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## REPRODUCING KERNEL PARTICLE METHOD AND ITS MODIFICATION

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*Abstract.* Meshless methods have become an effective tool for solving problems from engineering practice in last years. They have been successfully applied to problems in solid and fluid mechanics. One of their advantages is that they do not require any explicit mesh in computation. This is the reason why they are useful in the case of large deformations, crack propagations and so on. Reproducing kernel particle method (RKPM) is one of meshless methods. In this contribution we deal with some modifications of the RKPM. The construction of the methods considered is given together with simple examples of their applications to solving boundary value problems.

*Keywords:* meshless method, partition of unity, reproducing kernel particle method, reproducing kernel hierarchical partition of unity, enriched reproducing kernel particle method

*MSC 2010:* 65N30, 65L60

### 1. INTRODUCTION

The FEM is the most popular method that is used for solving boundary value problems. No wonder because this method is simple, quick and reliable. Essentially, it represents the Galerkin method where the solution is constructed in the form of a linear combination of the basis functions that are chosen in a special way (they satisfy the condition of partition of unity) on the chosen mesh.

However, meshing at the beginning of or remeshing during the computational process can be a very time consuming matter especially in the 3D case. This is the reason why a lot of attention has been paid during the past years to developing new approximations in the Galerkin framework, which do not require a mesh for their construction. The SPHM, RKPM, MLSM, EFG, XFEM or hp-clouds are some of them. These methods have been successfully applied in the case of large oscillations of solution, large deformations and crack propagation. The term “meshless” stems

from the fact that they require no explicitly given mesh at the beginning of the computation. Approximation of solutions of BVP's is here constructed from a set of nodal data and from the associated weight functions with compact support.

One group of the meshless methods—the reproducing kernel particle method (RKPM) together with its modifications reproducing kernel hierarchical partition of unity method (RKHPUM) and enriched reproducing kernel particle method (ERKPM)—are studied in this contribution. The next two sections are devoted to the basic principle of construction of the RKPM and its two modifications together with some simple test examples of their application.

## 2. CONSTRUCTION OF MESHLESS INTERPOLANTS

The following notation will be used in this text:

$\Omega \subset \mathbb{R}^n$ —a domain where approximation is constructed,

$p^T(x) = (p_1(x), \dots, p_l(x))$ , where  $l = \binom{s+n}{n}$ ,—the complete monomial basis of order  $s$  in  $\mathbb{R}^n$  (for instance, if  $n = 1$  then  $p(x) = (1, x, \dots, x^s)$  and if  $n = 2$  then  $p(x_1, x_2) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2, \dots, x_1x_2^{s-1}, x_2^s)$ ),

$r^T(x) = (p_1(x), \dots, p_l(x), u^\varepsilon(x))$ , where  $u^\varepsilon(x)$ —a suitable level set function,

$w$ —a weight function with compact support; we obtain a weight function of  $n$  variables as the tensor product of 1D weight functions,

$\varrho \in \mathbb{R}$ —a dilatation parameter that corrects the size of  $\text{supp } w$ ,

$x_1, \dots, x_N$ —a set of particles in  $\mathbb{R}^n$ ,

$\alpha$ —a multiindex, its components are denoted  $[\alpha]$ .

We focus on the following three methods: reproducing kernel particle method, reproducing kernel hierarchical partition of unity method and enriched reproducing kernel particle method. We can understand these methods as specific cases of the Galerkin method. The approximations  $\tilde{u}$  are here built from the convolution

$$u^R(x) = \int_{\Omega} K(x-y)u(y) dy.$$

Here the kernel  $K$  is approximated by a function  $\Phi$ , i.e.

$$(2.1) \quad \tilde{u}(x) = \int_{\Omega} \Phi(x, x-y)u(y) dy,$$

which satisfies some “conditions of consistency”.

There are more possibilities how to obtain a meshless approximation with the required type of consistence. For instance, we can use the moving least square method (MLSM) (see [5]). Also the approach with help of a Taylor series (see [2]) is possible.

Our task now is to derive the form of RKP, RKHPU and ERKP interpolants from suitable conditions of consistency.

