Jianqing Chen; Eugénio M. Rocha

Positive solutions for elliptic problems with critical nonlinearity and combined singularity

Mathematica Bohemica, Vol. 135 (2010), No. 4, 413--422

Persistent URL: http://dml.cz/dmlcz/140832

POSITIVE SOLUTIONS FOR ELLIPTIC PROBLEMS WITH CRITICAL NONLINEARITY AND COMBINED SINGULARITY

Jianqing Chen, Fuzhou, Eugénio M. Rocha, Aveiro

(Received October 15, 2009)

Abstract. Consider a class of elliptic equation of the form

$$-\Delta u-\frac{\lambda}{|x|^2}u=u^{2^*-1}+\mu u^{-q}\quad\text{in }\Omega\setminus\{0\}$$

with homogeneous Dirichlet boundary conditions, where $0\in\Omega\subset\mathbb{R}^N\,(N\geqslant 3)$, $0<q<1$, $0<\lambda<(N-2)^2/4$ and $2^*=2N/(N-2)$. We use variational methods to prove that for suitable μ, the problem has at least two positive weak solutions.

Keywords: multiple positive solutions, singular nonlinearity, critical nonlinearity, Hardy term

MSC 2010: 35J20, 35J65

1. Introduction

In this note we study the existence of multiple positive weak solutions of the equation

$$(P_{\lambda,\mu})\qquad \begin{cases}-\Delta u-\dfrac{\lambda}{|x|^2}u=u^{2^*-1}+\mu u^{-q} & \text{in }\Omega\setminus\{0\},\\ u(x)>0\quad\text{in}\quad\Omega\setminus\{0\},\quad u(x)=0 & \text{on }\partial\Omega,\end{cases}$$

where $0\in\Omega$ and $\Omega\subset\mathbb{R}^N(N\geqslant 3)$ is a bounded domain with smooth boundary, $2^*=2N/(N-2)$ is the critical Sobolev exponent, $0<\lambda<\Lambda=((N-2)/2)^2$ and $0<q<1$. We say $u\in H_0^1(\Omega)$ is a weak solution of $(P_{\lambda,\mu})$ if for any $\varphi\in H_0^1(\Omega)$, we have

$$\int\Big(\nabla u\nabla\varphi-\frac{\lambda}{|x|^2}u\varphi-\mu u^{-q}\varphi-|u|^{2^*-2}u\varphi\Big)=0.$$

Due to the Sobolev embedding theorem and the Hardy inequality (for any $u \in H_0^1(\Omega)$, $\int_\Omega |x|^{-2}|u|^2\,\mathrm{d}x \leqslant \Lambda^{-1}|\nabla u|^2$), $(P_{\lambda,\mu})$ is variational in nature. Finding weak solutions of $(P_{\lambda,\mu})$ is equivalent to seeking critical points of the functional

$$I(u) = \frac{1}{2}\int \Big(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\Big) - \frac{\mu}{1-q}\int |u|^{1-q} - \frac{1}{2^*}\int |u|^{2^*}, \quad u \in H_0^1(\Omega)).$$

Problems like $(P_{\lambda,\mu})$ have attracted great interests in the last two decades. When $\lambda = 0$ and u^{2^*-1} is replaced by u^p with $1 < p < 2^* - 1$, Coclite et al. [6] proved that there is μ_1 such that the problem has at least one positive solution for $0 < \mu < \mu_1$ and has no positive solution for $\mu > \mu_1$. Sun et al. [8] proved the existence of two positive solutions if $0 < q < 1$, $\lambda = 0$, $\mu > 0$ suitably small and u^{2^*-1} replaced by u^p with $1 < p < 2^* - 1$. Hirano et al. [7] proved that there is $\mu_2 > 0$ such that the problem has at least two positive solutions in the case $0 < q < 1$, $\lambda = 0$ and $0 < \mu < \mu_2$. The purpose here is to get two positive solutions of $(P_{\lambda,\mu})$ for $\lambda \neq 0$. Our main result is

