Applications of Mathematics

Eduard Feireisl; Šárka Matušů-Nečasová

The effective boundary conditions for vector fields on domains with rough boundaries: Applications to fluid mechanics

Applications of Mathematics, Vol. 56 (2011), No. 1, 39--49

Persistent URL: http://dml.cz/dmlcz/141405

THE EFFECTIVE BOUNDARY CONDITIONS FOR VECTOR FIELDS ON DOMAINS WITH ROUGH BOUNDARIES: APPLICATIONS TO FLUID MECHANICS*

Eduard Feireisl, Šárka Nečasová, Praha

Dedicated to Professor K. R. Rajagopal on the occasion of his 60th birthday

Abstract. The Navier-Stokes system is studied on a family of domains with rough boundaries formed by oscillating riblets. Assuming the complete slip boundary conditions we identify the limit system, in particular, we show that the limit velocity field satisfies boundary conditions of a mixed type depending on the characteristic direction of the riblets.

Keywords: Navier-Stokes system, rough boundary, slip boundary condition

MSC 2010: 35Q35

1. Introduction

Consider a viscous incompressible fluid occupying a bounded domain $\Omega \subset \mathbb{R}^3$. In the Eulerian reference system, the motion of the fluid is completely determined by the velocity field $\mathbf{u} = \mathbf{u}(t, x)$—a vector valued function of the time t and the spatial position $x \in \Omega$. Under the hypothesis of impermeability of the boundary, the velocity satisfies

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1}$$

* The work of E. Feireisl was supported by Grant 201/05/0164 GA CR, moreover the work of E. Feireisl was supported in the framework of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503. The work of Š. Nečasová was supported by Grant No. IAA10019505 and the final version by No. IAA100190804 of GA AS CR and in the framework of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

where the symbol $\mathbf{n}$ stands for the outer normal vector. In addition to (1), the widely accepted hypothesis asserts there is no relative motion between the *viscous* fluid and the rigid wall represented by $\partial\Omega$, meaning

$$[\mathbf{u}]_\tau|_{\partial\Omega} = 0, \tag{2}$$

where $[\mathbf{u}]_\tau$ denotes the tangential component of $\mathbf{u}$. The *no-slip* boundary conditions (1), (2) are the most frequently accepted because of their enormous success in reproducing the velocity profiles for macroscopic flows.

There have been several attempts to justify the no-slip boundary conditions as an inevitable consequence of fluid trapping by surface roughness (see Amirat et al. [1], Casado-Díaz et al. [6]). On the other hand, in order to simplify the complicated description of the fluid behavior in a boundary layer, the Navier boundary conditions or other so-called wall laws have been used instead of (2) to facilitate numerical computations (see Jaeger and Mikelić [7]).

Following the programme originated in [2], [4] we consider a family of bounded domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$,

$$\Omega_\varepsilon = \{(x_1, x_2, x_3)\colon\ (x_1, x_2) \in \mathcal{T}^2,\ 0 < x_3 < 1 + \Phi_\varepsilon(x_1, x_2)\}, \tag{3}$$

where the symbol $\mathcal{T}^2 = ([0,1]|_{\{0,1\}})^2$ denotes the two-dimensional torus. In other words, all quantities defined on Ω_ε are periodic with respect to the "horizontal" variables (x_1, x_2). We assume that the functions Φ_ε depend only on a single spatial variable, say, $\Phi_\varepsilon = \Phi_\varepsilon(x_1)$, mimicking a ribbed surface, where the amplitude as well as a typical wavelength of oscillations are small for ε approaching zero.

We suppose that the time evolution of the fluid velocity is governed by the Navier-Stokes system

$$\operatorname{div}_x \mathbf{u} = 0 \quad \text{in } (0,T) \times \Omega_\varepsilon, \tag{4}$$

$$\partial_t \mathbf{u} + \operatorname{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x P = \operatorname{div}_x \mathbb{S} \quad \text{in } (0,T) \times \Omega_\varepsilon, \tag{5}$$

where P is the pressure and the viscous stress tensor $\mathbb{S}$ is given by the classical Newton's rheological law

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) \tag{6}$$

with a constant viscosity coefficient $\mu > 0$. System (4)–(6) is supplemented with the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\{x_3=0\}} = 0, \quad [\mathbb{S}\mathbf{n}]_\tau|_{\{x_3=0\}} = 0, \tag{7}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\{x_3=1+\Phi_\varepsilon(x_1,x_2)\}} = 0, \quad [\mathbb{S}\mathbf{n}]_\tau|_{\{x_3=1+\Phi_\varepsilon(x_1,x_2)\}} = 0. \tag{8}$$

