Applications of Mathematics

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Applications of Mathematics, Vol. 56 (2011), No. 1, 137--160

Persistent URL: http://dml.cz/dmlcz/141410

56 (2011) APPLICATIONS OF MATHEMATICS No. 1, 137–160

WEAK SOLUTIONS FOR STEADY COMPRESSIBLE NAVIER-STOKES-FOURIER SYSTEM IN TWO SPACE DIMENSIONS*

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Dedicated to Professor K. R. Rajagopal on the occasion of his 60th birthday

Abstract. We consider steady compressible Navier-Stokes-Fourier system in a bounded two-dimensional domain. We show the existence of a weak solution for arbitrarily large data for the pressure law $p(\varrho,\vartheta) \sim \varrho^\gamma + \varrho\vartheta$ if $\gamma > 1$ and $p(\varrho,\vartheta) \sim \varrho \ln^\alpha(1+\varrho) + \varrho\vartheta$ if $\gamma = 1$, $\alpha > 0$, depending on the model for the heat flux.

Keywords: steady compressible Navier-Stokes-Fourier system, weak solution, entropy inequality, Orlicz spaces, compensated compactness, renormalized solution

MSC 2010: 76N10, 35Q30

1. Introduction, main result

We study the following system of partial differential equations which models the flow of a compressible heat conducting fluid:

$$\operatorname{div}(\varrho\mathbf{u}) = 0, \tag{1.1}$$

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p = \varrho\mathbf{f}, \tag{1.2}$$

$$\operatorname{div}(\varrho E\mathbf{u}) = \varrho\mathbf{f}\cdot\mathbf{u} - \operatorname{div}(p\mathbf{u}) + \operatorname{div}(\mathbb{S}\mathbf{u}) - \operatorname{div}\mathbf{q}. \tag{1.3}$$

The first equation expresses the balance of mass, the second the balance of momentum and the last one the balance of total energy. Here and in the sequel, the

* The work of M. Pokorný is a part of the research project MSM 0021620839 financed by MŠMT and partly supported by the grant of the Czech Science Foundation No. 201/08/0315 and by the project LC06052 (Jindřich Nečas Center for Mathematical Modeling).

scalar quantity ϱ is the density of the fluid, the vector $\mathbf{u}$ is the velocity field, the tensor $\mathbb{S}$ is the viscous part of the stress tensor, p is the pressure, $\mathbf{f}$ the (given) external force, E the specific total energy and $\mathbf{q}$ the heat flux.

We consider system (1.1)–(1.3) in a bounded domain $\Omega \subset \mathbb{R}^2$ together with the boundary conditions at $\partial\Omega$

$$\mathbf{u} = \mathbf{0}, \tag{1.4}$$

$$-\mathbf{q}\cdot\mathbf{n} + L(\vartheta - \Theta_0) = 0, \tag{1.5}$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega$, Θ_0 and the positive constant L are given. Note that (1.5) expresses the fact that the heat flux through the boundary is proportional to the difference of the temperature ϑ inside and the (known) temperature Θ_0 outside.

Finally, the total mass of the fluid M is given, i.e.

$$\int_\Omega \varrho \,\mathrm{d}x = M > 0. \tag{1.6}$$

In order to complete system (1.1)–(1.3) we have to specify the constitutive laws for the quantities $\mathbb{S}$, p, E, and $\mathbf{q}$. First, the fluid will be assumed newtonian, i.e.

$$\mathbb{S} = \mathbb{S}(\vartheta,\mathbf{u}) = \mu(\vartheta)[\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \operatorname{div}\mathbf{u}\mathbb{I}] + \xi(\vartheta)\operatorname{div}\mathbf{u}\mathbb{I}, \tag{1.7}$$

where $\mu(\vartheta)$ and $\xi(\vartheta)$ are the viscosity coefficients, $\mathbb{I}$ is the unit tensor. We assume the functions to be globally Lipschitz such that

$$c_1(1+\vartheta) \leqslant \mu(\vartheta), \quad 0 \leqslant \xi(\vartheta). \tag{1.8}$$

Due to the global Lipschitz continuity we also have

$$\mu(\vartheta), \xi(\vartheta) \leqslant c_2(1+\vartheta). \tag{1.9}$$

The heat flux obeys the Fourier law

$$\mathbf{q} = \mathbf{q}(\vartheta,\nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{1.10}$$

with $\kappa(\cdot) \in C([0,\infty))$ such that

$$c_3(1+\vartheta^m) \leqslant \kappa(\vartheta) \leqslant c_4(1+\vartheta^m), \tag{1.11}$$

$m > 0$. The specific total energy E has the form

$$E = E(\varrho,\vartheta,\mathbf{u}) = \frac{1}{2}|\mathbf{u}|^2 + e(\varrho,\vartheta) \tag{1.12}$$

with $e(\cdot,\cdot)$ the specific internal energy. Note at this moment that for sufficiently smooth solutions, (1.2)–(1.3) is equivalent to (1.2) and "the balance of internal energy"

$$\operatorname{div}(\varrho e\mathbf{u}) + \operatorname{div}\mathbf{q} = \mathbb{S} : \nabla\mathbf{u} - p\operatorname{div}\mathbf{u}. \tag{1.13}$$

Finally, we have to specify the pressure and the internal energy. We will assume two cases. For simplicity, in the case $\gamma > 1$ we consider

$$p = p(\varrho,\vartheta) = \varrho^\gamma + \varrho\vartheta. \tag{1.14}$$

Recall that, in agreement with the second law of thermodynamics, there exists a function of ϱ and ϑ, called entropy, such that

$$\frac{1}{\vartheta}\Big(De(\varrho,\vartheta) + p(\varrho,\vartheta)D\Big(\frac{1}{\varrho}\Big)\Big) = Ds(\varrho,\vartheta) \tag{1.15}$$

which, due to (1.13), obeys the equation

$$\operatorname{div}(\varrho s\mathbf{u}) + \operatorname{div}\Big(\frac{\mathbf{q}}{\vartheta}\Big) = \frac{\mathbb{S} : \nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}. \tag{1.16}$$

The Gibbs relation (1.15) immediately implies that the internal energy fulfils the Maxwell relation

$$\frac{\partial e(\varrho,\vartheta)}{\partial\varrho} = \frac{1}{\varrho^2}\Big(p(\varrho,\vartheta) - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\Big). \tag{1.17}$$

Assuming the pressure law (1.14), we have

$$e = e(\varrho,\vartheta) = \frac{\varrho^{\gamma-1}}{\gamma-1} + g(\vartheta). \tag{1.18}$$

For simplicity, we assume $g(\vartheta) = c_v\vartheta$ and thus

$$s(\varrho,\vartheta) = \ln\frac{\vartheta^{c_v}}{\varrho} + s_0. \tag{1.19}$$

Note that instead of (1.14) we can also treat, similarly to [10] or [11], the more general pressure law

$$p(\varrho,\vartheta) = (\gamma-1)\varrho e(\varrho,\vartheta), \quad \gamma > 1, \tag{1.20}$$

where due to (1.15) the pressure has the form

$$p(\varrho,\vartheta)=\vartheta^{\gamma/(\gamma-1)}P\Big(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\Big), \tag{1.21}$$

$P\in C^1(0,\infty)$. Assuming additionally

$$\begin{gathered}P(\cdot)\in C^1([0,\infty))\cap C^2(0,\infty),\\ P(0)=0,\quad P'(0)=p_0>0,\quad P'(Z)>0,\quad Z>0,\\ \lim_{Z\to\infty}\frac{P(Z)}{Z^\gamma}=p_\infty>0,\\ 0<\frac{1}{\gamma-1}\frac{\gamma P(Z)-ZP'(Z)}{Z}\leqslant K<\infty,\quad Z>0,\end{gathered} \tag{1.22}$$

we can get the same result as below. The main ideas are the same, however, many additional technicalities occur, similar to those presented in [10]. Therefore we will not consider this general model and restrict ourselves to (1.14).

