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ON THE ASYMPTOTIC BEHAVIOR AT INFINITY OF SOLUTIONS TO QUASI-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions are formulated for existence of non-oscillatory solutions to the equation

\[ y^{(n)} + \sum_{j=0}^{n-1} a_j(x)y^{(j)} + p(x)|y|^k \text{sgn } y = 0 \]

with \( n \geq 1 \), real (not necessarily natural) \( k > 1 \), and continuous functions \( p(x) \) and \( a_j(x) \) defined in a neighborhood of \( +\infty \). For this equation with positive potential \( p(x) \) a criterion is formulated for existence of non-oscillatory solutions with non-zero limit at infinity. In the case of even order, a criterion is obtained for all solutions of this equation at infinity to be oscillatory.

Sufficient conditions are obtained for existence of solution to this equation which is equivalent to a polynomial.

Keywords: quasi-linear ordinary differential equation of higher order, existence of non-oscillatory solution, oscillatory solution

MSC 2010: 34C15, 34C10

1. Introduction

Consider the differential equation

\[ y^{(n)} + \sum_{j=0}^{n-1} a_j(x)y^{(j)} + p(x)|y|^k \text{sgn } y = 0 \]

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with $n \geq 1$, real (not necessarily natural) $k > 1$ and continuous functions $p(x)$ and $a_j(x)$ defined in a neighborhood of $+\infty$.

A nontrivial solution to (1.1) is called oscillatory if it has arbitrarily large zeros.

A solution to (1.1) defined in a neighborhood of $+\infty$ is called non-oscillatory if it is ultimately one-signed.

The problem of existence of non-oscillatory solutions and of all solutions to be oscillatory was investigated in detail for equation (1.1) in the case $a_j(x) \equiv 0$, $j = 0, \ldots, n - 1$. For $n = 2$, F. Atkinson [1] proved the well-known criterion for all solutions to be oscillatory.

For more general non-linear second-order equations, theorems similar to that of F. Atkinson were obtained by S. A. Belohorec [5], I. T. Kiguradze [7], J. W. Masci and J. S. W. Wong [16], P. Waltman [21], J. S. W. Wong [22]. For third- and fourth-order non-linear equations, the oscillatory problem was investigated by I. V. Astashova [2], V. A. Kondratiev and V. S. Samovol [11], T. Kusano and M. Naito [12], D. L. Lovelady [15], V. R. Taylor, Jr. [19]. The result of F. Atkinson was generalized for the higher-order equation (1.1) in the case $a_j(x) \equiv 0$, $j = 0, \ldots, n - 1$, by I. T. Kiguradze [8]. Equations like (1.1) with some coefficients $a_j(x) \neq 0$ were investigated in [6], [10], [14]; some of these papers considered more general non-linearities.

Sufficient conditions were obtained by I. M. Sobol [18] which guarantee the existence of a solution to (1.1) with $p(x) = 0$ which is equivalent to a polynomial. I. T. Kiguradze [8] proved the same result for (1.1) with $a_j(x) \equiv 0$, $j = 0, \ldots, n - 1$.

2. Results

2.1. Oscillatory properties of solutions.

**Theorem 2.1.** Suppose the functions $p(x)$ and $a_j(x)$ in (1.1) satisfy the conditions

\begin{align}
(2.1) & \int_{x_0}^{\infty} x^{n-1} |p(x)| \, dx < \infty, \\
(2.2) & \int_{x_0}^{\infty} x^{n-j-1} |a_j(x)| \, dx < \infty, \quad j = 0, \ldots, n - 1.
\end{align}

Then for any $h \neq 0$ there exists, in a neighborhood of $+\infty$, a non-oscillatory solution $y(x)$ to (1.1) tending to $h$ as $x \to \infty$ and having derivatives satisfying the conditions

\begin{align}
(2.3) & \int_{x_0}^{\infty} x^{j-1} |y^{(j)}(x)| \, dx < \infty, \quad j = 1, \ldots, n.
\end{align}
**Theorem 2.2.** Let the function \( p(x) \) be positive and let the functions \( a_j(x) \), \( j = 0, \ldots, n-1 \), satisfy (2.2).

Then the following conditions are equivalent:

(i) \( p(x) \) satisfies (2.1),

(ii) there exists, in a neighborhood of \( +\infty \), a non-oscillatory solution to (1.1) that does not tend to 0 as \( x \to \infty \).