Theorem 1.1. *Let $0 < \lambda < \Lambda$ and $0 < q < 1$. Then there is $\mu_* > 0$ such that for any $\mu \in (0, \mu_*)$, $(P_{\lambda,\mu})$ possesses at least two positive solutions.*

To get the existence of multiple solutions, we use variational methods. Comparing $(P_{\lambda,\mu})$ with the previous works [6], [8], [7], we are facing three difficulties at the same time: (1) because of the critical nonlinearity u^{2^*-1}, the functional I does not satisfy a global Palais-Smale ((PS) in short) conditions; (2) since $(P_{\lambda,\mu})$ contains a Hardy term, we know that the solution does not belong to $L^\infty(\Omega)$; and (3) the functional I is not differentiable due to the singular nonlinearity u^{-q}. We need to use the methods recently developed in [4], [5] and some ideas of [1], [7] to overcome them.

The paper is organized as follows: in Section 2, we give some preliminaries; in Section 3, we prove Theorem 1.1.

Throughout this paper $\int_\Omega \cdot\,\mathrm{d}x$ is simply denoted by $\int \cdot$; $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|\cdot\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int |\cdot|^2$; and $H_0^1(\Omega)$ is the standard Sobolev space with the usual norm.

2. Preliminaries

The following proposition was taken from [3], [9] and will play an important role in what follows.

Proposition 2.1. *For $0 < \lambda < \Lambda = (N-2)^2/4$, equation*

$$-\Delta u - \frac{\lambda}{|x|^2}u = |u|^{2^*-2}u, \qquad x \in \mathbb{R}^N \setminus \{0\},\ u(x) \to 0 \text{ as } |x| \to +\infty, \tag{2.1}$$

has a family of solutions

$$U_\varepsilon(x) = \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{(N-2)/4}}{[\varepsilon|x|^{\gamma'/\sqrt{\Lambda}} + |x|^{\gamma/\sqrt{\Lambda}}]^{(N-2)/2}}, \quad \varepsilon > 0,$$

where $\Lambda = (\frac{1}{2}(N-2))^2$, $\gamma' = \sqrt{\Lambda} - \sqrt{\Lambda-\lambda}$, $\gamma = \sqrt{\Lambda} + \sqrt{\Lambda-\lambda}$. Moreover, $U_\varepsilon(x)$ is the unique positive radial symmetric solution of Eq. (2.1) up to a dilation, and $U_\varepsilon(x)$ is the extremal function of the minimization problem

$$S_\lambda = \inf\left\{\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2\right)\mathrm{d}x\colon\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^{2^*}\,\mathrm{d}x = 1\right\}.$$

Clearly,

$$\int_{\mathbb{R}^N}|U_\varepsilon(x)|^{2^*}\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(|\nabla U_\varepsilon|^2 - \frac{\lambda}{|x|^2}U_\varepsilon^2\right)\mathrm{d}x = S_\lambda^{N/2}.$$

According to the proof of [4, Theorem 1.1], we have the following exact local behavior of the solutions of $(P_{\lambda,\mu})$.

Proposition 2.2. *Let $0 < \lambda < \Lambda$. If $u \in H_0^1(\Omega)$ is a positive solution of $(P_{\lambda,\mu})$, then*

$$K_2|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})} \leqslant |u(x)| \leqslant K_1|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}, \qquad x \in B(0,r)\setminus\{0\} \tag{2.2}$$

for $r > 0$ sufficiently small and some positive constants K_1, K_2.

Define a cut-off function $\zeta(x) = 1$ if $|x| \leqslant \delta$, $\zeta(x) = 0$ if $|x| \geqslant 2\delta$, $\zeta(x) \in C_0^1(\Omega)$ and $|\zeta(x)| \leqslant 1$, $|\nabla\zeta(x)| \leqslant C$. Denote $v_\varepsilon(x) = \zeta(x)U_\varepsilon(x)$. Then using an argument similar to [5, Proposition 2.4], we have the following lemma.