The next case concerns $\gamma=1$. We are not able to set simply $\gamma=1$ as the resulting model does not have a good physical meaning. Moreover, due to the lack of a priori estimates, we will consider a slightly more regular model, i.e.

$$p(\varrho,\vartheta)=\frac{\varrho^2}{\varrho+1}\ln^\alpha(\varrho+1)+\varrho\vartheta,\quad \alpha>0. \tag{1.23}$$

Note that $\alpha=0$ corresponds to a possible generalization of the case $\gamma=1$ in (1.14). However, we will have to require $\alpha>0$. The pressure law (1.23) implies due to (1.17)

$$e(\varrho,\vartheta)=\frac{1}{\alpha+1}\ln^{\alpha+1}(\varrho+1)+c_v\vartheta, \tag{1.24}$$

where we did the same choice of the unknown function of temperature as above. Note that the entropy remains unchanged (cf. (1.19)). Again, a more general model with asymptotic behaviour as in (1.23)–(1.24) could be considered, however, in order to avoid additional technicalities, we restrict ourselves to the model presented above.

We consider weak solutions to our problem as follows:

Definition 1. The triple $(\varrho,\mathbf{u},\vartheta)$ is called a weak solution to system (1.1)–(1.12), (1.14)–(1.15) (or (1.15), (1.23)) if $\varrho\geqslant 0$ a.e. in Ω, $\varrho\in L^{s\gamma}(\Omega;\mathbb{R})$, $s>1$, $\int_\Omega\varrho\,\mathrm{d}x=M$, $\boldsymbol{u}\in W_0^{1,2}(\Omega;\mathbb{R}^2)$, $\vartheta>0$ a.e. in Ω, $\vartheta\in W^{1,r}(\Omega;\mathbb{R})$ $\forall\, r<2$, and

$$\int_\Omega\varrho\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x=0\quad\forall\,\psi\in C^\infty(\overline{\Omega};\mathbb{R}), \tag{1.25}$$

$$
\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} - p(\varrho,\vartheta)\operatorname{div}\boldsymbol{\varphi} + \mathbb{S}(\vartheta,\mathbf{u}) : \nabla\boldsymbol{\varphi}\big)\,\mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x \quad \forall\,\boldsymbol{\varphi}\in C_0^\infty(\Omega;\mathbb{R}^2), \tag{1.26}
$$

$$
\begin{aligned}
\int_\Omega -\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x
&= \int_\Omega (\varrho\mathbf{f}\cdot\mathbf{u}\psi + p(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi)\,\mathrm{d}x \\
&\quad - \int_\Omega \big((\mathbb{S}(\vartheta,\mathbf{u})\mathbf{u})\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta\cdot\nabla\psi\big)\,\mathrm{d}x \\
&\quad - \int_{\partial\Omega} L(\vartheta-\Theta_0)\psi\,\mathrm{d}\sigma \quad \forall\,\psi\in C^\infty(\overline{\Omega};\mathbb{R}).
\end{aligned} \tag{1.27}
$$

We will also need the notion of the renormalized solution to the continuity equation:

Definition 2. Let $\boldsymbol{u}\in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and $\varrho\in L^q_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R})$ solve

$$
\operatorname{div}(\varrho\boldsymbol{u}) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2).
$$

Then the pair $(\varrho,\mathbf{u})$ is called a renormalized solution to the continuity equation, if

$$
\operatorname{div}(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho))\operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2)
$$

for all $b\in C^1([0,\infty))\cap W^{1,\infty}(0,\infty)$ with $zb'(z)\in L^\infty(0,\infty)$.

We aim at proving the following two results:

Theorem 1. *Let $\Omega\in C^2$ be a bounded domain in $\mathbb{R}^2$, $\mathbf{f}\in L^\infty(\Omega;\mathbb{R}^2)$, $\Theta_0\geqslant K_0>0$ a.e. at $\partial\Omega$, $\Theta_0\in L^1(\partial\Omega)$, $L>0$.*

(i) *Let $\gamma>1$, $m>0$. Then there exists a weak solution to* (1.1)–(1.12), (1.14)–(1.15) *in the sense of Definition* 1.
(ii) *Let $\alpha>1$ and $\alpha\geqslant\max(2/m)$, $m>0$. Then there exists a weak solution to* (1.1)–(1.12), (1.15), (1.23) *in the sense of Definition* 1.

Moreover, $(\varrho,\mathbf{u})$, extended by zero outside of Ω, is a renormalized solution to the continuity equation in the sense of Definition 2.

The first existence result for arbitrarily large data in the theory of steady compressible Navier-Stokes equations goes back to P.-L. Lions (see [6]), where the existence was shown for $\gamma>\frac{5}{3}$ $(N=3)$ and $\gamma>1$ $(N=2)$. These results have been recently improved by Frehse, Steinhauer and Weigant (see [3], [4]) up to $\gamma>\frac{4}{3}$ $(N=3)$ and $\gamma\geqslant 1$ $(N=2)$; see also related papers by Plotnikov and Sokołowski [14], [15], [16].

These results concern the barotropic case, i.e. only (1.1)–(1.2) with $\vartheta = \mathrm{const}$. Note that the reason why it is possible (even relatively easy) to get the existence of a weak solution for $\gamma = 1$ in two space dimensions is connected with the fact that in order to pass to the limit from the approximation to the original system for $p(\varrho) = a\varrho$, the weak convergence of the density is sufficient, which simplifies the problem considerably. Note also that a problem with $\gamma = 1$ similar to that in this paper, for barotropic flow, was considered in the evolutionary case by Erban [1].

Concerning the heat conducting fluid, the first result for large data in the steady case goes back to Mucha and Pokorný [8], where the existence is shown for $\gamma > 3$, $m = l+1 > (3\gamma-1)/(3\gamma-7)$ (with $L(\vartheta) \sim (1+\vartheta)^l$) for Navier boundary conditions for the velocity. Note that the solution is more regular in this case, i.e. the density is bounded and the velocity and temperature belong to $W^{1,q}(\Omega)$ for any $q < \infty$. In [9] the same authors got the existence up to $\gamma > \frac{7}{3}$ for both the Dirichlet and Navier boundary conditions for the velocity. A similar result in two space dimensions is due to Pecharová and Pokorný; here $\gamma > 2$ and $m > (\gamma-1)/(\gamma-2)$, see [13]. All these results are proved for constant viscosities.

In [10], [11] the authors of this paper studied the problem for temperature dependent viscosity, similarly to the present paper. They proved the existence of a variational entropy solution (i.e. the balance of total energy being replaced by entropy inequality and global total energy balance) for $\gamma > \frac{1}{8}(3+\sqrt{41})$ and $m > \max\{\frac{2}{3}, \frac{2}{3}(\gamma-1)^{-1}, \frac{2}{9}\gamma(4\gamma-1)/(4\gamma^2-3\gamma-2)\}$. These solutions are weak solutions for $\gamma > \frac{4}{3}$, $m > \max\{1, \frac{2}{3}\gamma/(3\gamma-4)^{-1}\}$ or $\gamma > \frac{5}{3}$, $m > 1$. The aim of this paper is to extend these results to the twodimensional problem.

The plan of the paper is the following. First, we recall several useful results, mostly covering the properties of Orlicz and Sobolev-Orlicz spaces needed in the case of (1.23). Next we introduce the approximative system (based on the approach from [10]) and briefly recall the main steps in the existence proof and the first three limit passages. The last limit passage requires new a priori estimates which will be shown in Section 4. Section 5 is devoted to the proof of strong convergence of density, both for (1.14) and (1.23).

2. Preliminaries

In what follows, we use standard notation for the Lebesgue space $L^p(\Omega)$ endowed with the norm $\|\cdot\|_{p,\Omega}$ and the Sobolev spaces $W^{k,p}(\Omega)$ endowed with the norm $\|\cdot\|_{k,p,\Omega}$. If no confusion may arise, we skip the domain Ω in the norm. The vector-valued functions will be printed in boldface, the tensor-valued functions with a special font. We will use notation $\varrho \in L^p(\Omega;\mathbb{R})$, $\mathbf{u} \in L^p(\Omega;\mathbb{R}^2)$, and $\mathbb{S} \in L^p(\Omega;\mathbb{R}^{2\times 2})$. The generic constants are denoted by C and their values may vary even in the same formula or on the same line. We also use summation convention over twice repeated indices from 1 to 2; e.g. $u_i v_i$ means $\sum\limits_{i=1}^{2} u_i v_i$.