**Theorem 2.3.** Oscillatory criterium. Let \( n \) be even, the function \( p(x) \) positive, and let the functions \( a_j(x) \), \( j = 0, \ldots, n-1 \), satisfy (2.2).

Then the following conditions are equivalent:

(i) \( \int_0^\infty x^{n-1} p(x) \, dx = \infty \),

(ii) all solutions to (1.1) defined in a neighborhood of \( +\infty \) are oscillatory.

**Remark 1.** This theorem generalizes the results of works [1], [8]. Detailed proofs of Theorems 2.1, 2.2, 2.3 can be found in [4]. Note that Theorem 2.1 is an auxiliary result which can be also considered as a particular case of Corollary 8.2 from the monograph [9].

### 2.2. Existence of solution tending to polynomial.

**Theorem 2.4.** Suppose the functions \( p(x) \) and \( a_j(x) \) in (1.1) satisfy conditions (2.2) and

\[
(2.4) \quad \int_0^\infty x^{(n-1)(k+1)} |p(x)| \, dx < \infty.
\]

Then for any constants \( C_0, \ldots, C_{n-1} \) there exists, in a neighborhood of \( +\infty \), a non-oscillatory solution \( y(x) \) to (1.1) satisfying

\[
(2.5) \quad y(x) = \sum_{j=0}^{n-1} C_j \xi_j(x) + o(1) \quad \text{as } x \to +\infty,
\]

where \( \xi_j = x^j j!^{-1} (1 + o(1)) \) are fundamental solutions to (1.1) with \( p(x) \equiv 0 \).

**Remark 2.** Note that Theorem 1 in [8], for Equation (1.1) with \( a_j(x) \equiv 0 \) and \( p(x) \) satisfying some weaker conditions, in particular

\[
(2.6) \quad \int_0^\infty x^{(n-1)k} |p(x)| \, dx < \infty,
\]

provides existence of solutions equivalent to \( x^j \), \( j = 0, \ldots, n-1 \). However, solutions \( y(x) = \sum_{j=0}^{n-1} C_j x^j + o(1) \) with arbitrary \( C_j \) need not exist in this case.
Example. Consider the equation
\[ y'' = \frac{y^2}{\sqrt{x}}. \]

We have
\[ \int_{x_0}^{\infty} x^{(n-1)k} |p(x)| \, dx = \int_{x_0}^{\infty} x^{-3/2} \, dx < \infty. \]

So, according to [8] there exist, near \(+\infty\), solutions \( y_1(x) \sim 1 \) and \( y_2(x) \sim x \).

However, Theorem 2.4 cannot guarantee existence of a solution \( y(x) = x + 1 + o(1) \), since
\[ \int_{x_0}^{\infty} x^{(n-1)(k+1)} |p(x)| \, dx = \int_{x_0}^{\infty} x^{-1/2} \, dx = \infty. \]

Suppose such a solution exists. Then \( y(x) \sim x \), whence \( y'' \sim x^{-3/2} \) and \( y' = C_1 - 2x^{-1/2} + o(x^{-1/2}) \) with \( C_1 = 1 \) due to \( y(x) \sim x \).

So, \( y(x) = C_0 + x - 4x^{1/2} + o(x^{1/2}) \), which contradicts to \( y(x) = x + 1 + o(1) \).

Remark 3. Note that for Equation (1.1) with \( a_j(x) \neq 0 \), existence of a solution, admitting the asymptotic representation

\[ (2.7) \quad y(x) = \sum_{j=0}^{n-1} C_j x^j (1 + o(1)) \]

can be proved by using Corollary 8.2 from the monograph [9] if conditions (2.6), (2.2) are fulfilled, and \( \sum_{j=0}^{n-1} |C_j| \neq 0 \).

Properties (2.7) and (2.5) differ. For example, in the case \( n = 2 \), the solutions behaving as \( -\xi_1(x) + \xi_2(x) + o(1) \) and \( \xi_1(x) + \xi_2(x) + o(1) \), which exist by Theorem 2.4, must be different. On the contrary, the solutions behaving as \( (x + x^2)(1 + o(1)) \) and \( (-x + x^2)(1 + o(1)) \), which are particular cases of (2.7), may occur to be just the same function.

