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MODIFICATION OF UNFOLDING APPROACH TO
TWO-SCALE CONVERGENCE

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Abstract. Two-scale convergence is a powerful mathematical tool in periodic homogenization developed for modelling media with periodic structure. The contribution deals with the classical definition, its problems, the “dual” definition based on the so-called periodic unfolding. Since in the case of domains with boundary the unfolding operator introduced by D. Cioranescu, A. Damlamian, G. Griso does not satisfy the crucial integral preserving property, the contribution proposes a modified unfolding operator which satisfies the property and thus simplifies the theory. The properties of two-scale convergence are surveyed.

Keywords: homogenization, two-scale convergence, periodic unfolding

MSC 2010: 35B27, 49J45

1. INTRODUCTION

Two-scale convergence became a powerful tool in homogenization theory. It enables us to overcome the problem of passing to the limit in a product of two weakly converging sequences: If $u_n \rightharpoonup u^*$ and $v_n \rightharpoonup v^*$ weakly, then what is the limit of $u_n v_n$? It can differ from $u^* v^*$ as the following simple example shows: In $L^2(0, 2\pi)$ both sequences $\{u_n\}$ and $\{v_n\}$ given by $u_n(x) = v_n(x) = \sin(nx)$ converge weakly to the zero function, but their product $u_n v_n$ converges weakly to the constant function $\frac{1}{2}$. It is caused by the fact that the local behavior of u_n and v_n is lost in the weak limit.

The problem appears in homogenization, which studies the behavior of solutions u^ε to a sequence of equations of type $-\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = f$ with periodic coefficients

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$a^\varepsilon(x) = a(x/\varepsilon)$ while the period $\varepsilon \rightarrow 0$. Indeed, the weak formulation of the problem

$$(1.1) \quad \text{Find } u^\varepsilon \in W_0^{1,2}(\Omega) \text{ s.t. } \int_{\Omega} a^\varepsilon \nabla u^\varepsilon \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in W_0^{1,2}(\Omega)$$

contains coefficients a^ε that weakly converge and since the sequence of solutions u^ε is bounded in $W_0^{1,2}(\Omega)$, it contains a subsequence $u^{\varepsilon'}$ such that $\nabla u^{\varepsilon'}$ also weakly converge. The problem is to find a^* such that $\lim_{\varepsilon} (a^\varepsilon \nabla u^\varepsilon) = a^* \nabla (\lim_{\varepsilon} u^\varepsilon)$.

The problems was first solved by a special choice of test functions v^ε , see e.g. [3], and its substance was generalized to the so-called “div-rot” lemma by Murat and Tartar. The two-scale convergence introduced by Nguetseng in [12] and further developed in [1], [10], [9] and other, gives a straightforward approach; it simplifies the proofs and derives the form of the homogenized problem simultaneously with proving the convergence.

The two-scale limit of a sequence $u^\varepsilon(x)$ of one variable $x \in \mathbb{R}^N$ is a function $u^0(x, y)$ of two variables $x, y \in \mathbb{R}^N$, where the additional variable y contains the local behavior of u^ε . The classical definition converts the two-scale two-variable test function $v(x, y)$ into a one-variable function $v(x, x/\varepsilon)$ and tests the convergence in $L^p(\Omega)$, see Definition 3.1, which causes problems with the choice of the test function space, see Section 3; thus two additional conditions for test functions were added.

The so-called “dual” approach to two-scale convergence based on the unfolding operator, called in [2], [4] the dilation operator, helps to solve the problem of the test function space. This alternative approach was announced in [5], described in detail in [7] and further developed with proofs in [6]. It was also introduced in [11]. In contrast to the previous definition, where the test function was transformed, here using the unfolding operator \mathcal{T}_ε the functions $u^\varepsilon(x)$ are transformed into two-variable functions $\widehat{u}^\varepsilon(x, y)$ and the convergence \widehat{u}^ε to u^0 is tested in $L^p(\Omega \times Y)$, see Section 4. This approach yields also a natural definition of strong two-scale convergence and simplifies the proofs by using the known properties of the L^p spaces.