We will need the following version of the Korn inequality:

Lemma 1. *For* $\boldsymbol{u} \in W_0^{1,2}(\Omega;\mathbb{R}^2)$, $\vartheta > 0$ *and* $\mathbb{S}(\vartheta,\mathbf{u})$ *satisfying* (1.7)–(1.8) *we have*

$$\int_\Omega \frac{\mathbb{S}(\vartheta,\mathbf{u}):\nabla\mathbf{u}}{\vartheta}\,\mathrm{d}x \geqslant C\|\mathbf{u}\|_{1,2}^2 \quad \textit{and} \quad \int_\Omega \mathbb{S}(\vartheta,\mathbf{u}):\nabla\mathbf{u}\,\mathrm{d}x \geqslant C\|\mathbf{u}\|_{1,2}^2. \tag{2.1}$$

Proof. As

$$\int_\Omega \frac{\mathbb{S}(\vartheta,\mathbf{u}):\nabla\mathbf{u}}{\vartheta}\,\mathrm{d}x \geqslant \int_\Omega c_1(\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\operatorname{div}\mathbf{u}\mathbb{I}):\nabla\boldsymbol{u}\,\mathrm{d}x = c_1\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x,$$

similarly in the second case, (2.1) follows from the standard Friedrichs inequality. □

Next we recall basic properties of a certain class of Orlicz spaces. Let Φ be the Young function. We denote by $E_\Phi(\Omega)$ the set of all measurable functions u such that

$$\int_\Omega \Phi(|u(x)|)\,\mathrm{d}x < +\infty.$$

Now we introduce the Luxemburg norm $\|u\|_\Phi$, i.e.

$$\|u\|_\Phi = \inf\Big\{k>0\colon \int_\Omega \Phi\Big(\frac{1}{k}|u(x)|\Big)\,\mathrm{d}x \leqslant 1\Big\}. \tag{2.2}$$

The Orlicz space $L_\Phi(\Omega)$ is the set of all measurable functions u such that $\|u\|_\Phi$ is finite. Note that for any $u \in E_\Phi(\Omega)$ we have (see e.g. [5])

$$\|u\|_\Phi \leqslant \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x + 1. \tag{2.3}$$

Recall that a Young function is said to satisfy the Δ_2-condition if there exist $k > 0$ and $c \geqslant 0$ such that

$$\Phi(2t) \leqslant k\Phi(t) \quad \forall\, t \geqslant c.$$

If $c = 0$, we speak about the global Δ_2-condition.

We will work with the following classes of Young functions. For $\alpha \geqslant 0$ and $\beta \geqslant 1$ we denote by $L_{z^\beta \ln^\alpha(1+z)}(\Omega)$ the class of Orlicz spaces generated by $\Phi(z) = z^\beta \ln^\alpha(1+z)$. Note that for our choice of α and β this Young function fulfils the global Δ_2-condition and we have

$$E_{z^\beta \ln^\alpha(1+z)}(\Omega) = L_{z^\beta \ln^\alpha(1+z)}(\Omega).$$

Note further that the definition of the Luxemburg norm yields for $\beta \geqslant 1$ and $\alpha \geqslant 0$

$$\|u\|_{z^\beta \ln^\alpha(1+z)} \leqslant \left(1 + \int_\Omega |u(x)|^\beta \ln^\alpha(1+|u(x)|)\,\mathrm{d}x\right)^{1/\beta}. \tag{2.4}$$

Next, recall that the complementary function to $z\ln^\alpha(1+z)$ behaves as $\mathrm{e}^{z^{1/\alpha}}$ for z large. We denote by $E_{\mathrm{e}(1/\alpha)}(\Omega)$ and $L_{\mathrm{e}(1/\alpha)}(\Omega)$ the corresponding sets of functions. Note that this Young function does not satisfy the Δ_2-condition.

Due to the generalized Hölder inequality we have

$$\|uv\|_1 \leqslant \|u\|_{z\ln^\alpha(1+z)}\|v\|_{\mathrm{e}(1/\alpha)}. \tag{2.5}$$

Further, for $\alpha \geqslant 0$, we also have

$$\|uv\|_{z\ln^\alpha(1+z)} \leqslant C\|u\|_{z^2\ln^\alpha(1+z)}\|v\|_{z^2\ln^\alpha(1+z)}, \tag{2.6}$$

which is a direct consequence of Theorem 10.4 from [7].

As we work in two space dimensions, recall (see e.g. [5]) that

$$W_0^{1,2}(\Omega) \hookrightarrow L_{\mathrm{e}^{z^2}-1}(\Omega),$$

i.e. for $u \in W_0^{1,2}(\Omega)$

$$\|u\|_{\mathrm{e}(2)} \leqslant C(\|u\|_{1,2} + 1). \tag{2.7}$$

We also need to estimate $\||u|^\delta\|_\Phi$. The definition of the Luxemburg norm immediately yields for any $\delta > 0$

$$\||u|^\delta\|_{\Phi(z)} = \|u\|^\delta_{\Phi(z^\delta)}; \tag{2.8}$$

in particular

$$\||u|^\delta\|_{\mathrm{e}(\alpha)} \leqslant C(\|u\|^\delta_{\mathrm{e}(\delta\alpha)} + 1), \tag{2.9}$$

and for $\delta \geqslant 1$

$$\||u|^\delta\|_{z\ln^\alpha(1+z)} \leqslant C(\|u\|^\delta_{z^\delta\ln^\alpha(1+z)} + 1). \tag{2.10}$$

Next we consider the problem

$$\begin{aligned} \operatorname{div}\boldsymbol{\varphi} &= f \quad \text{in } \Omega, \\ \boldsymbol{\varphi} &= \mathbf{0} \quad \text{at } \partial\Omega. \end{aligned} \tag{2.11}$$

If $f \in L^p(\Omega)$, $1 < p < \infty$ and $\int_\Omega f \, \mathrm{d}x = 0$, then there exists a solution to (2.11) such that

$$\|\boldsymbol{\varphi}\|_{1,p} \leqslant C\|f\|_p, \tag{2.12}$$

see e.g. [12]. A similar result holds also in a certain class of Orlicz spaces. For the Young function Φ satisfying the global Δ_2-condition such that for certain $\gamma \in (0,1)$ the function Φ^γ is quasiconvex we have that for $f \in L_\Phi(\Omega)$ satisfying the same compatibility condition as above

$$\||\nabla \boldsymbol{\varphi}|\|_\Phi \leqslant C\|f\|_\Phi, \tag{2.13}$$

see [17]. In particular, for $\alpha \geqslant 0$ and $\beta > 1$

$$\||\nabla \boldsymbol{\varphi}|\|_{z^\beta \ln^\alpha(1+z)} \leqslant C\|f\|_{z^\beta \ln^\alpha(1+z)}. \tag{2.14}$$

3. Approximative system

For the sake of simplicity, set

$$p_\gamma(\varrho,\vartheta) = \begin{cases} \varrho^\gamma + \varrho\vartheta, & \gamma > 1, \\ \dfrac{\varrho^2}{\varrho+1}\ln^\alpha(1+\varrho) + \varrho\vartheta, & \gamma = 1. \end{cases} \tag{3.1}$$

Similarly, $e_\gamma(\varrho,\vartheta)$ denotes the corresponding specific internal energy (cf. (1.14) or (1.18)), and $s_\gamma(\varrho,\vartheta)$ the corresponding specific entropy (1.19).