3. Proofs

Lemma 3.1. The operator
\[ L = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} a_j(x) \frac{d^j}{dx^j} \]
with all functions \( a_j(x) \) satisfying (2.2) can be represented in a neighborhood of \(+\infty\) as the \( n \)th quasi-derivative operator, i.e.

\[
L: \ y \mapsto \left( r_n \frac{d}{dx} \left( r_{n-1} \frac{d}{dx} \left( \ldots r_1 \frac{d}{dx} (r_0 y) \ldots \right) \right) \right),
\]

with positive functions \( r_0, \ldots, r_n \) all tending to 1 as \( x \to +\infty \).

By the lemma, equation (1.1) can be rewritten in a neighborhood of \(+\infty\) as

\[
y^{[n]}(x) + p(x)|y|^k \operatorname{sgn} y = 0
\]

with \( y^{[j]} \) denoting the \( j \)-th quasi-derivative of a function \( y(x) \):

\[
y^{[j]} = \left( r_j \frac{d}{dx} \left( r_{j-1} \frac{d}{dx} \left( \ldots r_1 \frac{d}{dx} (r_0 y) \ldots \right) \right) \right).
\]

Thus, \( y^{[0]}(x) = r_0(y(x) \right) \) and \( y^{[i]}(x) = r_i(x)(y^{[i-1]}(x))' \), \( i = 1, \ldots, n \).

Such a representation for linear operators is described by G. Polya [17], Ch. I. de la Vallée-Poussin [20], A. Levin [13].

Now, the coefficients of the quasi-derivative operator are constructed so that their limits, as \( x \to +\infty \), are equal to 1, which is used in the proof of Theorem 2.4. Similar representation on finite segments was obtained and used in [3].

**Lemma 3.2.** There exist fundamental solutions \( \xi_j(x) \), \( j = 0, \ldots, n - 1 \), to the equation \( y^{[n]} = 0 \) satisfying the following properties:

\[
\begin{align*}
\xi^{[i]}_j(x) &= 0 \text{ if } j < i < n, \\
\xi^{[i]}_j(x) &= 1 \text{ if } i = j, \\
\xi^{[i]}_j(x) &= \frac{x^{j-i}}{(j-i)!} (1 + o(1)) \text{ as } x \to +\infty \text{ if } i < j.
\end{align*}
\]

**Proof.** Trying to solve the equation \( y^{[n]} = 0 \), let us prove by backward induction over \( i = n - 1, \ldots, 0 \) that the \( i \)-th quasi-derivative of its general solution is

\[
y^{[i]}(x) = \sum_{j=i}^{n-1} C_j \xi_{ij}(x)
\]

with arbitrary constants \( C_j \) and functions \( \xi_{ij}(x) \), \( i \leq j < n \), such that

\[
\begin{align*}
\xi_{ii}(x) &\equiv 1, \\
\xi_{ij}(x) &= \frac{x^{j-i}}{(j-i)!} (1 + o(1)) \text{ as } x \to +\infty, \\
r_{i+1}(x)(\xi_{ij}(x))' &= \xi_{i+1,j}(x).
\end{align*}
\]
Since \( y^{[n]}(x) = r_n(x)(y^{[n-1]}(x))' = 0 \), we obtain that \( y^{[n-1]}(x) \) must be constant. This provides the first induction step.

If for some \( i > 0 \) the statement needed is proved, then due to the equality \( y^{[i]}(x) = r_i(x)(y^{[i-1]})'(x) \) we have, with some \( a \in \mathbb{R} \),

\[
y^{[i-1]}(x) = C_{i-1} + \int_a^x \frac{\sum_{j=1}^{n-1} C_j \xi_{ij}(t)}{r_i(t)} \, dt = C_{i-1} \cdot 1 + \sum_{j=i}^{n-1} C_j \int_a^x \frac{\xi_{ij}(t) \, dt}{r_i(t)} = \sum_{j=i-1}^{n-1} C_j \xi_{i-1,j}(x),
\]

where \( \xi_{i-1,i-1}(x) \equiv 1 \) and, for \( j \geq i \), \( \xi_{i-1,j}(x) = \int_a^x \xi_{ij}(t) \, dt/r_i(t) \). The last function satisfies

\[
\lim_{x \to +\infty} \frac{\xi_{i-1,j}(x)}{x^{j-(i-1)}} = \lim_{x \to +\infty} \frac{\xi_{ij}(x)}{r_i(x)(j - i + 1)x^{j-i}} = \frac{1}{(j - i + 1)(j - i)!} = \frac{1}{(j - (i - 1))!},
\]

thus completing the induction step. To prove the lemma, it remains just to put \( \xi_j(x) = \xi_{0,j}(x)/r_0(x) \) and to notice that \( \xi_j^{[i]}(x) = \xi_{ij}(x) \) if \( i \leq j \) and \( \xi_j^{[i]}(x) = 0 \) otherwise. \( \square \)