Lemma 2.1. *If $u \in H_0^1(\Omega)$ is a positive solution of $(P_{\lambda,\mu})$, then for $\varepsilon > 0$ sufficiently small,*

$$\int u^{2^*-1}v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}}), \qquad \int u v_\varepsilon^{2^*-1} = O(\varepsilon^{\frac{N-2}{4}}).$$

Next, we define some Nehari type sets, which are relevant in getting multiple positive solutions. Denote $\|u\|_\lambda^2 = \int(|\nabla u|^2 - \lambda|x|^{-2}u^2)$ and set

$$\begin{aligned}\mathcal{M} &:= \Big\{u \in H_0^1(\Omega)\colon\ \|u\|_\lambda^2 = \mu\int|u|^{1-q} + \int|u|^{2^*}\Big\},\\ \mathcal{M}^+ &:= \Big\{u \in \mathcal{M}\colon\ (1+q)\|u\|_\lambda^2 > (2^*-1+q)\int|u|^{2^*}\Big\},\\ \mathcal{M}^0 &:= \Big\{u \in \mathcal{M}\colon\ (1+q)\|u\|_\lambda^2 = (2^*-1+q)\int|u|^{2^*}\Big\} \quad\text{and}\\ \mathcal{M}^- &:= \Big\{u \in \mathcal{M}\colon\ (1+q)\|u\|_\lambda^2 < (2^*-1+q)\int|u|^{2^*}\Big\}.\end{aligned}$$

Define also the minimization problems

$$d_+ = \inf_{u\in\mathcal{M}^+} I(u). \tag{2.3}$$

It is easy to see that $d_+ < 0$ for $\mu > 0$ and $d_+ \to 0$ as $\mu \to 0$. Take $\mu_3 > 0$ such that $d_+ + N^{-1}S_\lambda^{N/2} > 0$ for any $\mu \in (0, \mu_3)$. Denote

$$\mu_4 = \frac{2^*-2}{2^*-1+q}\Big(\frac{1+q}{2^*-1-q}\Big)^{\frac{N-2}{4}(1+q)} S_\lambda^{\frac{N}{4}(1+q)+\frac{1-q}{2}}|\Omega|^{\frac{1-q-2^*}{2^*}}.$$

Set

$$\mu_* = \min\{\mu_3,\ \mu_4\}.$$

Lemma 2.2. *If $\mu \in (0, \mu_*)$, then $\mathcal{M}^0 = \{0\}$. Moreover, for any $u \neq 0$ there exists a unique $t^+ = t^+(u) > 0$ such that $t^+(u)u \in \mathcal{M}^-$ and*

$$t^+ > T_m := \Big(\frac{\|u\|_\lambda^2}{(2^*-1)\int|u|^{2^*}}\Big)^{\frac{1}{2^*-2}}$$

and

$$I(t^+u) = \max_{t\geqslant T_m} I(tu),$$

and there exists a unique $t^- = t^-(u) > 0$ such that $t^-(u)u \in \mathcal{M}^+$, $t^- < T_{\max}$ and

$$I(t^-u) = \inf_{0\leqslant t\leqslant T_m} I(tu).$$

P r o o f. The proof is similar to [5, Lemma 3.2]. We omit the details. □

3. Proof of theorem 1.1

In this section we will prove Theorem 1.1. The proof of Theorem 1.1 is based on solving the minimization problem (2.3) and the minimization problem

$$d_- = \inf_{u \in \mathcal{M}^-} I(u). \tag{3.1}$$

We divide the proof into two steps. In the first step, we prove that if there is $w \in \mathcal{M}^+$ such that $d_+ = I(w)$ and there is $v \in \mathcal{M}^-$ such that $d_- = I(v)$, then w and v are two positive weak solutions of $(P_{\lambda,\mu})$. In the second step, we prove that the minima d_+ in (2.3) and d_- in (3.1) are achieved, respectively.

Step 1. Let $w \in \mathcal{M}^+$ be such that $d_+ = I(w)$ and $v \in \mathcal{M}^-$ such that $d_- = I(v)$.

