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ON THE ASYMPTOTIC BEHAVIOR AT INFINITY OF SOLUTIONS TO QUASI-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions are formulated for existence of non-oscillatory solutions to the equation

$$y^{(n)}+\sum_{j=0}^{n-1}a_j(x)y^{(j)}+p(x)|y|^k\operatorname{sgn}y=0$$

with $n\geqslant 1$, real (not necessarily natural) $k>1$, and continuous functions $p(x)$ and $a_j(x)$ defined in a neighborhood of $+\infty$. For this equation with positive potential $p(x)$ a criterion is formulated for existence of non-oscillatory solutions with non-zero limit at infinity. In the case of even order, a criterion is obtained for all solutions of this equation at infinity to be oscillatory.

Sufficient conditions are obtained for existence of solution to this equation which is equivalent to a polynomial.

Keywords: quasi-linear ordinary differential equation of higher order, existence of non-oscillatory solution, oscillatory solution

MSC 2010: 34C15, 34C10

1. Introduction

Consider the differential equation

$$y^{(n)}+\sum_{j=0}^{n-1}a_j(x)y^{(j)}+p(x)|y|^k\operatorname{sgn}y=0 \tag{1.1}$$

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with $n \geqslant 1$, real (not necessarily natural) $k > 1$ and continuous functions $p(x)$ and $a_j(x)$ defined in a neighborhood of $+\infty$.

A nontrivial solution to (1.1) is called *oscillatory* if it has arbitrarily large zeros.

A solution to (1.1) defined in a neighborhood of $+\infty$ is called *non-oscillatory* if it is ultimately one-signed.

The problem of existence of non-oscillatory solutions and of all solutions to be oscillatory was investigated in detail for equation (1.1) in the case $a_j(x) \equiv 0$, $j = 0, \ldots, n-1$. For $n = 2$, F. Atkinson [1] proved the well-known criterion for all solutions to be oscillatory.

For more general non-linear second-order equations, theorems similar to that of F. Atkinson were obtained by S. A. Belohorec [5], I. T. Kiguradze [7], J. W. Masci and J. S. W. Wong [16], P. Waltman [21], J. S. W. Wong [22]. For third- and fourth-order non-linear equations, the oscillatory problem was investigated by I. V. Astashova [2], V. A. Kondratiev and V. S. Samovol [11], T. Kusano and M. Naito [12], D. L. Lovelady [15], V. R. Taylor, Jr. [19]. The result of F. Atkinson was generalized for the higher-order equation (1.1) in the case $a_j(x) \equiv 0$, $j = 0, \ldots, n-1$, by I. T. Kiguradze [8]. Equations like (1.1) with some coefficients $a_j(x) \neq 0$ were investigated in [6], [10], [14]; some of these papers considered more general non-linearities.

Sufficient conditions were obtained by I. M. Sobol [18] which guarantee the existence of a solution to (1.1) with $p(x) = 0$ which is equivalent to a polynomial. I. T. Kiguradze [8] proved the same result for (1.1) with $a_j(x) \equiv 0$, $j = 0, \ldots, n-1$.

2. Results

2.1. Oscillatory properties of solutions.

Theorem 2.1. *Suppose the functions $p(x)$ and $a_j(x)$ in (1.1) satisfy the conditions*

$$\int_{x_0}^{\infty} x^{n-1}\,|p(x)|\,\mathrm{d}x < \infty, \tag{2.1}$$

$$\int_{x_0}^{\infty} x^{n-j-1}\,|a_j(x)|\,\mathrm{d}x < \infty, \quad j = 0, \ldots, n-1. \tag{2.2}$$

Then for any $h \neq 0$ there exists, in a neighborhood of $+\infty$, a non-oscillatory solution $y(x)$ to (1.1) tending to h as $x \to \infty$ and having derivatives satisfying the conditions

$$\int_{x_0}^{\infty} x^{j-1}\bigl|y^{(j)}(x)\bigr|\,\mathrm{d}x < \infty, \quad j = 1, \ldots, n. \tag{2.3}$$

Theorem 2.2. *Let the function* $p(x)$ *be positive and let the functions* $a_j(x)$, $j = 0, \dots, n-1$, *satisfy* (2.2).

Then the following conditions are equivalent:

(i) $p(x)$ *satisfies* (2.1),

(ii) *there exists, in a neighborhood of* $+\infty$, *a non-oscillatory solution to* (1.1) *that does not tend to 0 as* $x \to \infty$.