**Lemma 3.3.** Suppose \( f(x) \) is a continuous function defined in a neighborhood of \( +\infty \). Then the general solution to the equation \( y^{[n]}(x) = f(x) \) is

\[
y(x) = \sum_{j=0}^{n-1} \left( C_j + \int_a^x f(t)b_j(t)t^{n-j-1} \, dt \right) \xi_j(x)
\]

with some \( a \in \mathbb{R} \), arbitrary constants \( C_0, \ldots, C_{n-1} \), the fundamental solutions \( \xi_j(x) \) to the homogeneous equation described in Lemma 3.2, and bounded functions \( b_j(x) \) expressible in terms of the coefficients \( r_i(x) \) and the quasi-derivatives of \( \xi_i(x) \).

**Proof.** By variation of constants, the function

\[
y(x) = \sum_{j=0}^{n-1} g_j(x)\xi_j(x)
\]

is a solution to the equation considered if the functions \( g_j(x) \) satisfy the system

\[
\sum_{j=0}^{n-1} g_j'(x)\xi_j^{[i-1]}(x) = 0, \quad i = 1, \ldots, n - 1,
\]

\[
\sum_{j=0}^{n-1} g_j'(x)\xi_j^{[n-1]}(x) = \frac{f(x)}{r_n(x)}.
\]
In more detail, first we prove by induction over \( i = 0, \ldots, n - 1 \) that, due to (3.3), the quasi-derivatives of the function \( y(x) \) defined by (3.2) has the following form:

\[
y^{[i]}(x) = \sum_{j=0}^{n-1} g_j(x) \xi_j^{[i]}(x).
\]

The first step is trivial. If for some \( i < n - 1 \) the last equality is proved, then we have

\[
y^{[i+1]}(x) = r_{i+1}(x) \sum_{j=0}^{n-1} g_j'(x) \xi_j^{[i]}(x) + \sum_{j=0}^{n-1} g_j(x) r_{i+1}(x) (\xi_j^{[i]}(x))'
\]

with the first sum vanishing due to (3.3) and the second coinciding with the needed expression \( \sum_{j=0}^{n-1} g_j(x) \xi_j^{[i+1]}(x) \).

In the same way, due to (3.3) and the equation \( \xi_j^{[n]}(x) = 0 \), we have

\[
y^{[n]}(x) = r_n(x) \sum_{j=0}^{n-1} g_j'(x) \xi_j^{[n-1]}(x) + \sum_{j=0}^{n-1} g_j(x) \xi_j^{[n]}(x) = f(x).
\]

Now, let us solve system (3.3). Since \( \xi_j^{[i]}(x) = 0 \) for \( j < i < n \), the system is triangular and the derivatives \( g_j'(x) \) can be proved to have the needed form \( f(x) b_j(x) x^{n-j-1} \), step by step for \( j = n - 1, \ldots, 0 \).

We begin from the last equation of (3.3), which gives \( g_{n-1}'(x) = f(x) / r_n(x) \). Thus, we can take \( 1/r_n(x) \) as the bounded function \( b_{n-1}(x) \).

If for some \( i \geq 0 \) the needed expressions for \( g_j'(x), j > i \), are already obtained, then

\[
g_j'(x) = - \sum_{j=i+1}^{n-1} g_j'(x) \xi_j^{[i]}(x) = - \sum_{j=i+1}^{n-1} f(x) b_j(x) x^{n-j-1} \xi_j^{[i]}(x)
\]

\[
= f(x) \left( - \sum_{j=i+1}^{n-1} b_j(x) \xi_j^{[i]}(x) x^{j-i} \right) x^{n-i-1}.
\]

Since \( \xi_j^{[i]}(x) = x^{j-i}(j-i)!^{-1}(1 + o(1)) \), the last expression in the big parentheses is bounded and may be taken as \( b_{i-1}(x) \). The rest of the proof is evident.

Now we can prove Theorem 2.4.