Lemma 3.1. *For each $\varphi \in H_0^1(\Omega)$ and $\varphi \geqslant 0$, we have*

(i) *there is $\varrho_0 > 0$ such that $I(w + \varrho_0\varphi) \geqslant I(w)$ for each $0 \leqslant \varrho < \varrho_0$;*

(ii) *$t_\varrho^- \to 1$ as $\varrho \to 0+$, where t_ϱ^- is the unique positive number satisfying $t_\varrho^- \times (v + \varrho\varphi) \in \mathcal{M}^-$.*

Proof. The proof follows exactly the scheme in the proof of Lemma 3 in [7]. □

Lemma 3.2. *For each $\varphi \in H_0^1(\Omega)$ and $\varphi \geqslant 0$ we have that $w^{-q}\varphi,\ v^{-q}\varphi \in L^1(\Omega)$. Moreover,*

$$\int \Big(\nabla w \nabla\varphi - \frac{\lambda}{|x|^2} w\varphi - \mu w^{-q}\varphi - w^{2^*-1}\varphi\Big) \geqslant 0 \tag{3.2}$$

and

$$\int \Big(\nabla v \nabla\varphi - \frac{\lambda}{|x|^2} v\varphi - \mu v^{-q}\varphi - v^{2^*-1}\varphi\Big) \geqslant 0. \tag{3.3}$$

In particular, $w,\ v > 0$ a.e. in $\Omega \setminus \{0\}$.

Proof. We only prove (3.2) since the proof of (3.3) is similar. Let $\varphi \geqslant 0$ and $\varepsilon > 0$. By (i) of Lemma 3.1 and simple computations we have that

$$\begin{aligned}\frac{\mu}{1-q}\int \frac{(w+\varepsilon\varphi)^{1-q} - w^{1-q}}{\varepsilon} \leqslant{}& \frac{1}{2\varepsilon}\left(\|w+\varepsilon\varphi\|_\lambda^2 - \|w\|_\lambda^2\right) \\ &- \frac{1}{2^*\varepsilon}\left(|w+\varepsilon\varphi|^{2*} - |w|^{2*}\right).\end{aligned}$$

Since the right hand side of the inequality has a finite limit value as $\varepsilon \downarrow 0$ for each $x \in \Omega \setminus \{0\}$, we conclude $\varepsilon^{-1}((w+\varepsilon\varphi)^{1-q} - w^{1-q})$ increases monotonically as $\varepsilon \downarrow 0$

and

$$\lim_{\varepsilon\downarrow 0}\frac{(w+\varepsilon\varphi)^{1-q}-w^{1-q}}{\varepsilon}=\begin{cases}0 & \text{if } \varphi(x)=0,\\ (1-q)w^{-q}\varphi & \text{if } \varphi(x)>0 \text{ and } w(x)>0,\\ \infty & \text{if } \varphi(x)>0 \text{ and } w(x)=0.\end{cases}$$

The monotone convergence theorem yields $w^{-q}\varphi\in L^1(\Omega)$ and we get (3.2). □

Proposition 3.1. *We have that w and v are positive weak solutions of $(P_{\lambda,\mu})$.*

Proof. We borrow some ideas from [6], [8]. For any $\varphi\in H_0^1(\Omega)$ and $\varrho>0$, we define $\psi=(w+\varrho\varphi)$ and $\psi^+=\max\{\psi,0\}$. Then $\psi^+\in H_0^1(\Omega)$. Since $w\in\mathcal{M}$, we obtain from (3.2) that

$$\begin{aligned}0&\leqslant\int\Big(\nabla w\nabla\psi^+-\frac{\lambda}{|x|^2}w\psi^+-\mu w^{-q}\psi^+-w^{2^*-1}\psi^+\Big)\\&=\int_{[w+\varrho\varphi>0]}\Big(\nabla w\nabla\psi^+-\frac{\lambda}{|x|^2}w\psi^+-\mu w^{-q}\psi^+-w^{2^*-1}\psi^+\Big)\\&=\int\Big(\nabla w\nabla\psi-\frac{\lambda}{|x|^2}w\psi-\mu w^{-q}\psi-w^{2^*-1}\psi\Big)\\&\quad-\int_{[w+\varrho\varphi\leqslant 0]}\Big(\nabla w\nabla\psi^+-\frac{\lambda}{|x|^2}w\psi^+-\mu w^{-q}\psi^+-w^{2^*-1}\psi^+\Big)\\&\leqslant\varrho\int\Big(\nabla w\nabla\varphi-\frac{\lambda}{|x|^2}w\varphi-\mu w^{-q}\varphi-w^{2^*-1}\varphi\Big)-\varrho\int_{[w+\varrho\varphi\leqslant 0]}\nabla w\nabla\varphi.\end{aligned}$$