Theorem 2.3. Oscillatory criterium. *Let* n *be even, the function* $p(x)$ *positive, and let the functions* $a_j(x)$, $j = 0, \dots, n-1$, *satisfy* (2.2).

Then the following conditions are equivalent:

(i)
$$\int_{x_0}^{\infty} x^{n-1} p(x) \, \mathrm{d}x = \infty,$$

(ii) *all solutions to* (1.1) *defined in a neighborhood of* $+\infty$ *are oscillatory.*

Remark 1. This theorem generalizes the results of works [1], [8]. Detailed proofs of Theorems 2.1, 2.2, 2.3 can be found in [4]. Note that Theorem 2.1 is an auxiliary result wich can be also considered as a particular case of Corollary 8.2 from the monograph [9].

2.2. Existence of solution tending to polynomial.

Theorem 2.4. *Suppose the functions* $p(x)$ *and* $a_j(x)$ *in* (1.1) *satisfy conditions* (2.2) *and*

$$\int_{x_0}^{\infty} x^{(n-1)(k+1)} |p(x)| \, \mathrm{d}x < \infty. \tag{2.4}$$

Then for any constants $C_0, \dots, C_{n-1}$ *there exists, in a neighborhood of* $+\infty$, *a non-oscillatory solution* $y(x)$ *to* (1.1) *satisfying*

$$y(x) = \sum_{j=0}^{n-1} C_j \xi_j(x) + o(1) \qquad \text{as } x \to +\infty, \tag{2.5}$$

where $\xi_j = x^j j!^{-1}(1 + o(1))$ *are fundamental solutions to* (1.1) *with* $p(x) \equiv 0$.

Remark 2. Note that Theorem 1 in [8], for Equation (1.1) with $a_j(x) \equiv 0$ and $p(x)$ satisfying some weaker conditions, in particular

$$\int_{x_0}^{\infty} x^{(n-1)k} |p(x)| \, \mathrm{d}x < \infty, \tag{2.6}$$

provides existence of solutions equivalent to x^j, $j = 0, \dots, n-1$. However, solutions $y(x) = \sum\limits_{j=0}^{n-1} C_j x^j + o(1)$ with arbitrary C_j need not exist in this case.

E x a m p l e. Consider the equation

$$y'' = \frac{y^2}{\sqrt{x^7}}.$$

We have

$$\int_{x_0}^{\infty} x^{(n-1)k}|p(x)|\,\mathrm{d}x = \int_{x_0}^{\infty} x^{-3/2}\,\mathrm{d}x < \infty.$$

So, according to [8] there exist, near $+\infty$, solutions $y_1(x) \sim 1$ and $y_2(x) \sim x$.

However, Theorem 2.4 cannot guarantee existence of a solution $y(x) = x+1+o(1)$, since

$$\int_{x_0}^{\infty} x^{(n-1)(k+1)}|p(x)|\,\mathrm{d}x = \int_{x_0}^{\infty} x^{-1/2}\,\mathrm{d}x = \infty.$$

Suppose such a solution exists. Then $y(x) \sim x$, whence $y'' \sim x^{-3/2}$ and $y' = C_1 - 2x^{-1/2} + o(x^{-1/2})$ with $C_1 = 1$ due to $y(x) \sim x$.

So, $y(x) = C_0 + x - 4x^{1/2} + o(x^{1/2})$, which contradicts to $y(x) = x + 1 + o(1)$.

R e m a r k 3. Note that for Equation (1.1) with $a_j(x) \neq 0$, existence of a solution, admitting the asymptotic representation

$$y(x) = \sum_{j=0}^{n-1} C_j x^j (1 + o(1)) \tag{2.7}$$

can be proved by using Corollary 8.2 from the monograph [9] if conditions (2.6), (2.2) are fulfilled, and $\sum\limits_{j=0}^{n-1} |C_j| \neq 0$.

Properties (2.7) and (2.5) differ. For example, in the case $n = 2$, the solutions behaving as $-\xi_1(x)+\xi_2(x)+o(1)$ and $\xi_1(x)+\xi_2(x)+o(1)$, which exist by Theorem 2.4, must be different. On the contrary, the solutions behaving as $(x + x^2)(1 + o(1))$ and $(-x + x^2)(1 + o(1))$, which are particular cases of (2.7), may occur to be just the same function.

