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OPERATORS APPROXIMATING PARTIAL DERIVATIVES AT VERTICES OF TRIANGULATIONS BY AVERAGING

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Abstract. Let $\mathcal{T}_h$ be a triangulation of a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, $\mathcal{L}_h$ the space of the functions from $C(\overline{\Omega})$ linear on the triangles from $\mathcal{T}_h$ and Π_h the interpolation operator from $C(\overline{\Omega})$ to $\mathcal{L}_h$. For a unit vector z and an inner vertex a of $\mathcal{T}_h$, we describe the set of vectors of coefficients such that the related linear combinations of the constant derivatives $\partial\Pi_h(u)/\partial z$ on the triangles surrounding a are equal to $\partial u/\partial z(a)$ for all polynomials u of the total degree less than or equal to two. Then we prove that, generally, the values of the so-called recovery operators approximating the gradient $\nabla u(a)$ cannot be expressed as linear combinations of the constant gradients $\nabla\Pi_h(u)$ on the triangles surrounding a.

Keywords: partial derivative, high-order approximation, recovery operator

MSC 2010: 65D25

1. Introduction

Let us assume that the values of a function u defined on a domain $\Omega \subset \mathbb{R}^2$ are known at the vertices of a triangulation $\mathcal{T}_h$ of Ω only and approximate this function by the interpolant $\Pi_h(u) \in \mathcal{L}_h$ at the vertices of $\mathcal{T}_h$. The paper is devoted to the following classical problem treated in Zienkiewicz, Cheung [8], Vacek [6] and in many other papers: Let z be a unit vector and u a function whose derivative $\partial u/\partial z$ exists on $\overline{\Omega}$. Find a vector f of coefficients such that the weighted averaging

$$\mathrm{A}_h[\Pi_h(u)](z,a,f) = f_1\partial\Pi_h(u)|_{T_1}/\partial z + \ldots + f_n\partial\Pi_h(u)|_{T_n}/\partial z$$

of the constant derivatives on the triangles $T_1,\ldots,T_n$ from $\mathcal{T}_h$ meeting a approximates $\partial u/\partial z(a)$ with an error of the size $O(h^2)$ for all u smooth enough. Originally, the problem was formulated with the aim to find a second-order approximation of the small strain tensor in the postprocessing of the elasticity problem. Today, the interest concentrates upon the linear extensions of the second-order approximations

$$\mathrm{A}_h[\Pi_h(u)](a) = (\mathrm{A}_h[\Pi_h(u)]([1,0],a,f_x), \mathrm{A}_h[\Pi_h(u)]([0,1],a,f_y)) \tag{1}$$

of the gradients $\nabla u(a)$ from the vertices to the triangles of $\mathcal{T}_h$. Especially, operators $G_h\colon \mathcal{L}_h \longrightarrow \mathcal{L}_h \times \mathcal{L}_h$ with the properties

(R1) (Consistency)

$$G_h[\Pi_h(u)] = \nabla u \quad \text{for all quadratic polynomials } u,$$

(R2) (Localization) for every triangle $T \in \mathcal{T}_h$ and for every $u \in \mathcal{L}_h$, the values of $G_h[u]$ on T depend on the values of u in the triangles from a local neighbourhood $\Omega_h(T)$ of T only,

(R3) (Linearity and boundedness) the operator G_h is linear and there exists a constant C independent of h such that

$$\|G_h[u]\|_{0,\infty,T} \leqslant C|u|_{1,\infty,\Omega_h(T)} \quad \forall T \in \mathcal{T}_h,\ u \in \mathcal{L}_h,$$

called *recovery operators* in Ainsworth, Craig [1], provide a general tool for the construction of a posteriori error estimators of solutions of the boundary value problems for second-order partial diferential equations in the plane. Constructions and applications of these estimators are summarized in Ainsworth, Oden [2]. The Zienkiewicz, Zhu operator from [9] and the Zhang, Naga operator from [7] are based on approximations by the least squares method. The former has been applied very successfully. It is a recovery operator on triangulations closely related to the homogeneous triangulations only. The accuracy of the latter is of second order. It is comparable with the accuracy of the averaging operator of the form (1) presented and proved to be a recovery operator on an extensive class of triangulations in Dalík [3]. Another second-order averaging operator has been presented in Hlaváček, Křížek, Pištora [5].