The method works well in the whole \mathbb{R}^N , but the case of a domain Ω with boundary $\partial\Omega$ causes problems. The equality published in Proposition 1 of [5]

$$(1.2) \quad \int_{\Omega} u(x) \, dx = \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(u)(x, y) \, dx \, dy$$

is not true in general due to boundary cells. This equality—the integral preserving property of the unfolding operator—plays a crucial role in the theory. In [7] and [6] the problem of the invalid equality (1.2) is solved by considering auxiliary sequences of domains Λ_ε containing the boundary cells only, extending \mathcal{T}_ε at Λ_ε by zero and introducing an “unfolding criterion for the integral”, see (4.4).

The aim of the contribution is to introduce a modified periodic unfolding operator (announced in [8] and called the two-scale transform) such that it satisfies the integral conserving property, i.e. equality (1.2). The definition of two-scale convergence based on the modified unfolding operator makes it possible to simplify the two-scale convergence theory and homogenization of particular equations. In the end basic properties of the two-scale convergence are surveyed.

2. PRELIMINARIES

In periodic homogenization the real parameter $\varepsilon > 0$ denotes the period of the coefficients. A decreasing sequence of ε_n tending to zero will be called a scale. In homogenization instead of $n = 1, 2, 3, \dots$, the sequences are indexed by ε_n , but the n in u^{ε_n} is usually omitted and the sequence is denoted simply by u^ε . Usually the scale is not mentioned in definitions of two-scale convergence, but the two-scale convergence strongly depends on the chosen scale.

The basic cell denoted by Y is usually the unit cube $(0, 1)^N$. More generally, it can be an N -dimensional interval or any parallelepiped. Then the cell Y is spanned by an N -tuple of independent vectors $b_1, \dots, b_N \in \mathbb{R}^N$, i.e.

$$Y = \{\lambda_1 b_1 + \dots + \lambda_N b_N : 0 \leq \lambda_i < 1, i = 1, \dots, N\}.$$

The cell Y has the *paving property*, i.e. the collection $\{Y_\xi \equiv Y + \xi : \xi \in \Xi\}$ of cells Y shifted by a vector ξ from a set of shifts

$$\Xi = \{\xi = k_1 b_1 + \dots + k_N b_N : k_1, \dots, k_N \in \mathbb{Z}\}$$

is a partition of \mathbb{R}^N : the shifted cells Y_ξ are disjoint and cover \mathbb{R}^N .

The basic cell Y and the corresponding countable set of shifts Ξ define the unique decomposition of a point $y \in \mathbb{R}^N$ into its “integral” part $[y]$ —the shift ξ of the cell Y_ξ containing y and its “fractional” part $\{y\}$ —the local position of y in the cell:

$$(2.1) \quad y = [y] + \{y\}, \quad \text{where } [y] \in \Xi, \{y\} \in Y.$$

In the case when the basic cell Y is the unit cube, we have $\Xi = \mathbb{Z}^N$ and the decomposition $y = [y] + \{y\}$ of $y \in \mathbb{R}^N$ is the standard decomposition of each of its components into the integral and fractional parts $y_i = [y_i] + \{y_i\}$ defined by $[y_i] \in \mathbb{Z}$ and $0 \leq \{y_i\} < 1$.

A function u is called Y -periodic, if $u(x + \xi) = u(x)$ for each $\xi \in \Xi$. In the case when Y is the unique cube the Y -periodic function is periodic in the standard sense:

$u(y_1 + k_1, \dots, y_N + k_N) = u(y_1, \dots, y_N)$ holds for any $k_i \in \mathbb{Z}$. Let us denote the space of Y -periodic functions by the subscript per, e.g. $L^p_{\text{per}}(Y)$.

Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$ and $1 \leq p < \infty$. Then $C(\overline{\Omega})$ denotes the space of functions continuous on $\overline{\Omega}$, $L^p(\Omega)$ the Lebesgue space of functions on Ω integrable with the p -th power and $W^{1,p}(\Omega)$ the Sobolev space. The spaces of abstract functions will be denoted as usual, e.g. $C(\overline{\Omega}, L^p_{\text{per}}(Y))$.

