

Efraim Shmerling

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ASYMPTOTIC STABILITY CONDITION FOR STOCHASTIC  
MARKOVIAN SYSTEMS OF DIFFERENTIAL EQUATIONS

EFRAIM SHMERLING, Ariel

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*Abstract.* Asymptotic stability of the zero solution for stochastic jump parameter systems of differential equations given by  $dX(t) = A(\xi(t))X(t) dt + H(\xi(t))X(t) dw(t)$ , where  $\xi(t)$  is a finite-valued Markov process and  $w(t)$  is a standard Wiener process, is considered. It is proved that the existence of a unique positive solution of the system of coupled Lyapunov matrix equations derived in the paper is a necessary asymptotic stability condition.

*Keywords:* jump parameter system, Markov process, asymptotic stability

*MSC 2010:* 34F05, 47B80, 60H25, 93E03

## 1. INTRODUCTION

In the last decades, there has been a growing number of publications on linear systems subject to abrupt changes in their structure, which are called systems with switching, or jump parameter systems. Most researchers have considered the case where these changes are modeled by continuous or discrete Markov processes. These publications are motivated by a broad field of applications of jump parameter systems in various areas of science and technology, for instance, robotic manipulation, aircraft control, flexible structures for space stations.

Stability conditions and optimal control for Markovian linear systems that do not incorporate additive disturbances are considered in [1]–[4], [6]–[8], [10]–[13], [15], [16] among numerous other publications.

More complicated stochastic systems of differential and difference equations with Markovian and semi-Markovian parameter jumps, which incorporate additive disturbances characterized by a Wiener process, have been considered in some recent publications ([5], [9], [14], [17]), and many important results have been achieved. Among them are stability conditions for the system  $dX(t) = A(\xi(t))X(t) dt + H(\xi(t)) dw(t)$ ,

where  $\xi(t)$  is a finite-valued Markov process and  $w(t)$  is a standard Wiener process, which were derived in [9]. In this paper we examine a stochastic linear Markovian jump parameter system in which additive disturbances are incorporated in a different, more complicated form.

## 2. FORMULATION AND PROOF OF MAIN RESULT

We consider the Markovian system of linear stochastic differential equations

$$(2.1) \quad dX(t) = A(\xi(t))X(t) dt + H(\xi(t))X(t) dw(t),$$

where  $\xi(t)$  is a finite-valued Markov process that takes values  $\xi_k$  ( $k = 1, \dots, m$ ) with probabilities

$$(2.2) \quad p_k(t) = P\{\xi(t) = \xi_k\} \quad (k = 1, \dots, m).$$

Assume that the probabilities satisfy the system of linear differential equations

$$(2.3) \quad \frac{dp_s(t)}{dt} = \sum_{k=1}^n a_{sk} p_s(t) \quad (s = 1, \dots, n),$$

where the constant coefficients  $a_{sk}$  ( $k, s = 1, \dots, n$ ) satisfy the conditions  $a_{sk} \geq 0$  ( $k \neq s$ ),  $a_{ss} \leq 0$  ( $s = 1, \dots, n$ ) and

$$(2.4) \quad \sum_{k=1}^n a_{sk} = 0 \quad (s = 1, \dots, n),$$

$dw(t)$  is the standard Wiener process which satisfies the conditions

$$(2.5) \quad \begin{aligned} \langle dw(t) \rangle &\equiv 0, \\ \langle dw(t) \cdot dw(t) \rangle &= dt. \end{aligned}$$

Here  $\langle \cdot \rangle$  designates mathematical expectation. The zero solution  $X(t) \equiv 0$  of the system (2.1) is asymptotically stable in mean square if for any solution  $X(t)$

$$(2.6) \quad \lim_{t \rightarrow \infty} \langle \|X(t)\|^2 \rangle = 0.$$

Here  $\|X\|$  designates the Euclidean norm of a vector  $X$ :  $\|X\|^2 = X^*X$ , where  $*$  is the symbol of transposition. The equality (2.6) is satisfied iff the matrix  $D(t) = \langle X(t)X^*(t) \rangle$  tends to the zero matrix at  $t \rightarrow \infty$ . Since the coefficients of (2.1) do

not depend directly on  $t$ , uniform exponential stability of solutions of (2.1) in mean square follows from (2.6), and the inequality

$$(2.7) \quad \langle \|X(t)\|^2 \rangle \leq C e^{-\lambda(t-t_0)} \leq \langle \|X(t_0)\|^2 \rangle \quad (t \geq t_0)$$

is satisfied for some constants  $C \geq 1$ ,  $\lambda > 0$ . If (2.1) is asymptotically stable in mean square, the improper integral

$$(2.8) \quad J = \int_0^\infty \langle \|X(t)\|^2 \rangle dt$$

converges for any solution of the system. Let  $B_s$ ,  $s = 1, \dots, m$  be some positive symmetric matrices, and let  $\omega(X, \xi_s) = X^* B_s X$  be the corresponding quadratic forms for any vector  $X \neq 0$ . Let us introduce the Lyapunov functions

$$(2.9) \quad v_s(t, X) = \int_t^{+\infty} \langle \omega(X(y), \xi(y)) \mid X(t) = X, \xi(t) = \xi_s \rangle dy \quad (s = 1, \dots, n).$$