We take $\eta, \varepsilon, \delta > 0$ and $N \in \mathbb{N}$. We denote $X_N = \operatorname{span}\{\mathbf{w}^1, \ldots, \mathbf{w}^N\} \subset W_0^{1,2}(\Omega;\mathbb{R}^2)$ with $\{\mathbf{w}^i\}_{i=1}^\infty$ a complete orthogonal system in $W_0^{1,2}(\Omega;\mathbb{R}^2)$ such that $\mathbf{w}^i \in W^{2,q}(\Omega;\mathbb{R}^2)$ for all $i \in \mathbb{N}$ and all $q < \infty$. We look for a triple $(\varrho_{N,\eta,\varepsilon,\delta}, \mathbf{u}_{N,\eta,\varepsilon,\delta}, \vartheta_{N,\eta,\varepsilon,\delta})$ (we skip the indices in what follows in this section) such that $\varrho \in W^{2,q}(\Omega;\mathbb{R})$, $\mathbf{u} \in X_N$ and $\vartheta \in W^{2,q}(\Omega;\mathbb{R})$, $1 \leqslant q < \infty$ arbitrary, where

$$\begin{aligned} &\int_\Omega \Big(\frac{1}{2}\varrho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}^i - \frac{1}{2}\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{w}^i + \mathbb{S}_\eta(\vartheta,\mathbf{u}) : \nabla\mathbf{w}^i\Big)\,\mathrm{d}x \\ &\qquad - \int_\Omega \big(p_\gamma(\varrho,\vartheta) + \delta(\varrho^\beta + \varrho^2)\big)\operatorname{div}\mathbf{w}^i\,\mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{w}^i\,\mathrm{d}x \end{aligned} \tag{3.2}$$

for all $i = 1, 2, \ldots, N$,

$$\varepsilon\varrho - \varepsilon\Delta\varrho + \operatorname{div}(\varrho\mathbf{u}) = \varepsilon h \quad \text{a.e. in } \Omega, \tag{3.3}$$

and

$$-\operatorname{div}\Big((\kappa_\eta(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\nabla\vartheta\Big)+\operatorname{div}(\varrho e_\gamma(\varrho,\vartheta)\mathbf{u}) = \mathbb{S}_\eta(\vartheta,\mathbf{u}):\nabla\mathbf{u}+\delta\vartheta^{-1}-p_\gamma(\varrho,\vartheta)\operatorname{div}\mathbf{u} + \delta\varepsilon|\nabla\varrho|^2(\beta\varrho^{\beta-2}+2) \quad \text{a.e. in } \Omega, \tag{3.4}$$

with β and B sufficiently large,

$$\mathbb{S}_\eta(\vartheta,\mathbf{u})=\frac{\mu_\eta(\vartheta)}{1+\eta\vartheta}[\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\operatorname{div}\mathbf{u}\mathbb{I}]+\frac{\xi_\eta(\vartheta)}{1+\eta\vartheta}\operatorname{div}\mathbf{u}\mathbb{I}, \tag{3.5}$$

$h = M/|\Omega|$, $\mu_\eta(\cdot)$, $\xi_\eta(\cdot)$, $\kappa_\eta(\cdot)$ the standard regularizations of functions $\mu(\cdot)$, $\xi(\cdot)$, $\kappa(\cdot)$ (extended by constants $\mu(0)$, $\xi(0)$, $\kappa(0)$ to the negative real line) together with the boundary conditions on $\partial\Omega$

$$\frac{\partial\varrho}{\partial\mathbf{n}}=0, \tag{3.6}$$

$$(\kappa_\eta(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1})\frac{\varepsilon+\vartheta}{\vartheta}\frac{\partial\vartheta}{\partial\mathbf{n}}+(L+\delta\vartheta^{B-1})(\vartheta-\Theta_0^\eta)+\varepsilon\ln\vartheta=0, \tag{3.7}$$

with Θ_0^η a smooth approximation of Θ_0 such that Θ_0^η is strictly positive at $\partial\Omega$.

Similarly to [10] we can prove

Theorem 2. *Let ε, δ, η and N be as above, β and B sufficiently large and let all assumptions formulated in Sections 1 and 3 be satisfied. Let $1 \leqslant \gamma < \infty$ and let ε be sufficiently small with respect to δ. Then there exists a solution to system (3.2)–(3.7) such that $\varrho \in W^{2,q}(\Omega;\mathbb{R})$ $\forall q<\infty$, $\varrho \geqslant 0$ in Ω, $\int_\Omega \varrho \,\mathrm{d}x = M$, $\mathbf{u}\in X_N$, and $\vartheta \in W^{2,q}(\Omega;\mathbb{R})$ $\forall\, q<\infty$, $\vartheta \geqslant C(N) > 0$.*

We will not go into details of the proof of Theorem 2 as it contains tedious and long computations. The proof is based on a priori estimates coming from the entropy (in)equality and on the application of the Schaeffer fixed point theorem (which is a version of the Schauder fixed point theorem). The details in the case of the $3D$ problem can be found in [10] and the proof in $2D$ follows precisely the same lines and is only slightly easier.

We also omit the details of the limit passages $N\to\infty$, $\eta\to 0^+$, and $\varepsilon\to 0^+$. Let us only mention that the approximative energy balance (3.4) is in fact only the balance of internal energy. From it and the balance of momentum we can deduce the entropy equality (on the level $N<\infty$) and due to sufficient a priori estimates (needed also for the existence result) we can pass with $N\to\infty$. At this step we immediately lose the entropy equality and get only an entropy inequality. In order to pass with

$\eta \to 0^+$ we also lose the balance of internal energy; we must switch before the limit passage to the balance of total energy which is (similarly as on the level of the original system) just a consequence of the balance of internal energy and momentum. At this moment we have sufficient regularity to verify this fact rigorously, which is not the case for weak solutions to the original system. The next limit passage, $\varepsilon \to 0^+$, is more delicate due to the loss of compactness of density. However, exactly as in the $3D$ case, using a technique based on compensated compactness, we can prove that also the density sequence is strongly convergent in a certain $L^p(\Omega;\mathbb{R})$ which is enough to pass to the limit. The proof is exactly the same as in the case $\gamma > 1$ for $\delta \to 0^+$, only somewhat easier; hence there is no need to repeat this procedure twice here. See also [10] for more details.

The above mentioned limit passages give us for any $\delta > 0$ the existence of $\varrho \in L^{s\beta}(\Omega;\mathbb{R})$, $s < 2$ arbitrary, $\varrho \geqslant 0$ a.e. in Ω, $\int_\Omega \varrho \,\mathrm{d}x = M$, $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^2)$, $\vartheta \in W^{1,2}(\Omega;\mathbb{R})$, $\vartheta > 0$ a.e. in Ω such that

$$\int_\Omega \varrho\mathbf{u} \cdot \nabla\psi \,\mathrm{d}x = 0 \tag{3.8}$$

for all $\psi \in W^{1,r}(\Omega;\mathbb{R})$, $r > 2\beta/(2\beta-1)$,

$$\begin{aligned}\int_\Omega \big(-\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + \mathbb{S}(\vartheta,\mathbf{u}) : \nabla\boldsymbol{\varphi} - (p_\gamma(\varrho,\vartheta) + \delta\varrho^\beta + \delta\varrho^2)\operatorname{div}\boldsymbol{\varphi}\big)\,\mathrm{d}x \\ = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\end{aligned} \tag{3.9}$$

for all $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega;\mathbb{R}^2)$, $r > 2$,

$$\begin{aligned}&\int_\Omega \Big(\Big(-\frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho e_\gamma(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla\psi + (\kappa(\vartheta) + \delta\vartheta^B + \delta\vartheta^{-1})\nabla\vartheta : \nabla\psi\Big)\,\mathrm{d}x \\ &\qquad + \int_{\partial\Omega} (L + \delta\vartheta^{B-1})(\vartheta - \Theta_0)\psi\,\mathrm{d}\sigma \\ &\quad = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x \\ &\qquad + \int_\Omega \big((-\mathbb{S}(\vartheta,\mathbf{u})\mathbf{u} + (p_\gamma(\varrho,\vartheta) + \delta\varrho^\beta + \delta\varrho^2)\mathbf{u})\cdot\nabla\psi + \delta\vartheta^{-1}\psi\big)\,\mathrm{d}x \\ &\qquad + \delta\int_\Omega \Big(\frac{1}{\beta-1}\varrho^\beta + \varrho^2\Big)\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x\end{aligned} \tag{3.10}$$

for all $\psi \in C^1(\overline{\Omega};\mathbb{R})$.

Finally, we also get the entropy inequality

$$\begin{aligned}(3.11)\quad \int_\Omega \Big(\vartheta^{-1}\mathbb{S}(\vartheta,\mathbf{u}):\nabla\mathbf{u}+\delta\vartheta^{-2}+(\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1})\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\,\mathrm{d}x \\ \leqslant \int_\Omega\Big((\kappa(\vartheta)+\delta\vartheta^B+\delta\vartheta^{-1})\frac{\nabla\vartheta:\nabla\psi}{\vartheta}-\varrho s_\gamma(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x \\ +\int_{\partial\Omega}\frac{L+\delta\vartheta^{B-1}}{\vartheta}(\vartheta-\Theta_0)\psi\,\mathrm{d}\sigma\end{aligned}$$

for all nonnegative $\psi\in C^1(\overline{\Omega};\mathbb{R})$.

In the next section we will investigate estimates of $(\varrho,\mathbf{u},\vartheta)$ independent of δ which will allow us to pass with $\delta\to 0^+$.