Following the approach developed in [4] we identify the limit problem associated with (4)–(8) for ε tending to zero. In particular, any accumulation point $\mathbf{u}$ of a family of solutions $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of problem (4)–(8) satisfies (4)–(6) on the limit domain $\Omega = \mathcal{T}^2 \times (0,1)$, together with the complete slip boundary condition (7) on the bottom part of the boundary $\{x_3 = 0\}$. In addition, the limit velocity $\mathbf{u}$ on the upper boundary is parallel to the riblets, specifically,

$$\mathbf{u}|_{\{x_3=1\}} = (0, u_2, 0), \quad \text{and} \quad S_{2,3}|_{\{x_3=1\}} = 0. \tag{9}$$

The main result obtained in this paper can be viewed as an extension of the theory developed in [2] to the time-dependent case. Similarly to [4], the main difficulty is to handle possible oscillations in time of the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ resulting in the lack of compactness of the convective terms $\{\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\}_{\varepsilon>0}$. In order to overcome this stumbling block, we introduce a local pressure in the spirit of Wolf [11] (cf. also Koch and Solonnikov [8]). Although strongly motivated by [11], our construction of the local pressure is different, based on the Riesz transform rather than on the biharmonic decomposition introduced in [11]. The main advantage of our approach lies in the fact that the norm of the local pressure is independent of the parameter ε.

Finally we would like to mention that in paper [3] the complete description of the asymptotic limit by means of Γ-convergence arguments was done, and was identified a general class of boundary conditions.

2. Main result

To begin, let us recall the concept of a *weak solution* to problem (4)–(8).

Definition 2.1. A function $\mathbf{u}_\varepsilon$ is termed a weak solution to problem (4)–(8) if

$$\mathbf{u}_\varepsilon \in L^\infty(0,T; L^2(\Omega_\varepsilon; \mathbb{R}^3)) \cap L^2(0,T; W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)); \tag{10}$$

$$\operatorname{div}_x \mathbf{u}_\varepsilon(t,\cdot) = 0, \quad \mathbf{u}_\varepsilon(t,\cdot) \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0 \quad \text{for a.a. } t \in (0,T); \tag{11}$$

the integral identity

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \partial_t \varphi + \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi + P_\varepsilon \operatorname{div}_x \varphi) \,\mathrm{d}x \,\mathrm{d}t \\ &\qquad = \int_0^T \int_{\Omega_\varepsilon} \mu(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon) : \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{12}$$

holds for a certain $P_\varepsilon \in L^q((0,T)\times\Omega_\varepsilon)$, $q>1$, and any test function $\varphi \in \mathcal{D}((0,T)\times \overline{\Omega}_\varepsilon;\mathbb{R}^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega_\varepsilon}=0$; and the energy inequality

$$\int_{\Omega_\varepsilon} \frac{1}{2}|\mathbf{u}_\varepsilon|^2(\tau)\,\mathrm{d}x + \int_0^\tau\int_{\Omega_\varepsilon}\frac{\mu}{2}|\nabla_x\mathbf{u}_\varepsilon+\nabla_x^t\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant E_{0,\varepsilon} \tag{13}$$

is satisfied for a.a. $\tau\in(0,T)$.

R e m a r k. Note that Definition 2.1 anticipates the existence of the pressure P_ε as an integrable function. On the other hand, the *existence* of weak solutions belonging to the class specified in Definition 2.1 can be established for a fairly general set of initial data by the method developed by Bulíček et al. [5].

Similarly, we introduce the concept of a *weak solution* of the limit problem as follows.

Definition 2.2. We say that a function $\mathbf{u}$ is a weak solution of problem (4)–(7), and (9) if

$$\mathbf{u}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))\cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)); \tag{14}$$

$$\operatorname{div}_x\mathbf{u}(t,\cdot)=0,\quad \mathbf{u}(t,\cdot)\cdot\mathbf{n}|_{\{x_3=0\}}=0 \text{ for a.a. } t\in(0,T), \tag{15}$$

$$u_1|_{\{x_3=1\}}=u_3|_{\{x_3=1\}}=0; \tag{16}$$

and the integral identity

$$\int_0^T\int_\Omega(\mathbf{u}\cdot\partial_t\varphi+(\mathbf{u}\otimes\mathbf{u}):\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \mu(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}):\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{17}$$

holds for any test function $\varphi\in\mathcal{D}((0,T)\times\overline{\Omega};\mathbb{R}^3)$,

$$\operatorname{div}_x\varphi=0,\quad \varphi\cdot\mathbf{n}|_{\{x_3=0\}}=0,\quad \varphi_1|_{\{x_3=1\}}=\varphi_3|_{\{x_3=1\}}=0. \tag{18}$$

At this stage, we are ready to state our main result.