### 2.1. Reproducing kernel particle method.

We want to construct the RKP interpolants in 1D such that the approximation  $\Phi$  of the kernel  $K$  reproduces the polynomials of order  $s$  exactly. We can write this condition successively for polynomials of order  $0, 1, \dots, s$ :

$$\begin{aligned} \int_{\Omega} a_0 \Phi(x, x-y) dy &= a_0 \Rightarrow \int_{\Omega} \Phi(x, x-y) dy = 1, \\ \int_{\Omega} (a_0 + a_1 y) \Phi(x, x-y) dy &= a_0 + a_1 x \Rightarrow \int_{\Omega} y \Phi(x, x-y) dy = x, \\ &\vdots \\ \int_{\Omega} (a_0 + \dots + a_s y^s) \Phi(x, x-y) dy &= a_0 + \dots + a_s x^s \Rightarrow \int_{\Omega} y^s \Phi(x, x-y) dy = x^s. \end{aligned}$$

We obtain the system that represents the consistency conditions

$$(2.2) \quad \int_{\Omega} p(y) \Phi(x, x-y) dy = p(x) \quad \forall x \in \Omega, \quad \forall p \in P^s.$$

We suppose that

$$(2.3) \quad \Phi(x, x-y) = p^T(x-y) b(x) w(x-y),$$

where  $b$  is chosen such that the conditions (2.2) are satisfied. It means that

$$(2.4) \quad \int_{\Omega} p(y) p^T(x-y) b(x) w(x-y) dy = p(x),$$

where  $p(x) = (1, x, \dots, x^s)^T$ . Denote by

$$(2.5) \quad M(x) = \int_{\Omega} p(y) p^T(x-y) w(x-y) dy$$

the moment matrix. The unknown vector function  $b$  is a solution of the system

$$(2.6) \quad M(x) b(x) = p(x).$$

In the case when the moment matrix  $M$  is invertible we have

$$(2.7) \quad b(x) = M^{-1}(x) p(x).$$

If we put the (2.3) and (2.7) into the relation (2.1) we obtain

$$(2.8) \quad \tilde{u}(x) = \int_{\Omega} p^T(x-y)M^{-1}(x)p(x)w(x-y)u(y) dy.$$

If  $x_1, \dots, x_N \in \Omega$  are chosen particles and  $\Delta x_I$  are quadrature weights, we obtain the discretized form of the formulas (2.8) and (2.5)

$$(2.9) \quad \tilde{u}(x) = \sum_{I=1}^N p^T(x-x_I)M^{-1}(x)p(x)w(x-x_I)u_I\Delta x_I,$$

$$(2.10) \quad M(x) = \sum_{J=1}^N p(x_J)p^T(x-x_J)w(x-x_J)\Delta x_J.$$

**Remark.** If the conditions (2.2) are satisfied, we say that the function  $\Phi$  is consistent of order  $s$  or that the function  $\Phi$  forms a partition of unity of order  $s$ .

**Remark.** The unique solvability of (2.6), (2.10) depends on the size of  $\text{supp } w$  and on the distribution of the particles  $x_I$ ,  $I = 1, \dots, N$ . If we denote

$$A(x) = \{x_I: x \in \text{supp } w(x-x_I)\},$$

then a necessary condition for the unique solvability of (2.6), (2.10) is that  $\forall x \in \mathbb{R}^n$

$$\text{card } A(x) \geq \dim P^s.$$

(See [1].)

The relations (2.9) and (2.10) can be generalized to the  $n$ -dimensional case. If we shift the basis and regulate the size of  $\text{supp } \Psi_I$  by means of a dilatation parameter  $\varrho \in \mathbb{R}$ , then the results coincide with the results given in the next definition (the formulas were derived by means of the Taylor expansion in the article [2]). A translated and scaled basis can be used to get a better conditioning of the moment matrix.

**Definition 1.** Interpolants constructed by means of the RKPM in  $n$ D have the form

$$\tilde{u}(x) = \sum_{I=1}^N \Psi_I(x)u_I.$$

The shape functions

$$\Psi_I(x) = p(x-x_I)b(x) \frac{1}{\varrho^n} w\left(\frac{x-x_I}{\varrho}\right)\Delta x_I,$$

where  $p(x)$  is chosen a complete monomial basis of order  $s$  (see above) and the vector function  $b(x)$  is a solution of the system

$$M(x)b(x) = p(0).$$

The moment matrix is

$$M(x) = \sum_{J=1}^N p(x - x_J) p^T(x - x_J) \frac{1}{\varrho^n} w \left( \frac{x - x_J}{\varrho} \right) \Delta x_J.$$

**Remark.** If the shape functions are reproducing of order  $s$  and  $h = x_i - x_{i-1}$ ,  $i = 2, \dots, N$ , then for a smooth function  $u$  the error  $\|u - \tilde{u}\|$  has order  $O(h^s)$ . (See [1].)