4. A priori estimates

First, in (3.11), we set $\psi\equiv 1$. This yields

$$\begin{aligned}(4.1)\quad &\int_\Omega(\kappa(\vartheta_\delta)+\delta\vartheta_\delta^B+\delta\vartheta_\delta^{-1})\frac{|\nabla\vartheta_\delta|^2}{\vartheta_\delta^2}\,\mathrm{d}x \\ &+\int_\Omega\Big(\frac{1}{\vartheta_\delta}\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta):\nabla\mathbf{u}_\delta+\delta\vartheta_\delta^{-2}\Big)\,\mathrm{d}x+\int_{\partial\Omega}\frac{L+\delta\vartheta_\delta^{B-1}}{\vartheta_\delta}\Theta_0\,\mathrm{d}\sigma \\ &\leqslant\int_{\partial\Omega}(L+\delta\vartheta_\delta^{B-1})\,\mathrm{d}\sigma.\end{aligned}$$

Similarly, using the same test function in (3.10), we obtain

$$(4.2)\quad \int_{\partial\Omega}(L\vartheta_\delta+\delta\vartheta_\delta^B)\,\mathrm{d}\sigma=\int_\Omega\varrho_\delta\mathbf{u}_\delta\cdot\mathbf{f}\,\mathrm{d}x+\int_{\partial\Omega}(L+\delta\vartheta_\delta^{B-1})\Theta_0\,\mathrm{d}\sigma+\delta\int_\Omega\vartheta_\delta^{-1}\,\mathrm{d}x.$$

We plug in estimate (4.2) into (4.1). Thus we need to control the density, multiplied by δ. To this aim, we use as a test function in (3.9) the solution to (cf. [10, Section 5])

$$\begin{aligned}\operatorname{div}\boldsymbol{\varphi}&=\varrho_\delta-\frac{M}{|\Omega|}\quad\text{in }\Omega,\\ \boldsymbol{\varphi}&=0\quad\text{at }\partial\Omega\end{aligned}$$

with

$$\|\nabla\boldsymbol{\varphi}\|_{1,q}\leqslant C\|\varrho_\delta\|_q,\quad 1<q<\infty.$$

Hence, after relatively standard computations,

$$\delta\|\varrho_\delta\|_{\beta+1}^{\beta-3/2}\leqslant\text{const.}$$

Due to this additional a priori bound we have

$$\begin{aligned}(4.3)\qquad &\|\mathbf{u}_\delta\|_{1,2} + \|\nabla\vartheta_\delta^{m/2}\|_2 + \|\nabla \ln\vartheta_\delta\|_2 + \|\vartheta_\delta^{-1}\|_{1,\partial\Omega}\\ &+ \delta\big(\|\nabla\vartheta_\delta^{B/2}\|_2^2 + \|\nabla\vartheta_\delta^{-1/2}\|_2^2 + \|\vartheta_\delta\|_r^{B-2} + \|\vartheta_\delta^{-2}\|_1\big) \leqslant C,\end{aligned}$$

$r < \infty$ arbitrary, and from (4.2) also

$$(4.4)\quad \|\vartheta_\delta^{m/2}\|_{1,2}^{2/m} \leqslant C(1 + \|\vartheta_\delta\|_{1,\partial\Omega} + \|\nabla(\vartheta_\delta^{m/2})\|_2^{2/m}) \leqslant C\bigg(1 + \bigg|\int_\Omega \varrho_\delta \mathbf{f}\cdot\mathbf{u}_\delta\,\mathrm{d}x\bigg|\bigg).$$

At this moment, we need to establish estimates of the density which are independent of δ. We proceed separately for $\gamma > 1$ and $\gamma = 1$.

4.1. Estimates of the density for $\gamma > 1$

The aim is to use as a test function in (3.9) a suitable function which produces the estimates of the density. Similarly to the above, we take $\boldsymbol{\varphi}$, a solution to

$$(4.5)\qquad \begin{aligned} \operatorname{div}\boldsymbol{\varphi} &= \varrho_\delta^{(s-1)\gamma} - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta^{(s-1)\gamma}\,\mathrm{d}x \quad \text{a.e. in } \Omega,\\ \boldsymbol{\varphi} &= \mathbf{0} \quad \text{at } \partial\Omega,\\ \|\boldsymbol{\varphi}\|_{1,s/(s-1)} &\leqslant C\|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma},\end{aligned}$$

for s chosen suitably below. We have

$$\begin{aligned}(4.6)\quad &\int_\Omega (\varrho_\delta^{s\gamma} + \varrho_\delta^{(s-1)\gamma+1}\vartheta_\delta)\,\mathrm{d}x + \delta\int_\Omega (\varrho_\delta^\beta + \varrho_\delta^2)\varrho_\delta^{(s-1)\gamma}\,\mathrm{d}x\\ &= \frac{1}{|\Omega|}\int_\Omega (\varrho_\delta^\gamma + \varrho_\delta\vartheta_\delta + \delta(\varrho_\delta^\beta + \varrho_\delta^2))\,\mathrm{d}x \int_\Omega \varrho_\delta^{(s-1)\gamma}\,\mathrm{d}x\\ &\quad - \int_\Omega \varrho_\delta\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega \mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x - \int_\Omega \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Due to the fact that (4.4) implies

$$(4.7)\qquad \|\vartheta_\delta\|_q \leqslant C(q,\varepsilon)(1 + \|\varrho_\delta\|_{s\gamma}^\varepsilon)$$

for arbitrary small $\varepsilon > 0$ and arbitrary large $q < \infty$, there is no problem to estimate I_1 using interpolation between L^1 and $L^{s\gamma}$ for ϱ_δ. Further, assuming $s\gamma > 2$, we have

$$|I_2| \leqslant C\|\mathbf{f}\|_\infty \|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma+1}.$$

Next, for q sufficiently close to ∞,

$$|I_3| \leqslant C \int_\Omega (1+\vartheta_\delta)|\nabla \mathbf{u}_\delta||\nabla \boldsymbol{\varphi}| \,\mathrm{d}x \leqslant C(1+\|\vartheta_\delta\|_q)\|\nabla \mathbf{u}_\delta\|_2 \|\nabla \boldsymbol{\varphi}\|_{s/(s-1)},$$

i.e. we get the restriction $s < 2$. Then

$$|I_3| \leqslant C(1+\|\vartheta_\delta\|_q)\|\nabla \mathbf{u}_\delta\|_2 \|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma} \leqslant C\|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma+\varepsilon}.$$

Finally, taking s so close to 2 that $s\gamma > 2$ for any $\gamma > 1$, we have

$$|I_4| \leqslant \|\varrho_\delta\|_{s\gamma}\|\mathbf{u}_\delta\|_q^2 \|\nabla \boldsymbol{\varphi}\|_{s/(s-1)} \leqslant C\|\varrho_\delta\|_{s\gamma}^{(s-1)\gamma+1}.$$

Altogether, the estimates above read

$$\|\varrho_\delta\|_{s\gamma} \leqslant C. \tag{4.8}$$

Note that we can take any $s < 2$, hence for arbitrary $\gamma > 1$ we can ensure $s\gamma > 2$. Hence (4.3), (4.4) and (4.8) imply

$$\|\mathbf{u}_\delta\|_{1,2} + \|\nabla\vartheta_\delta\|_r + \|\nabla \ln \vartheta_\delta\|_2 + \|\vartheta_\delta\|_q + \|\ln \vartheta_\delta\|_q + \|\vartheta_\delta^{-1}\|_{1,\partial\Omega} + \|\varrho_\delta\|_{s\gamma} \leqslant C \tag{4.9}$$

with any $r < 2$, $q < \infty$, and any $s < 2$. Furthermore,

$$\delta\big(\|\nabla\vartheta_\delta^{B/2}\|_2^2 + \|\nabla\vartheta_\delta^{-1/2}\|_2^2 + \|\vartheta_\delta\|_q^B + \|\vartheta_\delta\|_{q,\partial\Omega}^B + \|\vartheta_\delta^{-2}\|_1 + \|\varrho_\delta\|_{\beta+1}^{\beta+1}\big) \leqslant C \tag{4.10}$$

for arbitrary $q < \infty$.