Theorem 2.1. *Let a family of domains* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be given by* (3), *with* $\Phi_\varepsilon=\Phi_\varepsilon(x_1)$ *such that*

$$\Phi_\varepsilon\in W^{1,\infty}(\mathcal{T}^1),\quad \mathcal{T}^1=[0,1]|_{\{0,1\}},\quad 0\leqslant\Phi_\varepsilon\leqslant\varepsilon,\quad |\Phi_\varepsilon'|\leqslant L, \tag{19}$$

$$\liminf_{\varepsilon\to0}\int_a^b|\Phi_\varepsilon'(z)|\,\mathrm{d}z\geqslant\lambda|a-b| \quad \text{for arbitrary } a\leqslant b,\ a,b\in\mathcal{T}^1, \tag{20}$$

for a certain $\lambda>0$.

Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions of problem* (4)–(8) *in the sense of Definition* 2.1 *such that*

$$\sup_{\varepsilon>0} E_{0,\varepsilon} = \overline{E} < \infty. \tag{21}$$

Then, passing to a subsequence as the case may be, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \textit{weakly-(*) in} \ L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{22}$$

where $\mathbf{u}$ *is a weak solution of problem* (4)–(7), (9) *in the sense specified in Definition* 2.2.

R e m a r k. The non-degeneracy condition (20) is satisfied in a number of particular cases discussed in [2].

The rest of the paper is devoted to the proof of Theorem 2.1.

3. Identifying the limit velocity field

In accordance with the energy inequality (13) and hypothesis (21), we have

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} \leqslant c \tag{23}$$

and

$$\int_0^T \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant c \tag{24}$$

uniformly for $\varepsilon \to 0$.

Estimates (23), (24), together with Korn's inequality, yield

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)} \,\mathrm{d}t \leqslant c. \tag{25}$$

Note that, by virtue of the result of Nitsche [9] and hypothesis (19), the bound established in (25) is independent of ε.

Consequently, in accordance with (23), (25), we can assume

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \text{weakly-(*) in} \ L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \tag{26}$$

passing to suitable subsequences as the case may be. Moreover, it is easy to check that

$$\operatorname{div}_x \mathbf{u} = 0 \quad \text{a.a. in } (0,T) \times \Omega,$$

and

$$\mathbf{u} \cdot \mathbf{n}|_{\{x_3=0\}} = u_3|_{\{x_3=0\}} = 0.$$

Finally, exactly as in [2, Section 3], we can show that hypotheses (19), (20) imply that the limit velocity field $\mathbf{u}$ satisfies

$$u_1|_{\{x_3=1\}} = u_3|_{\{x_3=1\}} = 0.$$

4. Identifying the limit equations

4.1. Pressure

Our ultimate goal is to identify the limit system of equations satisfied by $\mathbf{u}$. Here, the major problem is to control the pressure term P_ε in (12). In general, we do not expect to obtain any uniform bound on $\{P_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$, however, we claim the following result.

Lemma 4.1. *Under the hypotheses of Theorem* 2.1, *there exists a pair of functions* $p_{\mathrm{reg},\varepsilon}$, $p_{\mathrm{harm},\varepsilon}$ *such that*

$$\int_0^T \int_{\Omega_\varepsilon} P_\varepsilon \operatorname{div}_x \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Omega_\varepsilon} (p_{\mathrm{reg},\varepsilon} \operatorname{div}_x \varphi + p_{\mathrm{harm},\varepsilon} \partial_t \operatorname{div}_x \varphi) \,\mathrm{d}x\,\mathrm{d}t \tag{27}$$

for any $\varphi \in \mathcal{D}((0,T) \times \Omega_\varepsilon; \mathbb{R}^3)$, *where*

$$\|p_{\mathrm{reg},\varepsilon}\|_{L^2(0,T;L^{3/2}(\Omega_\varepsilon))} \leqslant c_1(\overline{E}), \tag{28}$$

$$\Delta_x p_{\mathrm{harm},\varepsilon} = 0 \text{ in } \mathcal{D}'((0,T) \times \Omega_\varepsilon), \quad \|p_{\mathrm{harm},\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_\varepsilon;\mathbb{R}^3))} \leqslant c_2(\overline{E}), \tag{29}$$

with the quantities c_1, c_2 *independent of the parameter* ε.