Dividing by ϱ and letting $\varrho\to 0$, since the measure of $[w+\varrho\varphi\leqslant 0]$ tends to 0 as $\varrho\to 0$, we get that $\int_{[w+\varrho\varphi\leqslant 0]}\nabla w\nabla\varphi\to 0$. Therefore

$$\int\Big(\nabla w\nabla\varphi-\frac{\lambda}{|x|^2}w\varphi-\mu w^{-q}\varphi-w^{2^*-1}\varphi\Big)\geqslant 0.$$

Since φ is arbitrary, we get that w is a solution of $(P_{\lambda,\mu})$. Similarly, we can prove that v is also a solution of $(P_{\lambda,\mu})$. □

Step 2. *The minima d_+ and d_- are achieved.* We only prove that d_- is achieved by some $v\in\mathcal{M}^-$ since proving that d_+ is achieved is similar but quite simpler. Since we are faced with critical nonlinearity and the Hardy term, the functional I does not satisfy (PS) conditions. We need some technique developed in [4], [5] and some ideas from [1], [7] to overcome them. We point out that v_ε and the exact local behavior of w (see Proposition 2.2) play essential roles. From Proposition 2.2, we also know that there is $m>0$ such that $w(x)\geqslant m$ for $x\in\operatorname{supp}w\setminus\{0\}$.

Lemma 3.3. *Under the assumptions of Theorem* 1.1,

$$d_- < I(w) + \frac{1}{N} S_\lambda^{N/2}.$$

P r o o f. First, using an argument similar to the proofs in [7, Lemma 8], we have $t_* > 0$ such that $w + t_* v_\varepsilon \in \mathcal{M}^-$. It remains to prove that

$$\sup\{I(w + tv_\varepsilon)\colon\ t > 0\} < I(w) + \frac{1}{N} S_\lambda^{N/2}. \tag{3.4}$$

Since w is a solution, we obtain by direct computation that

$$\begin{aligned} I(w + tv_\varepsilon) - I(w) &= \frac{t^2}{2} \|v_\varepsilon\|_\lambda^2 + t \int \Big(\nabla w \nabla v_\varepsilon - \frac{\lambda}{|x|^2} w v_\varepsilon \Big) \\ &\quad - \mu \int \Big(\frac{(w + tv_\varepsilon)^{1-q}}{1-q} - \frac{w^{1-q}}{1-q} \Big) - \int \Big(\frac{(w + tv_\varepsilon)^{2^*}}{2^*} - \frac{w^{2^*}}{2^*} \Big) \\ &= \frac{t^2}{2} \|v_\varepsilon\|_\lambda^2 - \mu \int \Big(\frac{(w + tv_\varepsilon)^{1-q}}{1-q} - \frac{w^{1-q}}{1-q} - w^{-q} t v_\varepsilon \Big) \\ &\quad - \int \Big(\frac{(w + tv_\varepsilon)^{2^*}}{2^*} - \frac{w^{2^*}}{2^*} - w^{2^*-1} t v_\varepsilon \Big). \end{aligned}$$

Note that the following inequality (see [7]) holds: there is $\alpha > 0$ and $0 < \delta < N/(N-2)$ such that

$$\mu \Big(\frac{(r+s)^{1-q}}{1-q} - \frac{r^{1-q}}{1-q} - r^{-q} s \Big) \geqslant -\alpha s^\delta \quad \text{for each } r \geqslant m \text{ and } s \geqslant 0.$$

Another useful inequality is: for $r,\ s > 0$ we have

$$\frac{(r+s)^{2^*}}{2^*} - \frac{r^{2^*}}{2^*} - \frac{s^{2^*}}{2^*} - r^{2^*-1} s \geqslant r s^{2^*-1}.$$