3. Proofs

Lemma 3.1. *The operator*

$$L = \frac{\mathrm{d}^n}{\mathrm{d}x^n} + \sum_{j=0}^{n-1} a_j(x)\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}$$

with all functions $a_j(x)$ *satisfying* (2.2) *can be represented in a neighborhood of* $+\infty$ *as the* n*th quasi-derivative operator, i.e.*

$$L\colon\ y \mapsto \Bigl(r_n \frac{\mathrm{d}}{\mathrm{d}x}\Bigl(r_{n-1}\frac{\mathrm{d}}{\mathrm{d}x}\Bigl(\dots r_1 \frac{\mathrm{d}}{\mathrm{d}x}(r_0 y)\dots\Bigr)\Bigr)\Bigr),$$

with positive functions $r_0, \dots, r_n$ *all tending to 1 as* $x \to +\infty$.

By the lemma, equation (1.1) can be rewritten in a neighborhood of $+\infty$ as

$$y^{[n]}(x) + p(x)|y|^k \operatorname{sgn} y = 0 \tag{3.1}$$

with $y^{[j]}$ denoting the j-th quasi-derivative of a function $y(x)$:

$$y^{[j]} = \Bigl(r_j \frac{\mathrm{d}}{\mathrm{d}x}\Bigl(r_{j-1}\frac{\mathrm{d}}{\mathrm{d}x}\Bigl(\dots r_1 \frac{\mathrm{d}}{\mathrm{d}x}(r_0 y)\dots\Bigr)\Bigr)\Bigr).$$

Thus, $y^{[0]}(x) = r_0(x)y(x)$ and $y^{[i]}(x) = r_i(x)(y^{[i-1]}(x))'$, $i = 1, \dots, n$.

Such a representation for linear operators is described by G. Polya [17], Ch. I. de la Vallée-Poussin [20], A. Levin [13].

Now, the coefficients of the quasi-derivative operator are constructed so that their limits, as $x \to +\infty$, are equal to 1, which is used in the proof of Theorem 2.4. Similar representation on finite segments was obtained and used in [3].

Lemma 3.2. *There exist fundamental solutions* $\xi_j(x)$, $j = 0, \dots, n-1$, *to the equation* $y^{[n]} = 0$ *satisfying the following properties:*

$$\xi_j^{[i]}(x) = 0 \quad \text{if} \quad j < i < n,$$
$$\xi_j^{[i]}(x) = 1 \quad \text{if} \quad i = j,$$
$$\xi_j^{[i]}(x) = \frac{x^{j-i}}{(j-i)!}(1 + o(1)) \quad \text{as} \quad x \to +\infty \quad \text{if} \quad i < j.$$

P r o o f. Trying to solve the equation $y^{[n]} = 0$, let us prove by backward induction over $i = n-1, \dots, 0$ that the i-th quasi-derivative of its general solution is

$$y^{[i]}(x) = \sum_{j=i}^{n-1} C_j \xi_{ij}(x)$$

with arbitrary constants C_j and functions $\xi_{ij}(x)$, $i \leqslant j < n$, such that

$$\xi_{ii}(x) \equiv 1,$$
$$\xi_{ij}(x) = \frac{x^{j-i}}{(j-i)!}(1 + o(1)) \text{ as } x \to +\infty,$$
$$r_{i+1}(x)(\xi_{ij}(x))' = \xi_{i+1,j}(x).$$

Since $y^{[n]}(x) = r_n(x)(y^{[n-1]}(x))' = 0$, we obtain that $y^{[n-1]}(x)$ must be constant. This provides the first induction step.

If for some $i > 0$ the statement needed is proved, then due to the equality $y^{[i]}(x) = r_i(x)(y^{[i-1]})'(x)$ we have, with some $a \in \mathbb{R}$,

$$\begin{aligned} y^{[i-1]}(x) &= C_{i-1} + \int_a^x \frac{\sum_{j=i}^{n-1} C_j \xi_{ij}(t)}{r_i(t)}\,\mathrm{d}t \\ &= C_{i-1}\cdot 1 + \sum_{j=i}^{n-1} C_j \int_a^x \frac{\xi_{ij}(t)\,\mathrm{d}t}{r_i(t)} = \sum_{j=i-1}^{n-1} C_j \xi_{i-1,j}(x), \end{aligned}$$

where $\xi_{i-1,i-1}(x) \equiv 1$ and, for $j \geqslant i$, $\xi_{i-1,j}(x) = \int_a^x \xi_{ij}(t)\,\mathrm{d}t/r_i(t)$. The last function satisfies