We study the property of consistency. First, we describe the vectors f such that

$$\mathrm{A}_h[\Pi_h(u)](z,a,f) = \partial u/\partial z(a) \quad \text{for all quadratic polynomials } u$$

for a unit vector z and an inner vertex a of a triangulation $\mathcal{T}_h$. Secondly, we show that any ring $r = (b^1,\dots,b^5)$ around an inner vertex a such that the value $\mathrm{G}_h[\Pi_h(u)](a)$ of a consistent operator is a linear combination of the constant gradients $\nabla\Pi_h(u)|_{T_1},\dots,\nabla\Pi_h(u)|_{T_5}$ has to satisfy certain symmetry conditions.

2. Lagrange six-tuples of vertices of triagulations

Let (x_1, x_2) be the cartesian coordinates of a point x (in $\mathbb{R}^2$). For different points a, b we denote by ab the straight line passing through a, b and by $d(x, ab)$ the distance of the point x from ab. Let us set

$$D(a,b,c) = \begin{vmatrix} a_1 - c_1 & a_2 - c_2 \\ b_1 - c_1 & b_2 - c_2 \end{vmatrix} \quad \text{for arbitrary points} \quad a, b, c,$$

reserving the symbol $\Pi_T(u)$ for the linear interpolant of the function u at the vertices of a triangle T and $\mathcal{P}^2$ for the space of (real) polynomials

$$P(x) = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 (x_1)^2 + \alpha_5 x_1 x_2 + \alpha_6 (x_2)^2$$

with (real) coefficients $\alpha_1, \dots, \alpha_6$. The points $c^1, \dots, c^6$ are called a *Lagrange six-tuple* when for arbitrary real $p_1, \dots, p_6$ there exists a unique $P \in \mathcal{P}^2$ such that

$$P(c^i) = p_i \quad \text{for } i = 1, \dots, 6.$$

It is easy to see that for arbitrary points $c^1, \dots, c^6$, the polynomial

$$\begin{aligned} l_1(x) = l(x, c^2, \dots, c^6) &= D(x, c^5, c^6) D(x, c^2, c^3) D(c^4, c^5, c^3) D(c^4, c^6, c^2) \\ &+ D(x, c^3, c^5) D(x, c^6, c^2) D(c^4, c^5, c^6) D(c^4, c^2, c^3) \end{aligned}$$

belongs to $\mathcal{P}^2$ and $l_1(c^j) = 0$ for $j = 2, \dots, 6$. Adopting the convention that "+" *and* "−" *on the set* $\{1, \dots, k\}$ *of indices mean addition and subtraction modulo* k, we set

$$l_i(x) = l(x, c^{i+1}, \dots, c^{i+5}) \quad \text{for } i = 2, \dots, 6.$$

Then, due to [4], $c^1, \dots, c^6$ is a Lagrange six-tuple whenever $l_1(c^1) \neq 0$ and we have

$$l_i(c^i) = (-1)^{i-1} l_1(c^1) \ \ \forall i, \qquad l_i(c^j) = 0 \ \ \forall j \neq i.$$

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. A non-empty finite set $\mathcal{T}$ of triangles is called a *triangulation* of Ω when $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} T$ and the intersection of any two different triangles T_1, T_2 from $\mathcal{T}$ is either a common side of T_1, T_2 or a common vertex of T_1, T_2 or an empty set. We denote by h the length of the longest side of the triangles from $\mathcal{T}$ and write $\mathcal{T}_h$ instead of $\mathcal{T}$. A vertex of a triangle from $\mathcal{T}_h$ situated in Ω is called an *inner vertex* (of $\mathcal{T}_h$).

