,\mathrm{d}u$ bounded and, in fact, R_t and R_s bounded so that $R(t,s) \to 0$ as $t \to \infty$ and $\int_0^T R(t,s)\,\mathrm{d}s \to 0$ as $t \to \infty$. This means we have totally left the Strauss format and are dealing with a very large class of kernels.

While the perturbed equation

$$x(t) = a(t) - \int_0^t C(t,s)[x(s) + G(s,x(s))]\,\mathrm{d}s$$

seems totally beyond help with the perturbed term $\int_0^t C(t,s)G(s,x(s))\,\mathrm{d}s$, when we write

$$x(t) = y(t) - \int_0^t R(t,s)G(s,x(s))\,\mathrm{d}s,$$

then everything is tamed and we see a totally reasonable problem. We have $\int_s^t R^2(u,s)\,\mathrm{d}u$ bounded, $R(t,s) \to 0$ as $t \to \infty$, and $\int_0^T R(t,s)\,\mathrm{d}s \to 0$ for every $T > 0$.

Growing memories are as fundamental and tractable as are fading memories. In fact, rapidly growing kernels are frequently found in applied problems. A very strong example is seen in the work of Reynolds [7] on the bending of viscoelastic rods. Classical requirements of L^1 kernels are not at all necessary.

But what can we say of stability? Do we dare make errors in measurements?

Notice that we have

$$R(t,s) = \ln[\mathrm{e}+t-s] - \int_s^t \ln[\mathrm{e}+t-u]R(u,s)\,\mathrm{d}u,$$

$$R_t = [\mathrm{e}+t-s]^{-1} - R(t,s) - \int_s^t [\mathrm{e}+t-u]^{-1}R(u,s)\,\mathrm{d}u.$$

We can get R_t and R_s bounded, as well as $\int_0^T |R(t,s)|\,\mathrm{d}s \to 0$ as $t \to \infty$ for each fixed $T > 0$.

Moreover, an integrable function, $D(t,s)$, can be added to the kernel without disturbing the boundedness properties.

But we are now dealing with differential equations rather than integral equations. Can we still add two Liapunov functionals together? We can. And in this case it is surprising. The error term, $D(t,s)$, can actually be as large as the main term, $C(t,s)$.

We proceed in a manner parallel to our above discussion with R_t using C_t and all its derivatives satisfying (1). If there is an $\alpha > 0$ with

$$2D(t,t) - \int_s^t |D_t(t,s)|\,\mathrm{d}s - \int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u \geqslant \alpha \tag{18}$$

then by using the Liapunov functional

$$W(t) = R^2(t,s) + \int_s^t \int_{t-v}^\infty |D_t(u+v,v)|\,\mathrm{d}u R^2(v,s)\,\mathrm{d}v$$

on the differential equation

$$R_t(t,s) = D_t(t,s) - D(t,t)R(t,s) - \int_s^t D_t(t,u)R(u,s)\,\mathrm{d}u$$

we will obtain $W'(t) \leqslant -AR^2(t,s) + KD_t^2(t,s)$ for positive constants A and K.

One may note that Miller [6] begins Chapter 6 by stating that if the integral equation can be differentiated then we can apply Liapunov's direct method to it. In fact, we have illustrated that the direct method works very well in the integral equation itself. Thus, by our differentiation we are not demanding anything particularly unusual. Moreover, conditions (1) extend naturally to it.

We now show precisely how these two problems fit together. Thus, we are considering again (6) which we differentiate and obtain

$$H_t(t,s) = [C_t(t,s) + D_t(t,s)] - [C(t,t) + D(t,t)]H(t,s) - \int_s^t [C_t(t,u) + D_t(t,u)]H(u,s)\,\mathrm{d}u. \tag{19}$$

Theorem 3.1. *Let*

$$C_t(t,s) \geqslant 0, \quad C_{ts}(t,s) \geqslant 0, \quad C_{tst}(t,s) \leqslant 0, \quad C_{tt}(t,s) \leqslant 0 \tag{20}$$

and let $\mu > 0$ *satisfy*

$$\int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u + \int_s^t |D_t(t,u)|\,\mathrm{d}u - 2C(t,t) - 2D(t,t) \leqslant -\mu. \tag{21}$$