$$\lim_{x\to+\infty} \frac{\xi_{i-1,j}(x)}{x^{j-(i-1)}} = \lim_{x\to+\infty} \frac{\xi_{ij}(x)}{r_i(x)(j-i+1)x^{j-i}} = \frac{1}{(j-i+1)(j-i)!} = \frac{1}{(j-(i-1))!},$$

thus completing the induction step. To prove the lemma, it remains just to put $\xi_j(x) = \xi_{0,j}(x)/r_0(x)$ and to notice that $\xi_j^{[i]}(x) = \xi_{ij}(x)$ if $i \leqslant j$ and $\xi_j^{[i]}(x) = 0$ otherwise. □

Lemma 3.3. *Suppose $f(x)$ is a continuous function defined in a neighborhood of $+\infty$. Then the general solution to the equation $y^{[n]}(x) = f(x)$ is*

$$y(x) = \sum_{j=0}^{n-1} \left(C_j + \int_a^x f(t) b_j(t) t^{n-j-1}\,\mathrm{d}t \right) \xi_j(x)$$

with some $a \in \mathbb{R}$, arbitrary constants $C_0, \dots, C_{n-1}$, the fundamental solutions $\xi_j(x)$ to the homogeneous equation described in Lemma 3.2, and bounded functions $b_j(x)$ expressible in terms of the coefficients $r_i(x)$ and the quasi-derivatives of $\xi_i(x)$.

Proof. By variation of constants, the function

$$y(x) = \sum_{j=0}^{n-1} g_j(x)\xi_j(x) \tag{3.2}$$

is a solution to the equation considered if the functions $g_j(x)$ satisfy the system

$$\begin{aligned} &\sum_{j=0}^{n-1} g_j'(x)\xi_j^{[i-1]}(x) = 0, \qquad i = 1, \dots, n-1, \\ &\sum_{j=0}^{n-1} g_j'(x)\xi_j^{[n-1]}(x) = \frac{f(x)}{r_n(x)}. \end{aligned} \tag{3.3}$$

In more detail, first we prove by induction over $i = 0, \dots, n-1$ that, due to (3.3), the quasi-derivatives of the function $y(x)$ defined by (3.2) has the following form:

$$y^{[i]}(x) = \sum_{j=0}^{n-1} g_j(x)\ \xi_j^{[i]}(x).$$

The first step is trivial. If for some $i < n-1$ the last equality is proved, then we have

$$y^{[i+1]}(x) = r_{i+1}(x) \sum_{j=0}^{n-1} g_j'(x)\xi_j^{[i]}(x) + \sum_{j=0}^{n-1} g_j(x)\ r_{i+1}(x)\big(\xi_j^{[i]}(x)\big)'$$

with the first sum vanishing due to (3.3) and the second coinciding with the needed expression $\sum\limits_{j=0}^{n-1} g_j(x)\xi_j^{[i+1]}(x)$.

In the same way, due to (3.3) and the equation $\xi_j^{[n]}(x) = 0$, we have

$$y^{[n]}(x) = r_n(x) \sum_{j=0}^{n-1} g_j'(x)\xi_j^{[n-1]}(x) + \sum_{j=0}^{n-1} g_j(x)\xi_j^{[n]}(x) = f(x).$$

Now, let us solve system (3.3). Since $\xi_j^{[i]}(x) = 0$ for $j < i < n$, the system is triangular and the derivatives $g_j'(x)$ can be proved to have the needed form $f(x)b_j(x)x^{n-j-1}$, step by step for $j = n-1, \dots, 0$.

We begin from the last equation of (3.3), which gives $g_{n-1}'(x) = f(x)/r_n(x)$. Thus, we can take $1/r_n(x)$ as the bounded function $b_{n-1}(x)$.

If for some $i \geqslant 0$ the needed expressions for $g_j'(x)$, $j > i$, are already obtained, then

$$\begin{aligned} g_i'(x) &= -\sum_{j=i+1}^{n-1} g_j'(x)\xi_j^{[i]}(x) = -\sum_{j=i+1}^{n-1} f(x)b_j(x)x^{n-j-1}\ \xi_j^{[i]}(x) \\ &= f(x)\Bigg(-\sum_{j=i+1}^{n-1} b_j(x)\xi_j^{[i]}(x)x^{i-j}\Bigg)x^{n-i-1}. \end{aligned}$$

Since $\xi_j^{[i]}(x) = x^{j-i}(j-i)!^{-1}(1+o(1))$, the last expression in the big parentheses is bounded and may be taken as $b_{i-1}(x)$. The rest of the proof is evident. $\square$

Now we can prove Theorem 2.4.