Let us recall that a Y -periodic function $a \in L^p_{\text{per}}(Y)$ ($p < \infty$) with the scale $\{\varepsilon\}$ defines a sequence of functions $a^\varepsilon(x) = a(x/\varepsilon)$ which weakly converge to the constant function $\bar{a} = |Y|^{-1} \int_{\Omega} a(y) \, dy$.

Let us consider the domain Ω and an ε -scaled paving, i.e. a collection of ε -scaled $\varepsilon\xi$ -shifted cells $Y_\xi^\varepsilon \equiv \varepsilon Y_\xi \equiv \varepsilon(Y + \xi)$. The intersection of Ω with the ε -scaled paving determines the “inner cells” denoted by Y_ξ^ε , which are subsets of $\overline{\Omega}$, and the boundary cells Y_ξ^ε , whose interiors intersect the boundary $\partial\Omega$; their parts in Ω will be denoted by $\widetilde{Y}_\xi^\varepsilon = Y_\xi^\varepsilon \cap \Omega$. In accordance with [6] the union of these “uncomplete” boundary cells will be denoted by Λ^ε .

3. CLASSICAL DEFINITION

The standard definition of two-scale convergence can be stated as follows:

Definition 3.1. A sequence of functions u^ε is said to (weakly) two-scale converge to a limit $u^0 \in L^p(\Omega \times Y)$ ($1 < p < \infty$) with respect to the scale $\{\varepsilon\}$ if

$$(3.1) \quad \int_{\Omega} u^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y u^0(x, y) \varphi(x, y) \, dy \, dx$$

as $\varepsilon \rightarrow 0$ for all admissible test functions $\varphi(x, y)$ from a space \mathcal{V} of functions Y -periodic in the variable y which is a subspace of $L^q(\Omega \times Y)$, $q = p/(p-1)$.

A sequence of functions $u^\varepsilon(x)$ is said to strongly two-scale converge to a limit $u^0(x, y) \in L^p(\Omega \times Y)$ with respect to the scale $\{\varepsilon\}$ if it converges two-scale weakly and moreover

$$(3.2) \quad \|u^\varepsilon\|_{L^p(\Omega)} \rightarrow \frac{1}{|Y|^{1/p}} \|u^0\|_{L^p(\Omega \times Y)}.$$

The specification “with respect to the scale $\{\varepsilon\}$ ” is mostly omitted.

The space of admissible functions \mathcal{V} can be e.g. $L^p(\Omega, C^0_{\text{per}}(\overline{Y}))$ or $C^0(\overline{\Omega}, L^p_{\text{per}}(Y))$. The proper choice of \mathcal{V} is a problem. It cannot be the whole $L^p(\Omega \times Y)$, since on the left hand side of (3.1) the two-variable test function $\varphi(x, y)$ is transformed into one-variable function $\varphi(x, x/\varepsilon)$. Taking into account that φ is periodic in y , the set of points $(x, x/\varepsilon) \in \mathbb{R}^{2N}$ consists of a countable system of N -dimensional “segments”

in $\Omega \times Y$. Even if a countable sequence of periods ε_k is taken, it is still a set of measure zero in $\Omega \times Y$. But the elements of $L^p(\Omega \times Y)$ are classes of functions which may differ on zero measure subsets. Thus some continuity of test functions must be assumed.

The problem of choosing the optimal space of test functions has not been satisfactorily solved yet. Let us mention that if it is too small, e.g. $\mathcal{V} = C_0^\infty(\Omega \times Y)$, then boundedness of u_ε in $L^p(\Omega)$ must be added, otherwise even an unbounded sequence is admitted. For the test functions the Carathéodory conditions are often assumed and the following two conditions are added:

$$\left\| \varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{L^p(\Omega)} \leq \|\varphi\|_{\mathcal{V}}, \quad \left\| \varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{L^p(\Omega)} \longrightarrow \frac{1}{|Y|^{1/p}} \|v\|_{L^p(\Omega \times Y)}.$$

The sequence $\{u^\varepsilon\}$ is often supposed to be bounded in the L^p -norm.