Then there are positive constants J *and* K *with*

$$\begin{aligned} &H^2(t,s) + J\int_s^t H^2(u,s)\,\mathrm{d}u \\ &\leqslant [C(s,s) + D(s,s)]^2 + K\int_s^t [C_t^2(u,s) + D_t^2(u,s)]\,\mathrm{d}u =: Q(t,s). \end{aligned} \tag{22}$$

If, in addition,

$$\begin{gathered} Q(t,s), \quad C_t(t,s) + D_t(t,s), \quad C(t,t) + D(t,t), \\ \int_s^t [C^2(t,u) + D^2(t,u)]\,\mathrm{d}u, \quad \int_s^t [C_t^2(t,u) + D_t^2(t,u)]\,\mathrm{d}u \end{gathered} \tag{23}$$

are all bounded for fixed $s \geqslant 0$, *then*

$$\lim_{t\to\infty} H(t,s) = 0. \tag{24}$$

If, in addition,

$$C_s(t,s) + D_s(t,s), \quad C(t,s) + D(t,s), \quad \int_s^t |B_s(u,s)|\,\mathrm{d}u \tag{25}$$

are bounded, then for each $T > 0$ *we have* $\int_0^T |H(t,s)|\,\mathrm{d}s \to 0$ *as* $t \to \infty$.

P r o o f. We define a Liapunov functional for (19) of the form

$$\begin{aligned} V(t) = H^2(t,s) &+ \int_s^t \int_{t-v}^\infty |D_t(u+v,v)|\,\mathrm{d}u H^2(v,s)\,\mathrm{d}v \\ &+ C_t(t,s)\Big(\int_s^t H(u,s)\,\mathrm{d}u\Big)^2 + \int_s^t C_{tv}(t,v)\Big(\int_v^t H(u,s)\,\mathrm{d}u\Big)^2\,\mathrm{d}v. \end{aligned}$$

Taking into account that $C_{tt}(t,s) \leqslant 0$ and $C_{tvt}(t,s) \leqslant 0$ we have

$$\begin{aligned}
V'(t) \leqslant \int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u H^2(t,s) - \int_s^t |D_t(t,v)|H^2(v,s)\,\mathrm{d}v \\
+ 2H(t,s)C_t(t,s)\int_s^t H(u,s)\,\mathrm{d}u + 2H(t,s)[C_t(t,s)+D_t(t,s)] \\
- 2[C(t,t)+D(t,t)]H^2(t,s) \\
- 2H(t,s)\int_s^t [C_t(t,u)+D_t(t,u)]H(u,s)\,\mathrm{d}u \\
+ 2H(t,s)\int_s^t C_{tv}(t,v)\int_v^t H(u,s)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Integration by parts of the last term yields

$$\begin{aligned}
&= 2H(t,s)\left[C_t(t,v)\int_v^t H(u,s)\,\mathrm{d}u\Big|_s^t + \int_s^t C_t(t,v)H(v,s)\,\mathrm{d}v\right] \\
&= 2H(t,s)\left[-C_t(t,s)\int_s^t H(u,s)\,\mathrm{d}u + \int_s^t C_t(t,v)H(v,s)\,\mathrm{d}v\right].
\end{aligned}$$

We now have

$$\begin{aligned}
V'(t) &\leqslant \int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u H^2(t,s) - \int_s^t |D_t(t,v)|H^2(v,s)\,\mathrm{d}v \\
&\quad + 2H(t,s)[D_t(t,s)+C_t(t,s)] - 2[C(t,t)+D(t,t)]H^2(t,s) \\
&\quad + \int_s^t |D_t(t,u)|(H^2(t,s)+H^2(u,s))\,\mathrm{d}u \\
&= \left[\int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u + \int_s^t |D_t(t,u)|\,\mathrm{d}u - 2C(t,t) - 2D(t,t)\right]H^2(t,s) \\
&\quad + 2H(t,s)[D_t(t,s)+C_t(t,s)] \\
&\leqslant -\mu H^2(t,s) + 2H(t,s)[D_t(t,s)+C_t(t,s)] \\
&\leqslant K[D_t^2(t,s)+C_t^2(t,s)] - JH^2(t,s)
\end{aligned}$$

for some positive numbers K and J. If we integrate V' and take into account that $V(s) = H^2(s,s) = [D(s,s)+C(s,s)]^2$ then we obtain

$$\begin{aligned}
H^2(t,s) \leqslant V(t) \leqslant [D(s,s)+C(s,s)]^2 \\
+ K\int_s^t [D_t^2(u,s)+C_t^2(u,s)]\,\mathrm{d}u - J\int_s^t H^2(u,s)\,\mathrm{d}u.
\end{aligned}$$