P r o o f. Consider the set V_{ac} of all continuous functions $v(x)$ defined on $[a, \infty)$ such that $\sup\big\{|v(x)|\ x^{1-n}\colon\ x \geqslant a\big\} \leqslant c$. If we define the norm $\|v(x)\|$ by the left-hand side of the last inequality, then V_{ac} becomes a Banach space.

Consider the mapping $F\colon V_{ac} \to V_{ac}$ such that

$$F(v)(x) = \sum_{j=0}^{n-1} \left(C_j - \int_x^{+\infty} p(t)|v|^k (\operatorname{sgn} v) b_j(t) t^{n-j-1} \, \mathrm{d}t \right) \xi_j(x)$$

with the bounded functions $b_j(x)$ participating in Lemma 3.3.

The integrals converge since their integrands are $O(|p(t)|t^K)$ with $K = (n-1)k + n - j - 1 \leqslant (n-1)(k+1)$.

As for the inclusion $F(V_{ac}) \subset V_{ac}$, it holds if $a > 1$ and $n(c^k B\delta + C_{\max}) \leqslant c$ with

$$\begin{aligned} B &= \sup\{|b_j(x)|\colon\ x \geqslant a, j = 0, \ldots, n-1\}, \\ \delta &= \int_a^{\infty} |p(t)| \ t^{(n-1)(k+1)} \, \mathrm{d}t, \\ C_{\max} &= \max\{|C_j|\colon\ j = 0, \ldots, n-1\}. \end{aligned}$$

The last inequality holds if we put $c = (n{+}1)\, C_{\max}$ and choose a big enough making δ sufficiently small to provide $n(n+1)^k \ C_{\max}^k \ B \ \delta \leqslant C_{\max}$. Furthermore, we can make F become a contraction mapping, i.e. provide the inequality $\|F(v) - F(w)\| \leqslant \theta \|v - w\|$ for some $\theta < 1$ and all $v, w \in V_{ac}$.

Indeed, for $x \geqslant a$ and a big enough we have $|\xi_j(x)| < 2x^{n-1}$ and, since $\big||X|^k \operatorname{sgn} X - |Y|^k \operatorname{sgn} Y\big| \leqslant |X - Y| \cdot k \max\{|X|, |Y|\}^{k-1}$, we have

$$\begin{aligned} x^{1-n}|F(v)(x) - F(w)(x)| &\leqslant 2Bn \int_x^{+\infty} |v(t) - w(t)| \ k \left(ct^{n-1}\right)^{k-1} |p(t)| t^{n-1} \, \mathrm{d}t \\ &\leqslant 2Bnkc^{k-1}\|v - w\| \int_x^{+\infty} |p(t)| t^{(n-1)(k+1)} \, \mathrm{d}t \leqslant 2Bnkc^{k-1}\|v - w\|\delta. \end{aligned}$$

So, all we need to make F a contraction mapping is to increase a so that δ could become sufficiently small.

The unique fixed point of F, which must exist, is a solution to (3.1) having the form $y(x) = \sum\limits_{j=0}^{n-1} C_j \xi_j(x) + \varepsilon(x)$ with

$$\varepsilon(x) = -\sum_{j=0}^{n-1} \xi_j(x) \int_x^{+\infty} p(t)|y|^k (\operatorname{sgn} y) b_j(t) t^{n-j-1} \, \mathrm{d}t.$$

Now we have to prove that $\varepsilon(x) = o(1)$ as $x \to +\infty$. Since $y = O\left(x^{n-1}\right)$, we have

$$\varepsilon(x) = O\bigg(\sum_{j=0}^{n-1} \xi_j(x) \int_x^{+\infty} |p(t)| \ t^{(n-1)(k+1)-j} \, \mathrm{d}t \bigg).$$

Further, since $|t|^{-j} \leqslant |x|^{-j}$ for $t \geqslant x \geqslant a > 1$, we obtain

$$\varepsilon(x) = \int_x^{+\infty} |p(t)|\ t^{(n-1)(k+1)}\,\mathrm{d}t \cdot O\biggl(\sum_{j=0}^{n-1} \frac{\xi_j(x)}{x^j}\biggr) = o(1).$$

□

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