4.2. Estimates of the density for $\gamma = 1$

Now we consider the case $\gamma = 1$. We replace problem (4.5) by

$$\begin{aligned} \operatorname{div} \boldsymbol{\varphi} &= \varrho_\delta - \frac{M}{|\Omega|} \quad \text{a.e. in } \Omega, \\ \boldsymbol{\varphi} &= \mathbf{0} \quad \text{at } \partial\Omega, \\ \|\boldsymbol{\varphi}\|_{1,q} &\leqslant C\|\varrho_\delta\|_q, \end{aligned} \tag{4.11}$$

and we use this $\boldsymbol{\varphi}$ as a test function in (3.9). It reads

$$\begin{aligned} &\int_\Omega \Big(\frac{\varrho_\delta^3}{1+\varrho_\delta}\ln^\alpha(1+\varrho_\delta) + \varrho_\delta^2\vartheta_\delta\Big)\,\mathrm{d}x + \delta\int_\Omega (\varrho_\delta^{\beta+1} + \varrho_\delta^3)\,\mathrm{d}x \\ &\quad = \frac{M}{|\Omega|}\int_\Omega \Big(\frac{\varrho_\delta^2}{\varrho_\delta+1}\ln^\alpha(1+\varrho_\delta) + \varrho_\delta\vartheta_\delta + \delta(\varrho_\delta^\beta + \varrho_\delta^2)\Big)\,\mathrm{d}x \\ &\quad\quad - \int_\Omega \varrho_\delta \mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega \mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x \\ &\quad\quad - \int_\Omega \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x = J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{4.12}$$

Estimates of J_1 and J_2 are easy; thus we concentrate only on the last two terms. We have

$$|J_3| \leqslant C \int_\Omega (1+\vartheta_\delta)|\nabla \mathbf{u}_\delta||\nabla \boldsymbol{\varphi}| \,\mathrm{d}x \leqslant C\|\nabla \mathbf{u}_\delta\|_2 \Big(\int_\Omega (1+\vartheta_\delta)^2 |\nabla \boldsymbol{\varphi}|^2 \,\mathrm{d}x\Big)^{1/2}.$$

We now use (2.5) to get

$$|J_3| \leqslant C(1+\|\vartheta_\delta^2\|_{\mathrm{e}(m/2)})^{1/2} (\||\nabla \boldsymbol{\varphi}|^2\|_{z \ln^{2/m}(1+z)})^{1/2},$$

and thus (2.13), (2.9), (2.10), (2.7), (4.4), and (4.7) yield

$$|J_3| \leqslant C(1+\|\varrho_\delta\|_2^{\varepsilon})(1+\|\varrho_\delta\|_{z^2 \ln^{2/m}(1+z)});$$

hence (2.4) implies for $\varepsilon > 0$, arbitrarily small,

$$|J_3| \leqslant C\Big(1+\Big(\int_\Omega \varrho^2 \ln^{2/m}(1+\varrho)\Big)^{1/2+\varepsilon}\Big).$$

For J_4 we proceed similarly: we use (2.5), (2.6), (2.7) and get

$$\begin{aligned} |J_4| &\leqslant \int_\Omega \varrho_\delta |\mathbf{u}_\delta|^2 |\nabla \boldsymbol{\varphi}| \,\mathrm{d}x \leqslant \||\mathbf{u}_\delta|^2\|_{\mathrm{e}(1)} \|\varrho_\delta |\nabla \boldsymbol{\varphi}|\|_{z \ln(1+z)} \\ &\leqslant C(1+\|\mathbf{u}_\delta\|_{1,2}^2) \||\nabla \boldsymbol{\varphi}|\|_{z^2 \ln(1+z)} \|\varrho_\delta\|_{z^2 \ln(1+z)}. \end{aligned}$$

Now, due to (2.14) and (2.4) we finally have

$$|J_4| \leqslant C\|\varrho_\delta\|_{z^2\ln(1+z)}(1+\|\varrho_\delta\|_{z^2 \ln(1+z)}) \leqslant C\Big(1+\int_\Omega |\varrho_\delta|^2 \ln(1+\varrho_\delta) \,\mathrm{d}x\Big).$$

Hence, for $\alpha > 1$ we get

$$|J_4| \leqslant C + \frac{1}{2}\int_\Omega |\varrho_\delta|^2 \ln^\alpha(1+\varrho_\delta) \,\mathrm{d}x$$

(consider separately $\varrho_\delta \leqslant K$ and $\varrho_\delta > K$ for K sufficiently large) and the estimates above yield for $\alpha > 1$ and $\alpha \geqslant 2/m$

$$\int_\Omega \varrho_\delta^2 \ln^\alpha(1+\varrho_\delta) \,\mathrm{d}x \leqslant C. \tag{4.13}$$

We get (4.8)–(4.10) with $s = 2$ or, more precisely, (4.9)–(4.10) together with (4.13).

We may now pass to the limit in the weak formulation. Using (4.8)–(4.9) and (4.13) we get a subsequence (denoted again by the index δ) such that

$$\varrho_\delta \rightharpoonup \varrho \quad \text{in } L^{s\gamma}(\Omega;\mathbb{R}),\ \forall\, s<2\ (\gamma>1) \text{ or } s=2\ (\gamma=1), \tag{4.14}$$

$$\mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad \text{in } W_0^{1,2}(\Omega;\mathbb{R}^2), \quad \mathbf{u}_\delta \to \mathbf{u} \quad \text{in } L^q(\Omega;\mathbb{R}^2),\ \forall\, q<\infty, \tag{4.15}$$

$$\vartheta_\delta \rightharpoonup \vartheta \quad \text{in } W^{1,r}(\Omega;\mathbb{R}),\ \forall\, r<2, \quad \vartheta_\delta \to \vartheta \quad \text{in } L^q(\Omega;\mathbb{R}),\ \forall\, q<\infty. \tag{4.16}$$

Passing to the limit in the weak formulation (3.8)–(3.10) (using also (4.10)) we have

$$\int_\Omega \varrho\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x = 0 \tag{4.17}$$

for all $\psi \in W^{1,r}(\Omega;\mathbb{R})$, $r>2$;

$$\int_\Omega \big(-\varrho(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\boldsymbol{\varphi} + \mathbb{S}(\vartheta,\mathbf{u}):\nabla\boldsymbol{\varphi} - \overline{p_\gamma(\varrho,\vartheta)}\,\operatorname{div}\boldsymbol{\varphi}\big)\,\mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x \tag{4.18}$$

for all $\boldsymbol{\varphi} \in W_0^{1,r}(\Omega;\mathbb{R}^2)$, $r>2$,

$$\begin{aligned}\int_\Omega \Big(\Big(-\frac{1}{2}\varrho|\mathbf{u}|^2 - \overline{\varrho e_\gamma(\varrho,\vartheta)}\Big)\mathbf{u}\cdot\nabla\psi + \kappa(\vartheta)\nabla\vartheta:\nabla\psi\Big)\,\mathrm{d}x + \int_{\partial\Omega} L(\vartheta-\Theta_0)\psi\,\mathrm{d}\sigma \\ = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x + \int_\Omega \big(-\mathbb{S}(\vartheta,\mathbf{u})\mathbf{u} + \overline{p_\gamma(\varrho,\vartheta)}\mathbf{u}\big)\cdot\nabla\psi\,\mathrm{d}x\end{aligned}$$

for all $\psi \in C^1(\overline{\Omega};\mathbb{R})$. Note that we have also used the fact that $\delta\int_\Omega \varrho_\delta^{\beta+1}\,\mathrm{d}x \leqslant C$. Finally, we could also pass to the limit in the entropy inequality (3.11) to get

$$\begin{aligned}\int_\Omega \Big(\vartheta^{-1}\mathbb{S}(\vartheta,\mathbf{u}):\nabla\mathbf{u} + \kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2}\Big)\psi\,\mathrm{d}x \\ \leqslant \int_\Omega \Big(\kappa(\vartheta)\frac{\nabla\vartheta:\nabla\psi}{\vartheta} - \overline{\varrho s_\gamma(\varrho,\vartheta)}\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x + \int_{\partial\Omega} \frac{L}{\vartheta}(\vartheta-\Theta_0)\psi\,\mathrm{d}\sigma\end{aligned}$$

for all $\psi \in C^1(\overline{\Omega};\mathbb{R})$, nonnegative. However, we will not use this fact in the sequel. To complete the proof of Theorem 1, we have to verify that $\varrho_\delta \to \varrho$ strongly at least in $L^1(\Omega;\mathbb{R})$. We will show this in the last section, separately for $\gamma>1$ and $\gamma=1$.