Definition 1. An ordered n-tuple $r = (b^1, \ldots, b^n)$ of vertices of a triangulation $\mathcal{T}_h$ is called a *ring* around an inner vertex a whenever the *related* triangles $T_1 = \overline{ab^n b^1}, \ldots, T_n = \overline{ab^{n-1}b^n}$ are equally oriented and belong to $\mathcal{T}_h$. We set $\delta^i = \overrightarrow{ab^i}$ for $i = 1, \ldots, n$.

If z is a unit vector and $r = (b^1, \ldots, b^n)$ a ring around an inner vertex a then we denote by $\mathcal{F}_z(r)$ the set of vectors f such that

$$(2) \qquad f_1 \frac{\partial \Pi_1(u)}{\partial z} + \ldots + f_n \frac{\partial \Pi_n(u)}{\partial z} = \frac{\partial u}{\partial z}(a) \quad \forall u \in \mathcal{P}^2$$

for the constant derivatives of the interpolants $\Pi_i(u) = \Pi_{T_i}(u)$.

We describe the sets $\mathcal{F}_z(r)$ under the assumption that *for every ring* $r = (b^1, \ldots, b^n)$ *there exist indices* $1 = i_1 < i_2 < \ldots < i_5 \leqslant n$ *such that* $c^1 = b^{i_1}, \ldots, c^5 = b^{i_5}, c^6 = a$ *is a Lagrange six-tuple.*

Due to the notation introduced above, the functions $L_i(x) = l_i(x)/l_i(c^i)$, $i = 1, \ldots, 6$, form a Lagrange basis in $\mathcal{P}^2$ for every Lagrange six-tuple $c^1, \ldots, c^5, c^6 = a$. Then we call

$$L(x) = \sum_{i=1}^{6} u(c^i) L_i(x)$$

the (quadratic) *Lagrange interpolation polynomial* of a function $u \in C(\overline{\Omega})$ at the nodes $c^1, \ldots, c^6$. An extensive class of triangulations such that every ring $(b^1, \ldots, b^n)$ with $n \geqslant 5$ satisfies the above assumption has been studied in [3].

3. Description of the sets $\mathcal{F}_z(r)$

For a unit vector $z = [z_1, z_2]^\top$ and a ring $(b^1, \ldots, b^n)$ around an inner vertex a we set $z^\perp = [-z_2, z_1]^\top$ and $\varphi_i = \langle \delta^i, z \rangle$, $\zeta_i = \langle \delta^i, z^\perp \rangle$ for $i = 1, \ldots, n$. Of course, (φ_i, ζ_i) are the coordinates of the point b^i in the Cartesian coordinate system with origin a and axes φ, ζ in the directions of vectors z, $z^\perp$. The identity

$$(3) \qquad \partial D(x, a, b)/\partial z = z_1(a_2 - b_2) - z_2(a_1 - b_1) = \langle \overrightarrow{ba}, z^\perp \rangle$$

is valid for all points $a \neq b$ and all unit vectors z, obviously.

We find a formula for the constant z-derivative of the interpolant $\Pi_i(u)$ and express the vectors from $\mathcal{F}_z(r)$ as solutions of certain systems of linear equations.

Lemma 1. *For a unit vector z, a triangle $T = \overline{abc}$ and a function $u \in C(T)$,*

$$\partial \Pi_T(u)/\partial z = \frac{1}{D(a,b,c)} \langle u(a)\overrightarrow{cb} + u(b)\overrightarrow{ac} + u(c)\overrightarrow{ba}, z^{\perp} \rangle.$$

P r o o f. Using (3), we obtain Lemma 1 by the identity

$$\Pi_T(u)(x) = \frac{u(a)D(x,b,c) + u(b)D(x,c,a) + u(c)D(x,a,b)}{D(a,b,c)}.$$

□

In Theorem 1, we find a matrix form of the condition (2).