4. THE “ADJOINT” DEFINITION USING UNFOLDING OPERATOR

The alternative approach is based on the unfolding or the so-called two-scale transform which removes difficulties with the space of test functions: instead of transforming the two-variable test function $\varphi(x, y)$ to the one-variable function $\varphi(x, x/\varepsilon)$ the one-variable members $u^\varepsilon(x)$ of the sequence are transformed into two-variable functions $\widehat{u}^\varepsilon(x, y)$ and the limit is tested in $L^p(\Omega \times Y)$:

u^ε two-scale converges to u^0 if \widehat{u}^ε converges to u^0 in $L^p(\Omega \times Y)$ weakly.

Thus both the limit u^0 and the test function φ can be taken from the maximal spaces: u^0 in $L^p(\Omega \times Y)$ and φ in its dual space $L^q(\Omega \times Y)$. We need not take care of the space \mathcal{V} , of admissibility and compatibility of the test functions as in the classical definition. It also enables us to introduce a natural definition of strong two-scale convergence: if \widehat{u}^ε converge to u^0 in $L^p(\Omega \times Y)$ strongly.

The unfolding operator appeared in [2], where it was called the dilation operator and was used for homogenization of periodic porous materials, and later e.g. in [4]. The notion of two-scale convergence based on periodic unfolding appeared in [5] and [11]. It was further developed in [7] and [6].

The unfolding operator \mathcal{T}_ε converts a single variable function u on Ω into a two-variable function \widehat{u} on $\Omega \times Y$. Using the decomposition (2.1) of $y \in \mathbb{R}^N$ into the integral and fractional parts $y = [y] + \{y\}$ and its scaling, the mapping $t_\varepsilon: \mathbb{R}^N \times Y \rightarrow \mathbb{R}^N$ given by

$$(4.1) \quad t_\varepsilon(x, y) = \varepsilon \left[\frac{x}{\varepsilon} \right] + \varepsilon y \quad x \in \Omega = \mathbb{R}^N, \quad y \in Y$$

will be used for introducing the corresponding unfolding operator \mathcal{T}_ε :

$$(4.2) \quad \mathcal{T}_\varepsilon(u)(x, y) = u(t_\varepsilon(x, y)) \quad (x, y) \in \Omega \times Y.$$

The mapping t_ε conserves the Lebesgue measure as follows: for a bounded measurable set $M \in \mathbb{R}^N$ the measure of M equals to measure of the inverse image $t_\varepsilon^{-1}(M)$ divided by the measure of the cell Y :

$$|M| = \frac{1}{|Y|} |t_\varepsilon^{-1}(M)|.$$

Thus the case when $\partial\Omega$ does not intersect the interior of the scaled cells Y_ξ^ε , i.e. the union of the boundary cell Λ_ε is empty, the unfolding operator \mathcal{T}_ε conserves the integral, i.e. the identity (1.2).

Nevertheless, the equality (1.2) with (4.2), as is stated in Proposition 1 of [5], is not true in the case of a general domain when Λ_ε has positive measure. Then even it does not map $L^p(\Omega)$ into $L^p(\Omega \times Y)$. This is caused by the uncomplete boundary cells $\widetilde{Y}_\xi^\varepsilon$ of Λ_ε . In the paper [6] the definition (4.2) of the unfolding operator \mathcal{T}_ε was changed in the uncomplete boundary cells (for notation see Section 2):

$$(4.3) \quad \mathcal{T}_\varepsilon(u)(x, y) = \begin{cases} u(t_\varepsilon(x, y)) & \text{for } x \text{ in inner cells } Y_\xi^\varepsilon, \\ 0 & \text{for } x \text{ in boundary cells } \widetilde{Y}_\xi^\varepsilon. \end{cases}$$

The desired equality (1.2), which was true neither by unfolding defined neither by (4.1), (4.2) nor by (4.1), (4.3), was replaced by an ‘‘equality in the limit’’ which was valid for sequences u^ε satisfying the so-called *unfolding criterion for integrals*, see [6], Proposition 2.6:

If the sequence u^ε satisfies $\int_{\Lambda_\varepsilon} |u^\varepsilon| dx \rightarrow 0$ as $\varepsilon \rightarrow 0$, then

$$(4.4) \quad \int_{\Omega} u^\varepsilon dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(u^\varepsilon) dx dy \rightarrow 0.$$

5. TWO-SCALE CONVERGENCE WITH MODIFIED UNFOLDING

The aim of this paper is to introduce a modified unfolding operator $\mathcal{T}_\varepsilon^*$, see also [8], for which the desired equality (1.2) holds and thus the problems mentioned above disappear. Instead of (4.3), the function u^ε is not transformed in the uncomplete boundary cells:

$$(5.1) \quad \mathcal{T}_\varepsilon^*(u)(x, y) = \begin{cases} u(t_\varepsilon(x, y)) & \text{for } x \text{ in inner cells } Y_\xi^\varepsilon, \\ u(x) & \text{for } x \text{ in boundary cells } \widetilde{Y}_\xi^\varepsilon. \end{cases}$$

This modified unfolding operator conserves integral and even the norm in the case of any domain:

$$(5.2) \quad \int_\Omega u \, dx = \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon^*(u) \, dx \, dy, \quad \|u\|_{L^p(\Omega)} = \frac{1}{|Y|^{1/p}} \|\mathcal{T}_\varepsilon^*(u)\|_{L^p(\Omega \times Y)}.$$

Thus the “dual” definition of two-scale convergence can be written in the following form:

Definition 5.1. Let $\{\varepsilon\}$ be a scale, $\{u^\varepsilon\}$ a sequence in $L^p(\Omega)$, $1 < p < \infty$ and $\mathcal{T}_\varepsilon^*$ the unfolding operator defined by (4.1), (5.1).

(a) We say that the sequence u^ε (*weakly two-scale converges*) in $L^p(\Omega)$ (with respect to the scale $\{\varepsilon\}$) to the limit $u^0(x, y) \in L^p(\Omega \times Y)$ if

$$\widehat{u^\varepsilon} = \mathcal{T}_\varepsilon^*(u^\varepsilon) \text{ converges to } u^0 \text{ in } L^p(\Omega \times Y) \text{ weakly.}$$

(b) We say that the sequence u^ε (*strongly two-scale converges*) in $L^p(\Omega)$ (with respect to the scale $\{\varepsilon\}$) to $u^0(x, y) \in L^p(\Omega \times Y)$ if

$$\widehat{u^\varepsilon} = \mathcal{T}_\varepsilon^*(u^\varepsilon) \text{ converges to } u^0 \text{ in } L^p(\Omega \times Y) \text{ strongly.}$$

Let us mention that it can be proved that all the definitions introduced above are equivalent. The advantage of the modified definition is that due to the integral conserving property the theory can be simplified, the proofs of the next section follow directly from the L^p theory. No unfolding criterion for integrals is necessary, since with the unfolding $\mathcal{T}_\varepsilon^*$ the equality (1.2) holds.

6. SURVEY OF PROPERTIES OF THE TWO-SCALE CONVERGENCE

Let us survey the results which follow from Definition 5.1 and the L^p theory.

Example 6.1. Let $f, g \in L^p(\Omega)$ and let $\psi \in L^\infty(Y_{\text{per}})$ be such that $\int_Y \psi(y) \, dy = 0$. Then the sequence $\{u^\varepsilon\}$ defined by

$$u^\varepsilon(x) = f(x)\psi\left(\frac{x}{\varepsilon}\right) + g(x)$$

is bounded in $L^p(\Omega)$ and two-scale converges both weakly and strongly in $L^p(\Omega)$ with respect to the scale $\{\varepsilon\}$ to the limit

$$u^0(x, y) = f(x)\psi(y) + g(x).$$

In $L^p(\Omega)$ the sequence converges weakly to $g(x)$. The example shows that the local oscillations of u^ε , which are lost in the weak limit, are conserved in the two-scale limit. Nevertheless, if the scale $\{\varepsilon\}$ is not “in resonance” with the period of the sequence, e.g. $u^\varepsilon = f(x)\psi(x/\sqrt{2}\varepsilon) + g(x)$, then u^ε two-scale converge only weakly to the limit $u^0(x, y) = g(x)$, i.e. the local oscillations are also lost in the limit.