Before starting, we recall several lemmas which will be useful throughout this section. The proof of them can be found in [2, Appendix].

Lemma 2 (Renormalized continuity equation)**.** *Assume that*

$$\begin{gathered} b \in C([0,\infty)) \cap C^1((0,\infty)), \\ \lim_{s\to 0^+} (sb'(s) - b(s)) \in \mathbb{R}, \\ |b'(s)| \leqslant Cs^\lambda, \quad s \in (1,\infty), \quad \lambda \leqslant \frac{a}{2} - 1. \end{gathered} \tag{5.1}$$

Let $\varrho \in L^a(\Omega;\mathbb{R})$, $a \geqslant 2$, $\varrho \geqslant 0$ *a.e. in* Ω, $\boldsymbol{u} \in W_0^{1,2}(\Omega;\mathbb{R}^2)$ *be such that*

$$\int_{\mathbb{R}^2} \varrho\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x = 0$$

for all $\psi \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ *with* ϱ, $\mathbf{u}$ *extended by zero outside of* Ω. *Then the pair* $(\varrho,\mathbf{u})$ *is a renormalized solution to the continuity equation, i.e. we have for all* $b(\cdot)$ *as specified in* (5.1)

$$\int_{\mathbb{R}^2} \big(-b(\varrho)\mathbf{u}\cdot\nabla\psi + (\varrho b'(\varrho) - b(\varrho))\operatorname{div}\mathbf{u}\psi\big)\,\mathrm{d}x = 0 \tag{5.2}$$

for all $\psi \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$.

We will work with the following operators

$$\begin{gathered} \nabla\Delta^{-1}v \equiv \mathcal{F}^{-1}\Big[\frac{\mathrm{i}\xi}{|\xi|^2}\mathcal{F}(v)(\xi)\Big], \\ (\mathcal{R}[v])_{ij} \equiv (\nabla\otimes\nabla\Delta^{-1})_{ij}v = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v)(\xi)\Big] \end{gathered} \tag{5.3}$$

with $\mathcal{F}$ the Fourier transform. Denote also

$$(\mathcal{R}[\mathbf{v}])_i = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v_j)(\xi)\Big].$$

We have

Lemma 3 (Continuity properties of $\nabla\otimes\nabla\Delta^{-1}$ and $\nabla\Delta^{-1}$)**.** *The operator* $\mathcal{R}$ *is a continuous operator from* $L^p(\mathbb{R}^2;\mathbb{R})$ *to* $L^p(\mathbb{R}^2;\mathbb{R}^{2\times 2})$ *for any* $1 < p < \infty$.

The operator $\nabla\Delta^{-1}$ *is a continuous linear operator from the space* $L^1(\mathbb{R}^2;\mathbb{R}) \cap L^2(\mathbb{R}^2;\mathbb{R})$ *to* $L^2(\mathbb{R}^2;\mathbb{R}^2) + L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ *and from* $L^p(\mathbb{R}^2;\mathbb{R})$ *to* $L^{2p/(2-p)}(\mathbb{R}^2;\mathbb{R}^2)$ *for any* $1 < p < 2$.

Lemma 4 (Commutators I). *Let* $\mathbf{U}_\delta \rightharpoonup \mathbf{U}$ *in* $L^p(\mathbb{R}^2;\mathbb{R}^2)$, $v_\delta \rightharpoonup v$ *in* $L^q(\mathbb{R}^2;\mathbb{R})$, *where*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1.$$

Then

$$v_\delta \mathcal{R}[\mathbf{U}_\delta]-\mathcal{R}[v_\delta]\mathbf{U}_\delta \rightharpoonup v\mathcal{R}[\mathbf{U}]-\mathcal{R}[v]\mathbf{U}$$

in $L^r(\mathbb{R}^2;\mathbb{R}^2)$.

Lemma 5 (Commutators II). *Let* $w\in W^{1,r}(\mathbb{R}^2;\mathbb{R})$, $\mathbf{z}\in L^p(\mathbb{R}^2;\mathbb{R}^2)$, $1<r<2$, $1<p<\infty$, $1/r+1/p-1/2<1/s<1$. *Then for all such* s *we have*

$$\|\mathcal{R}[w\mathbf{z}]-w\mathcal{R}[\mathbf{z}]\|_{a,s,\mathbb{R}^2}\leqslant C\|w\|_{1,r,\mathbb{R}^2}\|\mathbf{z}\|_{p,\mathbb{R}^2},$$

where $a/2=1/s+1/2-1/p-1/r$.

5.1. Strong convergence for $\gamma>1$

Using as a test function in (3.9) the function $\boldsymbol{\varphi}=\zeta(x)\nabla\Delta^{-1}(1_\Omega\varrho_\delta)$ and in (4.18) the function $\boldsymbol{\varphi}=\zeta(x)\nabla\Delta^{-1}(1_\Omega\varrho)$, $\zeta\in C_0^\infty(\Omega;\mathbb{R})$, 1_Ω being the characteristic function of the set Ω, we get applying also (4.17) (or (3.8), respectively)

$$\begin{aligned}(5.4)\quad &\lim_{\delta\to0^+}\int_\Omega \zeta(x)\big(p_\gamma(\varrho_\delta,\vartheta_\delta)\varrho_\delta-\mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta):\mathcal{R}[1_\Omega\varrho_\delta]\big)\,\mathrm{d}x\\ &=\int_\Omega\zeta(x)(\overline{p_\gamma(\varrho,\vartheta)}\varrho-\mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[1_\Omega\varrho])\,\mathrm{d}x\\ &+\lim_{\delta\to0^+}\int_\Omega\zeta(x)\big(\varrho_\delta\mathbf{u}_\delta\cdot\mathcal{R}[1_\Omega\varrho_\delta\mathbf{u}_\delta]-\varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta):\mathcal{R}[1_\Omega\varrho_\delta]\big)\,\mathrm{d}x\\ &-\int_\Omega\zeta(x)(\varrho\mathbf{u}\cdot\mathcal{R}[1_\Omega\varrho\mathbf{u}]-\varrho(\mathbf{u}\otimes\mathbf{u}):\mathcal{R}[1_\Omega\varrho])\,\mathrm{d}x.\end{aligned}$$

Note that ϱ_δ is bounded in $L^s(\Omega;\mathbb{R})$ for a certain $s>2$ and thus $\nabla\Delta^{-1}(1_\Omega\varrho_\delta)\to\nabla\Delta^{-1}(1_\Omega\varrho)$ in $C(\overline{\Omega};\mathbb{R}^2)$.

Equality (5.4) together with the above lemmas implies the following effective viscous flux identity:

Lemma 6. *Under the above assumption, for* $\gamma>1$,

$$(5.5)\qquad \overline{p_\gamma(\varrho,\vartheta)\varrho}-(\mu(\vartheta)+\xi(\vartheta))\overline{\varrho\operatorname{div}\mathbf{u}}=\overline{p_\gamma(\varrho,\vartheta)}\varrho-(\mu(\vartheta)+\xi(\vartheta))\varrho\operatorname{div}\mathbf{u}$$

a.e. in Ω.