Definition 2. Let z be a unit vector, $(b^1, \dots, b^n)$ a ring around an inner vertex a of $\mathcal{T}_h$ and $L_1, \dots, L_6$ the Lagrange basis in $\mathcal{P}^2$ at the nodes $c^1, \dots, c^6$. Then we set $L_{ji} = L_j(b^i)$ for $j = 1, \dots, 6$, $i = 1, \dots, n$,

$$L = \begin{bmatrix} L_{1n} & L_{11} & \dots & L_{1,n-1} \\ L_{2n} & L_{21} & \dots & L_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ L_{5n} & L_{51} & \dots & L_{5,n-1} \end{bmatrix}, \quad E = \begin{bmatrix} \zeta_1 & 0 & \dots & 0 & -\zeta_{n-1} \\ -\zeta_n & \zeta_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \zeta_{n-1} & 0 \\ 0 & 0 & \dots & -\zeta_{n-2} & \zeta_n \end{bmatrix},$$

$d^1 = [\partial L_1/\partial z, \dots, \partial L_5/\partial z]^{\top}(a)$ and relate the vector $\tilde{f} = [\tilde{f}_1, \dots, \tilde{f}_n]^{\top}$ to a given $f = [f_1, \dots, f_n]^{\top}$ by putting $\tilde{f}_i = f_i/D_i$ and $D_i = D(a, b^{i-1}, b^i)$ for $i = 1, \dots, n$.

Theorem 1. *Let z be a unit vector, $r = (b^1, \dots, b^n)$ a ring around an inner vertex a and $L_1, \dots, L_6$ the Lagrange basis at the nodes $c^1, \dots, c^6 = a$. Then $f \in \mathcal{F}_z(r)$ if and only if*

$$LE\tilde{f} = d^1. \tag{4}$$

P r o o f. Because of linearity, (2) is valid for all $P \in \mathcal{P}^2$ if and only if (2) is valid for $P = L_1, \dots, L_6$. By means of Lemma 1, we obtain

$$\frac{1}{D_i}\left[\zeta_i(L_{j,i-1} - L_j(a)) - \zeta_{i-1}(L_{ji} - L_j(a))\right] = \partial \Pi_i(L_j)/\partial z$$

for $i = 1, \dots, n$, $j = 1, \dots, 6$. This and (2), applied to $P = L_j$, give

$$\sum_{i=1}^{n} \tilde{f}_i \left[\zeta_i(L_{j,i-1} - L_j(a)) - \zeta_{i-1}(L_{ji} - L_j(a))\right] = \partial L_j/\partial z(a) \tag{5}$$

for $j = 1, \dots, 6$. Since $L_1 + \dots + L_6 = 1$ and $L_1(a) = \dots = L_5(a) = 0$, we can see that the equation (5) for $j = 6$ is a linear combination of the other equations. If we omit this equation from (5) and substitute zeros for $L_1(a), \dots, L_5(a)$, we get the system (4). □

We find the set of solutions f of the system (4) in two steps. In the first step, see Lemma 2, we characterize the set

$$\mathcal{W} = \mathcal{W}_z(r) = \{w \in \mathbb{R}^n \mid Lw = d^1\}.$$

Definition 3. For a ring $(b^1, \ldots, b^n)$ around an inner vertex a, let us set $\varrho = \{i_1 + 1, \ldots, i_5 + 1\}$ and $\varrho^c = \{1, \ldots, n\} - \varrho$.

Lemma 2. *If z is a unit vector and $r = (b^1, \ldots, b^n)$ a ring around an inner vertex a then the following statements* (a), (b) *are valid:*

(a) *$w \in \mathcal{W}_z(r)$ if and only if $\langle [P(b^n), P(b^1), \ldots, P(b^{n-1})], w \rangle = \partial P/\partial z(a)$ for all $P \in \mathcal{P}^2$ such that $P(a) = 0$.*

(b) *$w \in \mathcal{W}_z(r)$ if and only if w_i is arbitrary for all $i \in \varrho^c$ and*

$$w_{i_j+1} = \partial L_j/\partial z(a) - \sum_{i \in \varrho^c} w_i L_{j,i-1} \text{ for } j = 1, \ldots, 5.$$