**2.2. Reproducing kernel hierarchical partition of unity method.** We ask whether the approximation  $\Phi$  of the kernel  $K$  in this case is constructed so that its derivatives, whose order is less or equal  $s$ , reproduce polynomials of order  $s$  exactly. We have for  $0 \leq \alpha \leq s$  in 1D

$$\begin{aligned} \int_{\Omega} a_0 \Phi^{(\alpha)}(x, x - y) \, dy = 0 &\Rightarrow \int_{\Omega} \Phi^{(\alpha)}(x, x - y) \, dy = 0, \\ \int_{\Omega} (a_0 + a_1 y) \Phi^{(\alpha)}(x, x - y) \, dy = 0 &\Rightarrow \int_{\Omega} y \Phi^{(\alpha)}(x, x - y) \, dy = 0, \\ &\vdots \\ \int_{\Omega} (a_0 + \dots + a_{\alpha} y^{\alpha}) \Phi^{(\alpha)}(x, x - y) \, dy = \alpha! a_{\alpha} &\Rightarrow \int_{\Omega} y^{\alpha} \Phi^{(\alpha)}(x, x - y) \, dy = \alpha!, \\ \int_{\Omega} (a_0 + \dots + a_{\alpha+1} y^{\alpha+1}) \Phi^{(\alpha)}(x, x - y) \, dy = \alpha! (a_{\alpha} + a_{\alpha+1} x) & \\ \Rightarrow \int_{\Omega} y^{\alpha+1} \Phi^{(\alpha)}(x, x - y) \, dy = \alpha! x, & \\ \int_{\Omega} (a_0 + \dots + a_s y^s) \Phi^{(\alpha)}(x, x - y) \, dy = \alpha! (a_{\alpha} + \dots + a_s x^{s-\alpha}) & \\ \Rightarrow \int_{\Omega} y^s \Phi^{(\alpha)}(x, x - y) \, dy = \alpha! x^{s-\alpha}. & \end{aligned}$$

This yields the system of conditions

$$(2.11) \quad \int_{\Omega} p(y) \Phi^{(\alpha)}(x, x - y) \, dy = \alpha! p^{(\alpha)}(x), \quad 0 \leq \alpha \leq s.$$

Here  $p^{(\alpha)}(x) = \underbrace{(0, \dots, 0)}_{\alpha}, 1, x, \dots, x^{s-\alpha})^T$ . Suppose that

$$(2.12) \quad \Phi^{(\alpha)}(x, x - y) = p^T(x - y) b^{\alpha}(x) w(x - y),$$

then the conditions of consistence (2.11) have the form

$$(2.13) \quad \int_{\Omega} p(y)p^T(x-y)b^\alpha(x)w(x-y)dy = \alpha!p^{(\alpha)}(x), \quad 0 \leq \alpha \leq s$$

and the vector  $b^\alpha(x)$  is the solution of the system

$$(2.14) \quad M(x)b^\alpha(x) = \alpha!p^{(\alpha)}(x), \quad 0 \leq \alpha \leq s.$$

The elements of the moment matrix  $M$  are

$$(2.15) \quad M(x) = \int_{\Omega} p(y)p^T(x-y)w(x-y)dy.$$

The discretized form of the RKHPU approximation for the particles  $x_1, \dots, x_N$  is

$$(2.16) \quad \tilde{u}(x) = \sum_{\alpha=0}^s \sum_{I=1}^N \alpha!p(x_I)p^T(x-x_I)M^{-1}(x)p^{(\alpha)}(x)w(x-x_I)u_I\Delta x_I,$$

$$(2.17) \quad M(x) = \sum_{J=1}^N p(x_J)p^T(x-x_J)w(x-x_J)\Delta x_J.$$

The generalization in the  $n$ -dimensional case is given in the next definition. The form of the RKHPU interpolants was derived by means of the moving least squares method in the article [5].