Thus we get that

$$I(w + tv_\varepsilon) - I(w) \leqslant \frac{t^2}{2} \|v_\varepsilon\|_\lambda^2 - \frac{t^{2^*}}{2^*} \int |v_\varepsilon|^{2^*} - t^{2^*-1} \int w v_\varepsilon^{2^*-1} + \alpha t^\delta \int v_\varepsilon^\delta.$$

So when $t \to 0$ and $t \to \infty$, then $I(w + tv_\varepsilon) \to 0$. Hence we only consider the right hand side of the above inequality in the case of $t \in [t_0, t_1]$ for some $0 < t_0 < t_1 < \infty$.

Hence, we obtain from Lemma 2.1 that

$$\begin{aligned}\sup_{t>0} I(w+tv_\varepsilon)-I(w) &\leqslant \frac{1}{N}\Big(\int(|\nabla v_\varepsilon|^2-\frac{\lambda}{|x|^2}|v_\varepsilon|^2)\Big)^{\frac{2^*}{2^*-2}} \\ &\quad -\Big(\int|v_\varepsilon|^{2^*}\Big)^{-\frac{2}{2^*-2}} - O(\varepsilon^{\frac{N-2}{4}})+O(\varepsilon^{\frac{N-2}{4}\delta}) \\ &= \frac{1}{N}S_\lambda^{\frac{N}{2}}+O(\varepsilon^{\frac{N-2}{2}})-O(\varepsilon^{\frac{N-2}{4}})+O(\varepsilon^{\frac{N-2}{4}\delta}) \\ &< \frac{1}{N}S_\lambda^{\frac{N}{2}} \quad \text{for} \quad \varepsilon>0 \quad \text{sufficiently small.}\end{aligned}$$

The proof is complete. □

Lemma 3.4. *The minimum d_- in* (3.1) *is achieved by $v\in\mathcal{M}^-$ with $I(v)=d_-$.*

Proof. Let $\{v_n\}_{n\in\mathbb{N}}\subset\mathcal{M}^-$ be such that $I(v_n)\to d_-$. It is easy to see that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. We may assume that $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$. Set $z_n=v_n-v$ and assume that

$$\|z_n\|_\lambda^2\to a^2 \quad \text{and} \quad \int|z_n|^{2^*}\to b^{2^*}.$$

Since $v_n\in\mathcal{M}$, by using the Brezis-Lieb lemma and the Sobolev embedding theorem we get that

$$a^2+\|v\|_\lambda^2=\mu\int|v|^{1-q}+b^{2^*}+\int|v|^{2^*}.$$

We claim that $v\geqslant 0$ and $v\neq 0$. Indeed, if $v=0$, then $a\neq 0$ (since for any $u\in\mathcal{M}^-$, $\|u\|_\lambda$ is bounded away from zero) and this means that

$$d_-=\lim_{n\to\infty}I(v_n)=I(0)+\frac{1}{2}a^2-\frac{b^{2^*}}{2^*}\geqslant\frac{1}{N}S_\lambda^{N/2},$$

which contradicts the previous lemma.

From the assumption on $\mu\in(0,\mu_*)$ we have $0<t^+<T_m<t^-$ such that $t^+v\in\mathcal{M}^+$ and $t^-v\in\mathcal{M}^-$. For $t>0$, we define

$$\eta(t)=\frac{a^2}{2}t^2-\frac{b^{2^*}}{2^*}t^{2^*} \quad \text{and} \quad g(t)=I(tv)+\eta(t).$$

Now, we consider the cases

(i) $t^-<1$;

(ii) $t^-\geqslant 1$ and $b>0$, and

(iii) $t^-\geqslant 1$ and $b=0$.

Case (i). From $t^- < 1$, $g'(1) = 0$ and $g'(t^-) > 0$ we can see that g is increasing on $[t^-, 1]$. Then we have

$$d_- = g(1) > g(t^-) \geqslant I(t^- v) + \frac{(t^-)^2}{2}(a^2 - b^{2^*}) > I(t^- v) \geqslant d_-,$$

which is a contradiction.