P r o o f. (a) We have $w \in \mathcal{W} \iff Lw = d^1$

$$\begin{aligned} &\iff \langle [P(b^n), P(b^1), \ldots, P(b^{n-1})], w \rangle = \partial P/\partial z(a) \quad \forall P \in \{L_1, \ldots, L_5\} \\ &\iff \langle [P(b^n), P(b^1), \ldots, P(b^{n-1})], w \rangle = \partial P/\partial z(a) \quad \forall P \in \mathcal{P}^2, \ P(a) = 0. \end{aligned}$$

(b) The statement is a consequence of the fact that for $j = 1, \ldots, 5$, the column $i_j + 1$ of matrix L has the form $[L_1(c^j), \ldots, L_5(c^j)]^\top$ with entry 1 in the j-th position and 0 in the other positions. $\square$

In the second step, we describe the vectors $\tilde{f} = [\tilde{f}_1, \ldots, \tilde{f}_n]^\top$ satisfying

$$E\tilde{f} = w \tag{6}$$

for some $w \in \mathcal{W}_z(r)$. We point out that, due to our convention, the additions in $j+1, j+2, \ldots$ are operations modulo n in $\{1, \ldots, n\}$.

Definition 4. Let z be a unit vector and $r = (b^1, \ldots, b^n)$ a ring around an inner vertex. Then we put $\sigma_{j,k} = \sigma_{j,k}(w) = \sum_{i=j+1}^{k} w_i \zeta_{i-1}$ for all $w \in \mathcal{W}_z(r)$ and $j, k \in \{1, \ldots, n\}$.

Lemma 3. *Let z be a unit vector and $r = (b^1, \ldots, b^n)$ a ring around an inner vertex a. Then $\sigma_{k,k}(w) = 0$ for all $w \in \mathcal{W}_z(r)$ and for $k = 1, \ldots, n$.*

P r o o f. If $w \in \mathcal{W}_z(r)$ and $k \in \{1, \ldots, n\}$ then

$$\sigma_{k,k}(w) = w_1\zeta_n + w_2\zeta_1 + \ldots + w_n\zeta_{n-1} = \big\langle [P(b^n), P(b^1), \ldots, P(b^{n-1})], w \big\rangle$$

for $P(x) = \langle \overrightarrow{ax}, z^{\perp} \rangle = z_1(x_2 - a_2) - z_2(x_1 - a_1)$. Now, due to Lemma 2 (a), we obtain $\sigma_{k,k}(w) = \partial P/\partial z(a) = -z_2 z_1 + z_1 z_2 = 0$. □

The properties of the set $\mathcal{F}_z(r)$, related to a ring $r = (b^1, \ldots, b^n)$, depend on the number of zeros among the coordinates $\zeta_1, \ldots, \zeta_n$ of $b^1, \ldots, b^n$. We choose the indices $1, \ldots, n$ so that one of the following cases (Z0)–(Z2) occurs:
(Z0) $\zeta_i \neq 0$ for all i,
(Z1) $\zeta_i = 0$ if and only if $i = 1$,
(Z2) there exists $p \in \{3, \ldots, p-1\}$ such that $\zeta_i = 0$ if and only if $i \in \{1, p\}$.

Theorem 2. *If z is a unit vector and $r = (b^1, \ldots, b^n)$ a ring around an inner vertex a then $f \in \mathcal{F}_z(r)$ if and only there is $w \in \mathcal{W}_z(r)$ such that*
- a) *$\tilde f_1$ is arbitrary and $\tilde f_i = [\zeta_n\zeta_1\tilde f_1 + \sigma_{1,i}(w)]/(\zeta_{i-1}\zeta_i)$ for $i = 2, \ldots, n$ in the case* (Z0),
- b) *$-\zeta_n\tilde f_1 + \zeta_2\tilde f_2 = w_2$ and $\tilde f_i = \sigma_{2,i}(w)/(\zeta_{i-1}\zeta_i)$ for $i = 3, \ldots, n$ in* (Z1),
- c) *$-\zeta_n\tilde f_1 + \zeta_2\tilde f_2 = w_2$, $-\zeta_{p-1}\tilde f_p + \zeta_{p+1}\tilde f_{p+1} = w_{p+1}$, $\tilde f_i = \sigma_{2,i}(w)/(\zeta_{i-1}\zeta_i)$ for $i = 3, \ldots, p-1$, $\tilde f_i = \sigma_{p+1,i}(w)/(\zeta_{i-1}\zeta_i)$ for $i = p+2, \ldots, n$ and the mutually equivalent conditions $\sigma_{2,p}(w) = 0$, $\sigma_{p+1,1}(w) = 0$ are satisfied under the condition* (Z2).