**Definition 2.** Interpolants constructed in  $n$ D by means of the RKHPUM have the form

$$\tilde{u}(x) = \sum_{I=1}^N \sum_{|\alpha|=0}^s \Psi_I^{[\alpha]}(x)u_I^{[\alpha]}.$$

The shape functions are

$$\Psi_I^{[\alpha]}(x) = \alpha!p\left(\frac{x_I-x}{\varrho}\right)b^{[\alpha]}(x)\frac{1}{\varrho^n}w\left(\frac{x_I-x}{\varrho}\right)\Delta x_I,$$

where  $p(x)$  is the chosen complete monomial basis of order  $s$  (see above) and the vector  $b^{[\alpha]}(x)$  is the solution of the system

$$M(x)b^{[\alpha]}(x) = p^{[\alpha]}(0),$$

where

$$M(x) = \sum_{J=1}^N p\left(\frac{x_J-x}{\varrho}\right)p^T\left(\frac{x_J-x}{\varrho}\right)\frac{1}{\varrho^n}w\left(\frac{x_J-x}{\varrho}\right)\Delta x_J$$

and  $p^{[\alpha]}(0) = \underbrace{(0, \dots, 0, 1, 0, \dots, 0)^T}_i$  in the case when  $[\alpha] = \alpha_i$ .

**2.3. Enriched reproducing kernel particle method.** The main reason for constructing this meshless method was the requirement to construct an approximation that has a discontinuous derivative on a fixed or a moving interface  $\Gamma \subset \Omega$ . (The interface  $\Gamma$  is a point in 1D, a curve in 2D, a surface in 3D.) The approximation will be modeled using a function  $u^\varepsilon$  that is defined in the neighborhood of  $\Gamma$ .

The approximation  $\Phi$  of the kernel  $K$  is constructed so that it reproduces exactly a general function

$$(2.18) \quad u(x) = p(x) + u^\varepsilon(x),$$

where

$$(2.19) \quad u^\varepsilon(x) = H_0(\Theta(x))\Theta(x),$$

$H_0$  represents the Heaviside function and  $\Theta(x)$  is defined as the signed distance from  $x \in \Omega$  to the interface  $\Gamma$ .

We can write the condition (2.18) in 1D as

$$\begin{aligned} \int_{\Omega} a_0 \Phi(x, x-y) \, dy = a_0 &\Rightarrow \int_{\Omega} \Phi(x, x-y) \, dy = 1, \\ \int_{\Omega} (a_0 + a_1 y) \Phi(x, x-y) \, dy = a_0 + a_1 x &\Rightarrow \int_{\Omega} y \Phi(x, x-y) \, dy = x, \\ &\vdots \\ \int_{\Omega} (a_0 + \dots + a_s y^s + u^\varepsilon(y)) \Phi(x, x-y) \, dy &= a_0 + \dots + a_s x^s + u^\varepsilon(x) \\ &\Rightarrow \int_{\Omega} u^\varepsilon(y) \Phi(x, x-y) \, dy = u^\varepsilon(x). \end{aligned}$$

This yields the system of consistency conditions

$$(2.20) \quad \int_{\Omega} r(y) \Phi(x, x-y) \, dy = r(x),$$

where  $r(x) = (1, x, \dots, x^s, u^\varepsilon(x))^T$ . If we put

$$(2.21) \quad \Phi(x, x-y) = r^T(x-y)b(x)w(x-y)$$

in (2.20), we obtain

$$(2.22) \quad \int_{\Omega} r(y)r^T(x-y)b(x)w(x-y) \, dy = r(x).$$



If we denote

$$(2.23) \quad M(x) = \int_{\Omega} r(y)r^T(y-x)w(x-y) dy$$

and the moment matrix  $M$  is invertible, then

$$(2.24) \quad b(x) = M^{-1}(x)r(x).$$

From (2.1), (2.21) and (2.24) we conclude

$$(2.25) \quad \tilde{u}(x) = \int_{\Omega} r^T(x-y)M^{-1}(x)r(x)w(x-y)u(y) dy.$$

If  $x_1, \dots, x_N \in \Omega$  are chosen particles, the discretized form of (2.25), (2.23) is

$$(2.26) \quad \tilde{u}(x) = \sum_{I=1}^N r^T(x-x_I)M^{-1}(x)r(x)w(x-x_I)u_I \Delta x_I,$$

$$(2.27) \quad M(x) = \sum_{J=1}^N r(x_J)r^T(x-x_J)w(x-x_J)\Delta x_J.$$

**Remark.** If we study the properties of the enriched moment matrix (2.27), we see that it can be singular. This is the reason why we work with the ERKP moment matrix (2.10) in such discontinuity neighborhoods where this matrix is nonsingular.