Proof. The "regular" component of the pressure $p_{\mathrm{reg},\varepsilon}$ is uniquely determined as

$$p_{\mathrm{reg},\varepsilon} = -\sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j [1_{\Omega_\varepsilon} T^\varepsilon_{i,j}], \tag{30}$$

where we have set

$$\mathbb{T}^\varepsilon = \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \mu(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon),$$

and where the symbol $\mathcal{R}_j$ stands for the standard Riesz transform in the x_j-variable.

Using the uniform bounds (23), (25) together with the continuity of the Riesz transform in the Lebesgue spaces $L^p(\mathbb{R}^3)$, $1 < p < \infty$, we deduce that $p_{\mathrm{reg},\varepsilon}$ satisfies

$$\|p_{\mathrm{reg},\varepsilon}\|_{L^2(0,T;L^{3/2}(\Omega_\varepsilon))} \leqslant c_1(\overline{E});$$

whence (28) follows. Note that we have used the Sobolev embedding relation $W^{1,2}(\Omega_\varepsilon) \hookrightarrow L^p(\Omega_\varepsilon)$, $1 \leqslant p \leqslant 6$, the norm of which is independent of ε.

As $\mathbf{u}_\varepsilon$ satisfies (12), we have

$$\mathbf{u}_\varepsilon \in C_{\mathrm{weak}}([0,T]; L^2(\Omega_\varepsilon;\mathbb{R}^3)),$$

in particular, it follows from (12) that

$$\int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon(\tau,\cdot) - \mathbf{u}_\varepsilon(0,\cdot)) \cdot \psi \,\mathrm{d}x - \int_{\Omega_\varepsilon} \left(\int_0^\tau \mathbb{T}^\varepsilon \,\mathrm{d}t \right) : \nabla_x \psi \,\mathrm{d}x = 0$$

for all $\psi \in \mathcal{D}(\Omega_\varepsilon;\mathbb{R}^3)$, $\mathrm{div}_x\, \psi = 0$ and all $\tau \in [0,T]$.

Thus, by virtue of Lemma 2.2.1 in Sohr [10], there exists a pressure p_ε such that $\int_{\Omega_\varepsilon} p_\varepsilon(\tau,\cdot)\,\mathrm{d}x = 0$, and

$$\begin{aligned}\int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon(\tau,\cdot) - \mathbf{u}_\varepsilon(0,\cdot)) \cdot \psi \,\mathrm{d}x - \int_{\Omega_\varepsilon} \left(\int_0^\tau \mathbb{T}^\varepsilon \,\mathrm{d}t \right) : \nabla_x \psi \,\mathrm{d}x \\ + \int_{\Omega_\varepsilon} p_\varepsilon(\tau,\cdot)\, \mathrm{div}_x\, \psi \,\mathrm{d}x = 0\end{aligned} \tag{31}$$

for all $\psi \in \mathcal{D}(\Omega_\varepsilon;\mathbb{R}^3)$ and all $\tau \in [0,T]$. Exactly as in Sections 4, 5 in [2], we can deduce from (31) that

$$\sup_{\tau\in[0,T]} \|p_\varepsilon(\tau,\cdot)\|_{L^2(\Omega_\varepsilon)} \leqslant c_2(\overline{E})$$

uniformly with respect to ε.

It follows from (31) that

$$\begin{aligned}\int_0^T \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \partial_t \varphi + (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi + p_\varepsilon \partial_t \,\mathrm{div}_x\, \varphi) \,\mathrm{d}x \,\mathrm{d}t \\ = \int_0^T \int_{\Omega_\varepsilon} \mu(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon) : \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}t\end{aligned}$$

for all $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon;\mathbb{R}^3)$; whence, in accordance with (12),

$$\int_0^T\int_{\Omega_\varepsilon} P_\varepsilon \operatorname{div}_x \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Omega_\varepsilon} p_\varepsilon \partial_t \operatorname{div}_x \varphi \,\mathrm{d}x\,\mathrm{d}t \tag{32}$$

for all $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon;\mathbb{R}^3)$.