P r o o f. $f \in \mathcal{F}_z(r)$ if and only if $\tilde f$ satisfies (6) for some $w \in \mathcal{W}_z(r)$. In the case (Z0), the first equation of (6) gives $\tilde f_n = (\zeta_n\zeta_1\tilde f_1 - \sigma_{n,1})/(\zeta_{n-1}\zeta_n)$ and the following equations lead to $\tilde f_i = (\zeta_n\zeta_1\tilde f_1 + \sigma_{1,i})/(\zeta_{i-1}\zeta_i)$ for $i = 2, \ldots, n$. As $\sigma_{1,n} = -\sigma_{n,1}$ due to Lemma 3, the first formula for $\tilde f_n$ is just the case $i = n$ of the second formula. Hence a) is valid.

In the case (Z1), $\zeta_1 = 0$ means that in (6) the unknowns $\tilde f_1$, $\tilde f_2$ appear in the single equation $-\zeta_n\tilde f_1 + \zeta_2\tilde f_2 = w_2$ and the system of the other $n-1$ equations for $n-2$ unknowns is of the form

$$\begin{bmatrix} \zeta_3 & & & \\ -\zeta_2 & \zeta_4 & & \\ & \ddots & \ddots & \\ & & -\zeta_{n-2} & \zeta_n \\ & & & -\zeta_{n-1} \end{bmatrix} \begin{bmatrix} \tilde f_3 \\ \tilde f_4 \\ \vdots \\ \tilde f_{n-1} \\ \tilde f_n \end{bmatrix} = \begin{bmatrix} w_3 \\ w_4 \\ \vdots \\ w_n \\ w_1 \end{bmatrix}.$$

After elimination, the first $n-2$ equations give the values $\tilde{f}_3, \dots, \tilde{f}_n$ from b). Then $\tilde{f}_n = \sigma_{2,n}(w)/(\zeta_{n-1}\zeta_n)$, so that the last equation $-\zeta_{n-1}\tilde{f}_n = w_1$ is equivalent to $\sigma_{2,1} = 0$. But $0 = \sigma_{2,2} = \sigma_{2,1} + w_2\zeta_1 = \sigma_{2,1}$ by Lemma 3 and $\zeta_1 = 0$.

The statement c) can be proved by analogous considerations. □

4. The consistent operators

In this section we show that for every unit vector z there exist rings $r = (b^1, \dots, b^5)$ around inner vertices a such that the value of any consistent operator $G_h[\Pi_h(u)]$ at vertex a cannot be expressed in form of a linear combination of the constant gradients $\nabla\Pi_h(u)|_{T_1}, \dots, \nabla\Pi_h(u)|_{T_n}$ on the triangles from $\mathcal{T}_h$ meeting vertex a.

Let us assume that there exists a vector $f = [f_1, \dots, f_n]^\top$ such that

$$G_h[\Pi_h(u)](a) = f_1\nabla\Pi_1(u) + \ldots + f_n\nabla\Pi_n(u) \quad \forall u \in \mathcal{P}^2. \tag{7}$$

Due to consistency, (7) means that $f \in \mathcal{F}_z(r) \cap \mathcal{F}_{z^\perp}(r)$ for $z = [1, 0]$. In the following example we find neighbourhoods with the property $\mathcal{F}_z(r) \cap \mathcal{F}_{z^\perp}(r) = \emptyset$.