This establishes (22), while the remainder of the conclusions follow just as in the proof of Theorem 2.1. □

We observe with surprise that the derivative conditions (1) produce the same property of R as do the integral conditions of (7). With similar surprise we observe that the growing memory kernel $\ln[\mathrm{e} + t - s]$ produces the same properties in R that the fading memory kernel $[1 + t - s]^{-3/4}$ does. It is also a surprise to see that $\int_s^t [C(t,u) + D(t,u)]H(u,s)\,\mathrm{d}u$ produces both an L^2 and an asymptotic pointwise copy of $C(t,s)+D(t,s)$ since $H(t,s) \to 0$ as $t \to \infty$ for fixed s. Under the conditions of Theorem 2.1, $\int_0^t H(t,s)a(s)\,\mathrm{d}s$ produces an L^2 and an asymptotic pointwise copy of every $a \in L^2$. Under the conditions of Theorem 3.1, $\int_0^t H(t,s)a(s)\,\mathrm{d}s$ produces an L^2 and an asymptotic pointwise copy of every a with $a' \in L^2$.

These results are all about non-convolution problems, but we can hardly resist the idea that when we say "The convolution of two L^1 functions, f and g, is an L^1 function," that convolution may, in fact, be pointwise asymptotic to either f or g and that can be an exceedingly useful idea.

4. First order Liapunov functionals

In this section we take $C(t,s) = 0$ and are concerned with the new resolvent

$$Q(t,s) = D(t,s) - \int_s^t D(t,u)Q(u,s)\,\mathrm{d}u \tag{26}$$

where D may be so large that (7) fails, possibly owing to an additive constant. Thus, we turn to

$$Q_t(t,s) = D_t(t,s) - D(t,t)Q(t,s) - \int_s^t D_t(t,u)Q(u,s)\,\mathrm{d}u \tag{27}$$

with a view to studying

$$z(t) = a(t) - \int_0^t D(t,s)[z(s) + G(s,z(s))]\,\mathrm{d}s \tag{28}$$

and

$$z(t) = w(t) - \int_0^t Q(t,s)G(s,z(s))\,\mathrm{d}s \tag{29}$$

where

$$w(t) = a(t) - \int_0^t D(t,s)w(s)\,\mathrm{d}s. \tag{30}$$

Theorem 4.1. *Suppose there are positive constants* β *and* μ *with*

$$-\int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u + D(t,t) \geqslant \beta \tag{31}$$

and

$$D(s,s) + \int_s^t |D_t(u,s)|\,\mathrm{d}u < \mu. \tag{32}$$

If Q *solves* (27) *then*

$$|Q(t,s)| + \beta\int_s^t |Q(u,s)|\,\mathrm{d}u \leqslant \mu \tag{33}$$

for $0 \leqslant s \leqslant t < \infty$.

P r o o f. Let

$$V(t) = |Q(t,s)| + \int_s^t \int_{t-v}^\infty |D_t(u+v,v)|\,\mathrm{d}u|Q(v,s)|\,\mathrm{d}v$$

so that

$$\begin{aligned} V'(t) &\leqslant |D_t(t,s)| - D(t,t)|Q(t,s)| + \int_s^t |D_t(t,u)Q(u,s)|\,\mathrm{d}u \\ &\quad + \int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u|Q(t,s)| - \int_s^t |D_t(t,v)Q(v,s)|\,\mathrm{d}v \\ &\leqslant |D_t(t,s)| - \left[D(t,t) - \int_0^\infty |D_t(u+t,t)|\,\mathrm{d}u\right]|Q(t,s)|, \end{aligned}$$

so with $V(s) = |Q(s,s)| = |D(s,s)|$ we have

$$\begin{aligned} |Q(t,s)| \leqslant V(t) &\leqslant V(s) + \int_s^t |D_t(u,s)|\,\mathrm{d}u - \beta\int_s^t |Q(u,s)|\,\mathrm{d}u \\ &\leqslant |D(s,s)| + \int_s^t |D_t(u,s)|\,\mathrm{d}u - \beta\int_s^t |Q(u,s)|\,\mathrm{d}u, \end{aligned}$$

as required. □

We would now like to say that $Q(t,s) \to 0$ as $t \to \infty$ for fixed s. Since $|Q(t,s)| \leqslant \mu$ we have

$$|Q_t(t,s)| \leqslant |D_t(t,s)| + \mu|D(t,t)| + \|D_t\|(\mu/\beta)$$

where $\|D_t\|$ denotes the supremum. Notice that (31) implies that $D(t,t) \geqslant 0$, while (32) implies that $|D(t,t)|$ is bounded.