We use the classical RKP moment matrix (2.10) in the rest of  $\Omega$  (see [4]).

In the next definition the  $n$ -dimensional analogues of (2.26) and (2.27) are given.

**Definition 3.** Interpolants constructed by means of the ERKPM in  $n$ D have the form

$$(2.28) \quad \tilde{u}(x) = \sum_I^N \Psi_I(x)u_I.$$

The shape functions

$$(2.29) \quad \Psi_I(x) = r(x-x_I)b(x) \frac{1}{\varrho^n} w\left(\frac{x-x_I}{\varrho}\right) \Delta x_I,$$

where  $r(x)$  is the complete monomial basis of order  $s$  enriched with a suitable level set function  $u^\varepsilon$  (see above) and a vector  $b(x)$  is the solution of the system

$$(2.30) \quad M(x)b(x) = r(0),$$

$$(2.31) \quad M(x) = \sum_{J=1}^N r(x-x_J)r^T(x-x_J) \frac{1}{\varrho^n} w\left(\frac{x-x_J}{\varrho}\right) \Delta x_J.$$

### 3. EXAMPLES

**3.1. BVP for Helmholtz equation.** Let  $\Omega = \langle 0, 1 \rangle \times \langle 0, 1 \rangle$ . Solve the boundary value problem

$$\begin{aligned} \Delta u(x, y) + 16^2 u(x, y) &= 1 \quad \text{in } \Omega, \\ \frac{\partial u(x, y)}{\partial n} &= 2 \quad \text{on } \partial\Omega. \end{aligned}$$

**Solution:** We develop the approximation for uniformly distributed particles  $(x_I, y_I)$ ,  $I = 1, \dots, 100$ , the polynomial basis  $p(x, y) = (1, x, y)$ , the conical weight function

$$w(x, y) = \begin{cases} ((1 - x^2)(1 - y^2))^2 & \text{for } |x| \leq 1, |y| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the dilatation parameter  $\varrho = 0.3$ .

The errors of the computed RKP and RKHPU approximations are

$$\|\tilde{u}_{\text{RKPM}} - u(x)\|_{L_2} = 0.0948 \quad \text{and} \quad \|\tilde{u}_{\text{RKHPUM}} - u(x)\|_{L_2} = 0.0203$$

in this case.

**3.2. BVP with discontinuous coefficient.**

We consider the problem

$$\begin{aligned} (k(x)T'(x))' &= 2 \quad \text{if } x \in (0, 1), \\ u(0) &= 0, \\ u(1) &= 0, \end{aligned}$$

where

$$k(x) = \begin{cases} 1 & \text{if } x \in (0, 0.5), \\ 10 & \text{if } x \in (0.5, 1). \end{cases}$$

**Solution:** The solution was constructed for  $N = 8$  particles, the polynomial basis  $p(x) = (1, x)$ , the function  $u^\varepsilon(x) = x - 0.5$  and the conical weight function

$$w(x) = \begin{cases} (1 - x^2)^2 & \text{if } |x| < 0.3, \\ 0 & \text{otherwise.} \end{cases}$$

The errors of the computed ERKP approximation and RKP approximation are

$$\|\tilde{u}_{\text{ERKPM}}(x) - u(x)\|_{L^2(0,1)} = 0.0003 \quad \text{and} \quad \|\tilde{u}_{\text{KPM}}(x) - u(x)\|_{L^2(0,1)} = 0.0021$$

in this case.

#### 4. CONCLUSION

Different reproduction conditions were used to construct different meshless methods in this text. We received the form of the RKP, RKHPU and ERKP interpolants. We can see that the methods presented differ from each other especially in the form of the moment matrix. The resulting shape functions have the same smoothness as the chosen weight functions. The size of  $\text{supp } w$  together with the distribution of particles  $x_1, \dots, x_n$  affect the convergence of our methods. The RKHPUM was presented in Example 1. It gives in this case of oscillatory solution better results than the RKPM. Also the results obtained using the ERKPM on the problem with discontinuity given in Example 2 were more accurate from the point of view of the associated level of error than the results received by means of the RKPM.

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