Example 1. For a ring $r = (b^1, \dots, b^5)$ around an inner vertex a we find conditions necessary for the existence of a vector f in $\mathcal{F}_z(r) \cap \mathcal{F}_{z^\perp}(r)$. Lemma 2 b) tells us that $\mathcal{W}_z(r)$ and $\mathcal{W}_{z^\perp}(r)$ are one-element sets with the elements $w^1 = [\partial L_5/\partial x_1(a), \dots, \partial L_4/\partial x_1(a)]^\top$ and $w^2 = [\partial L_5/\partial x_2(a), \dots, \partial L_4/\partial x_2(a)]^\top$, respectively. Then, due to Theorem 2 a), $\mathcal{F}_z(r)$ is the set of vectors f such that f_1 is arbitrary and

$$\tilde{f}_i = \Bigl(\zeta_5\zeta_1\tilde{f}_1 + \sum_{k=1}^{i-1}\frac{\partial L_k}{\partial x_1}(a)\zeta_k\Bigr)\Big/(\zeta_{i-1}\zeta_i) \quad \text{for} \quad i = 2, \dots, 5, \tag{8}$$

and $\mathcal{F}_{z^\perp}(r)$ is the set of vectors g such that g_1 is arbitrary and

$$\tilde{g}_i = \Bigl(\varphi_5\varphi_1\tilde{g}_1 - \sum_{k=1}^{i-1}\frac{\partial L_k}{\partial x_2}(a)\varphi_k\Bigr)\Big/(\varphi_{i-1}\varphi_i) \quad \text{for} \quad i = 2, \dots, 5. \tag{9}$$

It is easy to see that the formulas

$$\tilde{f}_i = \Bigl(\tilde{f}_{i-1}\zeta_{i-2} + \frac{\partial L_{i-1}}{\partial x_1}(a)\Bigr)\Big/\zeta_i \quad \text{for} \quad i = 2, \dots, 5 \tag{10}$$

and

$$\tilde{g}_i = \Bigl(\tilde{g}_{i-1}\varphi_{i-2} - \frac{\partial L_{i-1}}{\partial x_2}(a)\Bigr)\Big/\varphi_i \quad \text{for} \quad i = 2, \dots, 5 \tag{11}$$

are equivalent to (8) and (9), respectively. For $i = 2, \dots, 5$, the identities $\tilde{f}_{i-1} = \tilde{g}_{i-1}$ and $\tilde{f}_i = \tilde{g}_i$ can be shown to be equivalent to

$$\tilde{f}_{i-1} = \frac{\langle \nabla L_{i-1}(a), \delta^i \rangle}{D(b^{i-2}, b^i, a)} = \tilde{g}_{i-1} \quad \text{and} \quad \tilde{f}_i = \frac{\langle \nabla L_{i-1}(a), \delta^{i-2} \rangle}{D(b^{i-2}, b^i, a)} = \tilde{g}_i.$$

These identities lead to

$$\frac{\langle \nabla L_i(a), \delta^{i+1} \rangle}{D(b^{i-1}, b^{i+1}, a)} = \frac{\langle \nabla L_{i-1}(a), \delta^{i-2} \rangle}{D(b^{i-2}, b^i, a)} \tag{12}$$

for $i = 2, 3, 4$. If we use the explicit form of the polynomials $L_1, \dots, L_6$ from Section 2 then we can simplify the equalities (12) to the form

$$\frac{d(a, b^3b^4)}{d(b^2, b^3b^4)} = \frac{d(a, b^4b^5)}{d(b^1, b^4b^5)}, \quad \frac{d(a, b^4b^5)}{d(b^3, b^4b^5)} = \frac{d(a, b^5b^1)}{d(b^2, b^5b^1)}, \quad \frac{d(a, b^5b^1)}{d(b^4, b^5b^1)} = \frac{d(a, b^1b^2)}{d(b^3, b^1b^2)}.$$

Fig. 1 illustrates that the ring $(b^1, \dots, b^5)$ around the vertex a satisfies the first of these identities but, obviously, the second identity is invalid.

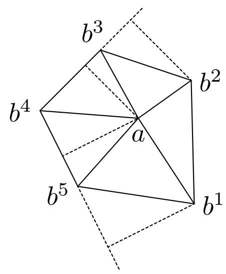

Figure 1

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