Finally, we set

$$p_{\mathrm{harm},\varepsilon}(\tau,\cdot) = p_\varepsilon(\tau,\cdot) - \int_0^\tau \left(p_{\mathrm{reg},\varepsilon} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p_{\mathrm{reg},\varepsilon}\,\mathrm{d}x \right)\mathrm{d}t. \tag{33}$$

As relation (27) follows from (32), it remains to show that $p_{\mathrm{harm},\varepsilon}$ is a harmonic function in the x-variable. In order to see this, we use (30) to obtain

$$\begin{aligned} &\int_{\Omega_\varepsilon} p_{\mathrm{reg},\varepsilon}\Delta\varphi\,\mathrm{d}x \\ &\quad= \int_{\Omega_\varepsilon} [\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon - \mu(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon)] : \nabla_x^2\varphi\,\mathrm{d}x \text{ a.a. in } (0,T) \end{aligned} \tag{34}$$

for any $\varphi \in \mathcal{D}(\Omega_\varepsilon)$. Consequently, taking $\psi = \nabla_x\varphi$ in (31) and comparing the resulting expression with (33), (34) we deduce the desired conclusion

$$\int_{\Omega_\varepsilon} p_{\mathrm{harm},\varepsilon}(\tau,\cdot)\Delta\varphi\,\mathrm{d}x = 0 \quad \text{for all } \varphi\in\mathcal{D}(\Omega_\varepsilon) \text{ and a.a. } \tau\in(0,T).$$

□

4.2. Limit equations

It follows from (27) that the quantities P_ε and $p_{\mathrm{reg},\varepsilon} - \partial_t p_{\mathrm{harm},\varepsilon}$ differ only by a spatially homogeneous time dependent function; in particular, the integral identity (12) can be replaced by

$$\begin{aligned} &\int_0^T\int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon\cdot\partial_t\varphi + \mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi + p_{\mathrm{reg},\varepsilon}\operatorname{div}_x\varphi + p_{\mathrm{harm},\varepsilon}\partial_t\operatorname{div}_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad= \int_0^T\int_{\Omega_\varepsilon} \mu(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon) : \nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{35}$$

to be satisfied for any test function $\varphi \in W_0^{1,\infty}((0,T)\times\overline{\Omega}_\varepsilon;\mathbb{R}^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0$.

Any test function φ for the limit problem in the sense specified in (18) can be extended to $(0,T)\times\Omega$ to be admissible in (35); specifically, we can take φ_1, φ_3 to be zero outside Ω_ε. In particular, taking relation (26) together with the uniform pressure estimates (28), (29) into account, we can let $\varepsilon \to 0$ in (35) in order to conclude that

$$\int_0^T \int_\Omega (\mathbf{u} \cdot \partial_t \varphi + (\overline{\mathbf{u} \otimes \mathbf{u}}) : \nabla_x \varphi) \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \mu (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any test function $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega}; \mathbb{R}^3)$,

$$\mathrm{div}_x \, \varphi = 0, \quad \varphi \cdot \mathbf{n}|_{\{x_3=0\}} = 0, \quad \varphi_1|_{\{x_3=1\}} = \varphi_3|_{\{x_3=1\}} = 0,$$

where the symbol $\overline{\mathbf{u} \otimes \mathbf{u}}$ stands for the weak limit of the sequence $\{\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ in the Lebesgue space $L^{3/2}((0,T) \times \Omega; \mathbb{R}^{3\times 3})$. Consequently, it remains to identify the quantity $\overline{\mathbf{u} \otimes \mathbf{u}}$. This will be done in the last section.

5. Convergence of the convective terms

In order to complete the proof of Theorem 2.1, we have to show that

$$\int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \ \text{ as } \varepsilon \to 0 \tag{36}$$

for any $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega}; \mathbb{R}^3)$,

$$\mathrm{div}_x \, \varphi = 0, \quad \varphi \cdot \mathbf{n}|_{\{x_3=0\}} = 0, \quad \varphi_1|_{\{x_3=1\}} = \varphi_3|_{\{x_3=1\}} = 0.$$

To begin, it is easy to observe that it is enough to show (36) for any $\varphi \in \mathcal{D}((0,T) \times \Omega; \mathbb{R}^3)$, $\mathrm{div}_x \, \varphi = 0$. Indeed we have

$$\int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t = - \int_0^T \int_\Omega \nabla_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t,$$

whence (36) implies

$$\int_0^T \int_\Omega \nabla_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \nabla_x \mathbf{u} \mathbf{u} \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{37}$$

as soon as $\varphi \in \mathcal{D}((0,T) \times \Omega; \mathbb{R}^3)$, $\mathrm{div}_x \, \varphi = 0$. On the other hand, relation (37) is easily extended to $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega}; \mathbb{R}^3)$, $\mathrm{div}_x \, \varphi = 0$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$.