Proof. Consider the set \( V_{ac} \) of all continuous functions \( v(x) \) defined on \([a, \infty)\) such that \( \sup \{|v(x)| x^{1-n} : x \geq a\} \leq c \). If we define the norm \( ||v(x)|| \) by the left-hand side of the last inequality, then \( V_{ac} \) becomes a Banach space.
Consider the mapping $F: V_{ac} \to V_{ac}$ such that
\[
F(v)(x) = \sum_{j=0}^{n-1} \left( C_j - \int_x^{+\infty} p(t) |v|^k (\text{sgn } v) b_j(t) t^{n-j-1} \, dt \right) \xi_j(x)
\]
with the bounded functions $b_j(x)$ participating in Lemma 3.3.

The integrals converge since their integrands are $O(|p(t)| t^K)$ with $K = (n-1)k + n - j - 1 \leq (n-1)(k + 1)$.

As for the inclusion $F(V_{ac}) \subset V_{ac}$, it holds if $a > 1$ and $n(c^k B \delta + C_{\text{max}}) \leq c$ with
\[
B = \sup \{|b_j(x)| : x \geq a, j = 0, \ldots, n - 1 \},
\]
\[
\delta = \int_a^{+\infty} |p(t)| t^{(n-1)(k+1)} \, dt,
\]
\[
C_{\text{max}} = \max \{|C_j| : j = 0, \ldots, n - 1 \}.
\]

The last inequality holds if we put $c = (n+1) C_{\text{max}}$ and choose $a$ big enough making $\delta$ sufficiently small to provide $n(n+1)^k C_{\text{max}} B \delta \leq C_{\text{max}}$. Furthermore, we can make $F$ become a contraction mapping, i.e. provide the inequality $\|F(v) - F(w)\| \leq \theta \|v - w\|$ for some $\theta < 1$ and all $v, w \in V_{ac}$.

Indeed, for $x \geq a$ and $a$ big enough we have $|\xi_j(x)| < 2x^{n-1}$ and, since $||X|^k \text{sgn } X - |Y|^k \text{sgn } Y| \leq |X - Y| \cdot k \max\{|X|, |Y|\}^{k-1}$, we have
\[
x^{1-n}|F(v)(x) - F(w)(x)| \leq 2Bn \int_x^{+\infty} |v(t) - w(t)| k \left( ct^{n-1} \right)^{k-1} |p(t)| t^{n-1} \, dt
\]
\[
\leq 2Bn k c^{k-1} \|v - w\| \int_x^{+\infty} |p(t)| t^{(n-1)(k+1)} \, dt \leq 2Bn k c^{k-1} \|v - w\| \delta.
\]

So, all we need to make $F$ a contraction mapping is to increase $a$ so that $\delta$ could become sufficiently small.

The unique fixed point of $F$, which must exist, is a solution to (3.1) having the form $y(x) = \sum_{j=0}^{n-1} C_j \xi_j(x) + \epsilon(x)$ with
\[
\epsilon(x) = -\sum_{j=0}^{n-1} \xi_j(x) \int_x^{+\infty} p(t) |y|^k (\text{sgn } y) b_j(t) t^{n-j-1} \, dt.
\]

Now we have to prove that $\epsilon(x) = o(1)$ as $x \to +\infty$. Since $y = O(x^{n-1})$, we have
\[
\epsilon(x) = O\left( \sum_{j=0}^{n-1} \xi_j(x) \int_x^{+\infty} |p(t)| t^{(n-1)(k+1)-j} \, dt \right).
\]
Further, since $|t|^{-j} \leq |x|^{-j}$ for $t \geq x \geq a > 1$, we obtain

$$\varepsilon(x) = \int_x^{+\infty} |p(t)| t^{(n-1)(k+1)} \, dt \cdot O \left( \sum_{j=0}^{n-1} \frac{\xi_j(x)}{x^j} \right) = o(1).$$

□

References

[7] Kiguradze I. T.: On conditions for oscillation of solutions of the equation $u'' + a(t)|u|^n \times \text{sign} u = 0$. Čas. Pěst. Mat. 87 (1962), 492–495. (In Russian.)
[8] Kiguradze I. T.: On the oscillation of solution of the equation $d^m/dt^m + a(t)|u|^n \times \text{sign} u = 0$. Mat. Sbornik 65 (1964), 172–187. (In Russian.)
[13] Levin A. Yu.: Nonoscillation of solutions of the equation $x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t) \times x = 0$. Usp. Mat. Nauk. 24 (1969), 43–96. (In Russian.)


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