Corollary 4.1. *If* (31) *and* (32) *hold and if* $\|D_t\|$ *is bounded then* $Q(t,s)\to 0$ *as* $t\to\infty$ *for fixed* s.

Next, we could write (26) instead as

$$Q(t,s) = D(t,s) - \int_s^t Q(t,u)D(u,s)\,\mathrm{d}u$$

and then

$$Q_s(t,s) = D_s(t,s) + Q(t,s)D(s,s) - \int_s^t Q(t,u)D_s(u,s)\,\mathrm{d}u,$$

so

$$|Q_s(t,s)| \leqslant |D_s(t,s)| + \mu^2 + \mu\int_s^t |D_s(u,s)|\,\mathrm{d}u.$$

Corollary 4.2. *If* (31) *and* (32) *hold and if there is an* $M>0$ *with*

$$|D_s(t,s)| + \int_s^t |D_s(u,s)|\,\mathrm{d}u \leqslant M \tag{34}$$

then $Q_s(t,s)$ *is bounded and the conditions of Proposition* 2.1 *hold so that for each* $T>0$ *we have* $\lim\limits_{t\to\infty}\int_0^T |Q(t,s)|\,\mathrm{d}s = 0$.

That result is, indeed, useful in the study of (29), as is seen in Theorem 2.2. But there are further results in the same spirit.

Theorem 4.2. *Suppose that* ψ *is continuous,* $\psi\geqslant 0$, $\psi\in L^1[0,\infty)$, *and that there is a* $\gamma>0$ *with* $\int_0^\infty |Q(u+t,t)|\,\mathrm{d}u\leqslant\gamma$. *Choose* $\Gamma>\gamma$. *Then there is a function* $w\in L^1[0,\infty)$ *with* $\int_0^t |Q(t,s)|\psi(s)\,\mathrm{d}s\leqslant\Gamma w(t)$.

Proof. Consider the equation

$$w(t) = \psi(t) + \frac{1}{\Gamma}\int_0^t |Q(t,s)|w(s)\,\mathrm{d}s.$$

We conclude that $w(t)\geqslant 0$ by considering the set of equations

$$q_n(t) = \varepsilon_n + \psi(t) + \frac{1}{\Gamma}\int_0^t |Q(t,s)|q_n(s)\,\mathrm{d}s$$

where $\varepsilon_n\downarrow 0$. It is readily argued that the $q_n(t)$ are all positive and that on any interval $[0,L]$ a subsequence converges uniformly to $w(t)$. Define

$$V(t) = \frac{1}{\Gamma}\int_0^t\int_{t-s}^\infty |Q(u+s,s)|\,\mathrm{d}u w(s)\,\mathrm{d}s$$

so that

$$\begin{aligned}V'(t) &= \frac{1}{\Gamma}\int_0^\infty |Q(u+t,t)|\,\mathrm{d}u w(t) - (1/\Gamma)\int_0^t |Q(t,s)w(s)|\,\mathrm{d}s \\ &\leqslant (1/\Gamma)\int_0^\infty |Q(u+t,t)|\,\mathrm{d}u w(t) + \psi(t) - w(t) \\ &\leqslant -\lambda w(t) + \psi(t)\end{aligned}$$

for some $\lambda > 0$ so that $\int_0^\infty w(t)\,\mathrm{d}t < \infty$. Moreover,

$$w(t) \geqslant w(t) - \psi(t) = \frac{1}{\Gamma}\int_0^t |Q(t,s)|w(s)\,\mathrm{d}s$$

or

$$\int_0^t |Q(t,s)|\psi(s)\,\mathrm{d}s \leqslant \int_0^t |Q(t,s)|w(s)\,\mathrm{d}s \leqslant \Gamma w(t).$$

□

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Author's address: *T. A. Burton*, Northwest Research Institute, 732 Caroline St., Port Angeles, Washington 98362, USA and University of Memphis, Memphis, TN, USA, e-mail: `taburton@olypen.com`; *D. P. Dwiggins*, University of Memphis, Memphis, TN, USA, e-mail: `ddwiggns@memphis.edu`.