Case (ii). We set $T_0 = (a^2/b^{2^*})^{(N-2)/4}$. We know that η attains the unique maximum at T_0 and $\eta(T_0) \geqslant N^{-1} S_\lambda^{N/2}$. Moreover, $\eta'(t) > 0$ for $0 < t < T_0$ and $\eta'(t) < 0$ for $t > T_0$.

By the assumption $\mu \in (0, \mu_*)$, we also know $g(1) \geqslant g(T_0)$. If $T_0 \leqslant 1$, we have

$$d_- = g(1) \geqslant g(T_0) = I(T_0 v) + \eta(T_0) \geqslant I(T_0 v) + \frac{1}{N} S_\lambda^{N/2},$$

which contradicts the previous lemma. Thus we have $T_0 > 1$. By virtue of $g'(t) \leqslant 0$ for $t \geqslant 1$, we obtain $\frac{\partial}{\partial t} I(tv) \leqslant -\eta'(t) \leqslant 0$ for $1 \leqslant t \leqslant T_0$ and

$$d_- = g(1) = I(v) + \frac{1}{2}a^2 - \frac{b^{2^*}}{2^*} \geqslant I(v) + \frac{1}{N} S_\lambda^{N/2},$$

which also contradicts the previous lemma.

Case (iii). If $a \neq 0$, then we obtain from the fact that $v_n \in \mathcal{M}^-$ by some computations that $(\partial/\partial t) I(tv)|_{t=1} < 0$ and $(\partial^2/\partial t^2) I(tv)|_{t=1} < 0$, which contradicts $t^- \geqslant 1$. Thus $a = 0$ and $v_n \to v$ strongly in $H_0^1(\Omega)$. Hence, we have $v \in \mathcal{M}^-$ and $I(v) = d_-$.

The proof of Lemma 3.4 is complete. □

Proof of Theorem 1.1. The proof follows directly from Lemma 3.4 and Proposition 3.1. □

Acknowledgments. The authors acknowledge the partial financial support from the Portuguese Foundation for Science and Technology (FCT) and the research unit *CIDMA*. The first author also acknowledges the partial financial support from NSF of China (No. 10971026) and NSF of Fujian (2009J06001). The second author also acknowledges the support of the project UTAustin/MAT/0035/2008.

References

[1] *A. Ambrosetti, H. Brezis, G. Cerami*: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Functional Analysis *122* (1994), 519–543.

[2] *H. Brezis, E. Lieb*: A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. *88* (1983), 486–490.

[3] *F. Catrina, Z. Q. Wang*: On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal functions. Comm. Pure. Appl. Math. *56* (2001), 229–258.

[4] *J. Chen*: Multiple positive solutions for a class of nonlinear elliptic equation. J. Math. Anal. Appl. *295* (2004), 341–354.

[5] *J. Chen, E. M. Rocha*: Four solutions of an inhomogeneous elliptic equation with critical exponent and singular term. Nonlinear Anal., Theory Methods Appl. *71* (2009), 4739–4750.

[6] *M. M. Coclite, G. Palmieri*: On a singular nonlinear Dirichlet problem. Comm. Partial Differential Equations *14* (1989), 1315–1327.

[7] *N. Hirano, C. Saccon, N. Shioji*: Existence of multiple positive solutions for singular elliptic problems with a concave and convex nonlinearities. Adv. Differential Equations *9* (2004), 197–220.

[8] *Y. Sun, S. Wu, Y. Long*: Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. J. Differential Equations *176* (2001), 511–531.

[9] *S. Terracini*: On positive entire solutions to a class of equations with singular coefficient and critical exponent. Adv. Differential Equations *1* (1996), 241–264.

Authors' addresses: *Jianqing Chen*, School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, P. R. China, e-mail: jqchen@fjnu.edu.cn; *Eugénio M. Rocha*, Department of Mathematics, University of Aveiro, Campus Universitrio de Santiago, 3810-193 Aveiro, Portugal, e-mail: eugenio@ua.pt.