In order to see that (36) holds for any $\varphi \in \mathcal{D}((0,T) \times \Omega; \mathbb{R}^3)$, $\mathrm{div}_x \, \varphi = 0$, we evoke the method developed in [4] based on the pressure decomposition established in Lemma 4.1. The reader may consult [4] for details.

It follows from (35) that

$$\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon} \to \mathbf{u} + \nabla_x p_{\mathrm{harm}} \ \text{ in } \ C_{\mathrm{weak}}([0,T]; L^2(V;\mathbb{R}^3)), \quad V \subset \overline{V} \subset \Omega,$$

where p_{harm} denotes the weak limit of $\{p_{\mathrm{harm},\varepsilon}\}_{\varepsilon>0}$. Here, we have used the fact that the harmonic part of the pressure is smooth in the x-variable on any set $V \subset \overline{V} \subset \Omega$. Consequently, a simple Lions-Aubin type argument yields

$$\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon} \to \mathbf{u} + \nabla_x p_{\mathrm{harm}} \ \text{ in } \ L^2(0,T; L^2(V;\mathbb{R}^3)).$$

Finally, we get

$$\begin{aligned}
\int_0^T \int_\Omega \overline{\mathbf{u}\otimes\mathbf{u}} : \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t &= \lim_{\varepsilon\to 0} \int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\varepsilon\to 0} \int_0^T \int_\Omega ((\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon}) \otimes \mathbf{u}_\varepsilon) : \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \lim_{\varepsilon\to 0} \int_0^T \int_\Omega (\nabla_x p_{\mathrm{harm},\varepsilon} \otimes (\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon})) : \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \lim_{\varepsilon\to 0} \int_0^T \int_\Omega (\nabla_x p_{\mathrm{harm},\varepsilon} \otimes \nabla_x p_{\mathrm{harm},\varepsilon}) : \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_\Omega (\mathbf{u}\otimes\mathbf{u}) : \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

whenever $\varphi \in \mathcal{D}((0,T)\times\Omega;\mathbb{R}^3)$, $\mathrm{div}_x\,\varphi = 0$. Thus we have shown relation (36). The proof of Theorem 2.1 is complete.

References

[1] *Y. Amirat, D. Bresch, J. Lemoine, J. Simon*: Effect of rugosity on a flow governed by stationary Navier-Stokes equations. Q. Appl. Math. *59* (2001), 768–785.

[2] *D. Bucur, E. Feireisl, Š. Nečasová*: On the asymptotic limit of flows past a ribbed boundary. J. Math. Fluid Mech. *10* (2008), 554–568.

[3] *D. Bucur, E. Feireisl, Š. Nečasová*: Boundary behavior of viscous fluids: Influence of wall roughness and friction-driven boundary conditions. Arch. Ration. Mech. Anal. *197* (2010), 117–138.

[4] *D. Bucur, E. Feireisl, Š. Nečasová, J. Wolf*: On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries. J. Differ. Equations *244* (2008), 2890–2908.

[5] *M. Bulíček, J. Málek, K. R. Rajagopal*: Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. Indiana Univ. Math. J. *56* (2007), 51–85.

[6] *J. Casado-Díaz, E. Fernández-Cara, J. Simon*: Why viscous fluids adhere to rugose walls: A mathematical explanation. J. Differ. Equations *189* (2003), 526–537.

[7] *W. Jaeger, A. Mikelić*: On the roughness-induced effective boundary conditions for an incompressible viscous flow. J. Differ. Equations *170* (2001), 96–122.
[8] *H. Koch, V. A. Solonnikov*: L_p estimates for a solution to the nonstationary Stokes equations. J. Math. Sci. *106* (2001), 3042–3072.
[9] *J. A. Nitsche*: On Korn's second inequality. RAIRO, Anal. Numér. *15* (1981), 237–248.
[10] *H. Sohr*: The Navier-Stokes Equations. An Elementary Functional Analytic Approach. Birkhäuser, Basel, 2001.
[11] *J. Wolf*: Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. J. Math. Fluid Mech. *9* (2007), 104–138.

Authors' address: *E. Feireisl*, *Š. Nečasová*, Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `feireisl@math.cas.cz`, `matus@math.cas.cz`.