Theorem 6.2 (Properties of the two-scale convergences). *Let $\{\varepsilon\}$ be a scale, $\{u^\varepsilon\}$ a sequence in $L^p(\Omega)$ and $u^0 \in L^p(\Omega \times Y)$. Then:*

- (a) *Each weakly or strongly two-scale converging sequence is bounded in $L^p(\Omega)$.*
- (b) *The weak or strong two-scale limit u^0 is unique as an element of $L^p(\Omega \times Y)$.*
- (c) *The weak or strong two-scale convergence of u^ε to $u^0(x, y)$ implies weak convergence in $L^p(\Omega)$ of u^ε to the limit $u^*(x) = |Y|^{-1} \int_Y u^0(x, y) \, dy$.*
- (d) *The relation between the convergences and the two-scale convergences in $L^p(\Omega)$ can be summarized in the following diagram of implications:*

$$\text{strong} \implies \text{strong two-scale} \implies \text{weak two-scale} \implies \text{weak}.$$

Theorem 6.3 (Compactness). *Let $\{\varepsilon\}$ be a scale and $\{u^\varepsilon\}$ a bounded sequence in $L^p(\Omega)$. Then there exist a subscale $\{\varepsilon'\} \subset \{\varepsilon\}$ and a limit $u^0 \in L^p(\Omega \times Y)$ such that $u^{\varepsilon'}$ (weakly) two-scale converge to u^0 with respect to the subscale $\{\varepsilon'\}$.*

Theorem 6.4 (Convergence result). *Let a sequence $\{u^\varepsilon\}$ strongly two-scale converge to u^0 and a sequence $\{v^\varepsilon\}$ two-scale converge to v^0 , both with respect to the same scale $\{\varepsilon\}$, the former in $L^p(\Omega)$ and the latter in $L^q(\Omega)$. The exponents $p, q, r \geq 1$ are supposed to satisfy $1/p + 1/q = 1/r < 1$. Then the product $u^\varepsilon v^\varepsilon$ two-scale converges to the limit $u^0 v^0 \equiv u^0(x, y)v^0(x, y)$ in $L^r(\Omega)$.*

In particular, for any $\varphi \in L^s(\Omega)$ with $s \in \langle 1, \infty \rangle$ satisfying $1/p + 1/q + 1/s = 1$ we have

$$(6.1) \quad \int_{\Omega} u^\varepsilon(x)v^\varepsilon(x)\varphi(x) \, dx \longrightarrow \iint_{\Omega \times Y} u^0(x, y)v^0(x, y)\varphi(x) \, dx \, dy.$$

Remark 6.5. The last result enables us in many cases to solve the problem introduced in the introduction: to pass to the limit of the product of two weakly converging sequences if one of them is strongly two-scale converging.

Let us sketch passing to the limit in the homogenization problem (1.1). Since the sequence of solutions u^ε is bounded in $W_0^{1,2}(\Omega)$, components of its gradient ∇u^ε are also bounded in $L^2(\Omega)$ and due to Theorem 6.3 it contains a subsequence $(\nabla u^{\varepsilon'})_i$ weakly two-scale converging with respect to the subscale $\{\varepsilon'\}$. Passing to the limit as $\varepsilon' \rightarrow 0$ in (1.1) is possible due to Theorem 6.4 since a^ε is converging two-scale strongly to $a(y)$. If we prove that the limit is unique, the whole sequence converges.

7. CONCLUSION

The modified unfolding operator \mathcal{T}^* enables us to introduce a natural definition of the weak and the strong two-scale convergence, this Definition 5.1 is equivalent to the previous ones. It simplifies the proofs of theorems on the two-scale convergence. In many cases it enables to pass to the limits of product of weakly converging sequences, see Theorem 6.4, particularly in the homogenization problems, where the coefficients strongly two-scale converge and the bounded solutions weakly two-scale converge. The modified definition simplifies proofs of the fundamental theorems of Section 6.

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