Here  $\langle \cdot | \cdot \rangle$  designates conditional mathematical expectation. It can be easily shown that the convergence of the improper integrals  $v_s(t, X)$ ,  $s = 1, \dots, n$  is equivalent to the convergence of (2.7), and therefore it is a necessary and sufficient condition for mean square asymptotic stability of the zero solution of (2.1). Let us find a necessary condition for the convergence of the improper integrals  $v_s(t, X)$ ,  $s = 1, \dots, n$ .

Assume that the integrals converge. They do not depend on  $t$ , and therefore can be expressed in the form

$$(2.10) \quad v_s(t, X) = X^* C_s X, \quad C_s > 0 \quad (s = 1, \dots, m).$$

Let us derive a system of matrix equations for defining the matrices  $C_s$  ( $s = 1, \dots, m$ ). Let  $\Delta t = h > 0$  be a small increment of the argument  $t$ . If the Markov process stays at  $\xi_s$  during the time interval  $[t; t+h]$ , we can utilize the equality

$$(2.11) \quad X(t+h) = X + A_s X h + H_s X dw(t),$$

where  $A_s$  designates  $A(\xi_s)$ ,  $H_s$  designates  $H(\xi_s)$ . In further discussion we will also utilize the well-known equalities for conditional probabilities:

$$(2.12) \quad \begin{aligned} P\{\xi(t+h) = \xi_s \mid \xi(t) = \xi_s\} &= 1 + h a_{ss} + O(h^2), \quad s = 1, \dots, m, \\ P\{\xi(t+h) = \xi_k \mid \xi(t) = \xi_s\} &= h a_{sk} + O(h^2), \quad k, s = 1, \dots, m, \quad k \neq s. \end{aligned}$$

The following equality follows from (2.11) and (2.12):

$$(2.13) \quad v_s(t, X) = hX^*B_sX + v_s(t + h, X + A_sXh + H_sX \, dw(t)) \\ + h \sum_{k=1}^m a_{sk}v_k(t + h, X) + O(h^2).$$

We derive the auxiliary equality

$$(2.14) \quad v_s(t + h, X + A_sXh + H_sX \, dw(t)) \\ = (X^* + X^*A_s^*h + X^*H_s^* \, dw(t))C_s(X + A_sXh + H_sX \, dw(t)) \\ = X^*C_sX + X^*A_s^*C_sXh + X^*C_sA_sXh \\ + X^*H_s^*C_sH_sXh + O(h^2), \quad s = 1, \dots, m.$$

In view of (2.14), (2.13) takes the form

$$v_s(t, X) = X^*C_sX = hX^*B_sX + X^*C_sX + hX^*C_sA_sX \\ + hX^*A_s^*C_sX + hX^*H_s^*C_sH_sX \\ + h \sum_{k=1}^m a_{sk}X^*C_kX + O(h^2), \quad s = 1, \dots, m.$$

Letting  $h$  tend to zero, we obtain the system of matrix equations

$$(2.15) \quad C_sA_s + A_s^*C_s + \sum_{k=1}^m a_{sk}C_k + H_s^*C_sH_s + B_s = 0 \quad (s = 1, \dots, m)$$

We have proved the following: if the improper integrals  $v_s(t, X)$  ( $s = 1, \dots, m$ ) converge, then the corresponding matrices  $C_s$  ( $s = 1, \dots, m$ ) satisfy the system of equations (2.15), which means that it has a unique positive solution. Let us formulate the obtained result as a theorem.

**Theorem 2.1.** *The existence of a unique solution  $C_s > 0$  ( $s = 1, \dots, m$ ) of the system (15) at some  $B_s > 0$  ( $s = 1, \dots, m$ ) is a necessary condition for mean square asymptotic stability of the zero solution of (1).*

### 3. CONCLUDING REMARK

The derived system of coupled Lyapunov matrix equations for a stochastic Markovian system of differential equations can be viewed as a generalization of the well-known Markovian system of coupled Lyapunov matrix equations

$$(3.1) \quad C_s A_s + A_s^* C_s + \sum_{k=1}^n a_{sk} C_k + B_s = 0, \quad B_s > 0, \quad s = 1, \dots, m$$

for a Markovian system without white noise  $dX(t) = A(\xi(t)) dt$  and as a generalization of the Lyapunov matrix equation

$$(3.2) \quad CA + A^*C + H^*CH + B = 0, \quad B > 0$$

for a non-jump stochastic system of differential equations

$$(3.3) \quad dX(t) = AX(t) + HX(t) dw(t).$$

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*Author's address: Efraim Shmerling, Ariel University Center of Samaria, Ariel, Israel, e-mail: efrainsh@ariel.ac.il, efrainsh@yahoo.com.*