P r o o f. We apply Lemma 4 with

$$v_\delta = \varrho_\delta \rightharpoonup \varrho \quad \text{in } L^s(\mathbb{R}^2;\mathbb{R}),\ s>2,$$
$$\mathbf{U}_\delta = \varrho_\delta \mathbf{u}_\delta \rightharpoonup \varrho\mathbf{u} \quad \text{in } L^2(\mathbb{R}^2;\mathbb{R}^2),$$

with all functions extended by zero outside Ω to the whole $\mathbb{R}^2$. Then

$$\varrho_\delta \mathcal{R}[1_\Omega \varrho_\delta \mathbf{u}_\delta] - \mathcal{R}[1_\Omega \varrho_\delta]\varrho_\delta \mathbf{u}_\delta \rightharpoonup \varrho\mathcal{R}[1_\Omega\varrho\mathbf{u}] - \mathcal{R}[1_\Omega\varrho]\varrho\mathbf{u}$$

in $L^r(\mathbb{R}^2;\mathbb{R}^2)$, for a certain $r>1$. Consequently,

$$\int_\Omega \zeta(x)\mathbf{u}_\delta \cdot (\varrho_\delta\mathcal{R}[1_\Omega\varrho_\delta\mathbf{u}_\delta] - \varrho_\delta\mathcal{R}[1_\Omega\varrho_\delta]\mathbf{u}_\delta)\,\mathrm{d}x \\ \to \int_\Omega \zeta(x)\mathbf{u}\cdot(\varrho\mathcal{R}[1_\Omega\varrho\mathbf{u}] - \varrho\mathcal{R}[1_\Omega\varrho]\mathbf{u})\,\mathrm{d}x.$$

Hence, equation (5.4) reduces to

$$\lim_{\delta\to0^+}\int_\Omega \zeta(x)\big(p_\gamma(\varrho_\delta,\vartheta_\delta)\varrho_\delta - \mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta):\mathcal{R}[1_\Omega\varrho_\delta]\big)\,\mathrm{d}x \\ = \int_\Omega \zeta(x)\big(\overline{p_\gamma(\varrho,\vartheta)}\varrho - \mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[1_\Omega\varrho]\big)\,\mathrm{d}x. \tag{5.6}$$

Next,

$$\int_\Omega \zeta(x)\mu(\vartheta_\delta)(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T):\mathcal{R}[1_\Omega\varrho_\delta]\,\mathrm{d}x \\ = \int_\Omega \mathcal{R}:[\zeta(x)\mu(\vartheta_\delta)(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T)]\varrho_\delta\,\mathrm{d}x,$$

and

$$\mathcal{R}:[\zeta(x)\mu(\vartheta_\delta)(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T)] \\ = \zeta(x)2\mu(\vartheta_\delta)\operatorname{div}\mathbf{u}_\delta + \mathcal{R}:[\zeta(x)\mu(\vartheta_\delta)(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T)] \\ - \zeta(x)\mu(\vartheta_\delta)\mathcal{R}:[\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T].$$

Similar formulas hold also for the limit term. Then, using Lemma 5 with $w = \zeta(x)\mu(\vartheta) \sim (1+\vartheta)$, $r<2$ and $z_i = \partial_j u_i + \partial_i u_j$, $j=1,2,3$, $p=2$ and recalling that $\mu(\cdot)$ is globally Lipschitz, we obtain

$$\|\mathcal{R}:[\zeta(x)\mu(\vartheta_\delta)(\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T)] - \zeta(x)\mu(\vartheta_\delta)\mathcal{R}:[\nabla\mathbf{u}_\delta + (\nabla\mathbf{u}_\delta)^T]\|_{a,s,\mathbb{R}^2} \leqslant C$$

with $1 < s < 2$, $a < (2-s)/s$. As $W^{a,s}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for any $q < 2$ and the density ϱ_δ is bounded in $L^s(\Omega;\mathbb{R})$, $s > 2$, we get

$$\begin{aligned}&\big(\mathcal{R} : [\zeta(x)\mu(\vartheta_\delta)(\nabla \mathbf{u}_\delta + (\nabla \mathbf{u}_\delta)^T)] - \zeta(x)\mu(\vartheta_\delta)\mathcal{R} : [\nabla \mathbf{u}_\delta + (\nabla \mathbf{u}_\delta)^T]\big)\varrho_\delta \\ &\quad\to \big(\mathcal{R} : [\zeta(x)\mu(\vartheta)(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] - \zeta(x)\mu(\vartheta)\mathcal{R} : \big[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\big]\big)\varrho\end{aligned}$$

in $L^1(\Omega;\mathbb{R})$. Lemma 6 is thus proved. $\square$

Recall that $\varrho \in L^s(\Omega;\mathbb{R})$ for $s > 2$ and thus it fulfils the renormalized continuity equation. We use also (see e.g. [2, Appendix])

Lemma 7 (Weak convergence, monotone functions)**.** *Let $(P, G) \in C(\mathbb{R}) \times C(\mathbb{R})$ be a couple of nondecreasing functions. Assume that $\varrho_n \in L^1(\Omega;\mathbb{R})$ is a sequence such that*

$$\left.\begin{aligned} P(\varrho_n) &\rightharpoonup \overline{P(\varrho)}, \\ G(\varrho_n) &\rightharpoonup \overline{G(\varrho)}, \\ P(\varrho_n)G(\varrho_n) &\rightharpoonup \overline{P(\varrho)G(\varrho)} \end{aligned}\right\} \text{ in } L^1(\Omega;\mathbb{R}).$$

(i) *Then*

$$\overline{P(\varrho)}\,\overline{G(\varrho)} \leqslant \overline{P(\varrho)G(\varrho)}$$

a.e. in Ω.

(ii) *If, in addition,*

$$G(z) = z, \quad P \in C(\mathbb{R}), \;\; P \text{ non-decreasing}$$

and

$$\overline{P(\varrho)}\varrho = \overline{P(\varrho)\varrho}$$

(where we have denoted $\varrho = \overline{G(\varrho)}$), then

$$\overline{P(\varrho)} = P(\varrho).$$

Using Lemma 2 with the renormalization function $b(\varrho) = \varrho \ln \varrho$ implies

$$\int_\Omega \varrho \operatorname{div} \mathbf{u} \,\mathrm{d}x = 0 \tag{5.7}$$

as well as

$$\int_\Omega \varrho_\delta \operatorname{div} \mathbf{u}_\delta \,\mathrm{d}x = 0,$$

i.e.

$$\int_\Omega \overline{\varrho \operatorname{div} \mathbf{u}} \, \mathrm{d}x = 0. \tag{5.8}$$

Thus formula (5.5) implies

$$\overline{p_\gamma(\varrho,\vartheta)\varrho} = \overline{p_\gamma(\varrho,\vartheta)}\varrho,$$

i.e. in particular,

$$\overline{\varrho^{\gamma+1}} = \overline{\varrho^\gamma}\varrho$$

which, due to Lemma 7 and the properties of the L^p-spaces, yields immediately the strong convergence of the density. Theorem 1 for $\gamma > 1$ is proved.

5.2. Strong convergence for $\gamma = 1$

This case is slightly more delicate as we are not able to get (5.4). However, we can replace the test function $\boldsymbol{\varphi} = \zeta(x)\nabla\Delta^{-1}(1_\Omega\varrho_\delta)$ by $\boldsymbol{\varphi} = \zeta(x)\nabla\Delta^{-1}(1_\Omega\varrho_\delta^\Theta)$ for $0 < \Theta < 1$; similarly for the limit problem. Thus, exactly as in Lemma 6 we can get

Lemma 8. *Under the above assumption, for $\gamma = 1$, $0 < \Theta < 1$ arbitrary we have*

$$\overline{p_1(\varrho,\vartheta)\varrho^\Theta} - (\mu(\vartheta)+\xi(\vartheta))\overline{\varrho^\Theta \operatorname{div} \mathbf{u}} = \overline{p_1(\varrho,\vartheta)}\,\overline{\varrho^\Theta} - (\mu(\vartheta)+\xi(\vartheta))\overline{\varrho^\Theta} \operatorname{div} \mathbf{u} \tag{5.9}$$

a.e. in Ω.

Proof. Using as a test function in (3.9) $\boldsymbol{\varphi} = \zeta(x)\nabla\Delta^{-1}(1_\Omega\varrho_\delta^\Theta)$ and in (4.18) $\boldsymbol{\varphi} = \zeta(x)\nabla\Delta^{-1}(1_\Omega\overline{\varrho^\Theta})$ we get as above (note that $\nabla\Delta^{-1}(1_\Omega\varrho_\delta^\Theta) \to \nabla\Delta^{-1}(1_\Omega\overline{\varrho^\Theta})$ in $C(\overline{\Omega};\mathbb{R}^2)$)

$$\begin{aligned}
&\lim_{\delta\to 0^+} \int_\Omega \zeta(x)\big(p_1(\varrho_\delta,\vartheta_\delta)\varrho_\delta^\Theta - \mathbb{S}(\vartheta_\delta,\mathbf{u}_\delta) : \mathcal{R}[1_\Omega\varrho_\delta^\Theta]\big)\,\mathrm{d}x \\
&\quad= \int_\Omega \zeta(x)\big(\overline{p_1(\varrho,\vartheta)}\,\overline{\varrho^\Theta} - \mathbb{S}(\vartheta,\mathbf{u}) : \mathcal{R}[1_\Omega\overline{\varrho^\Theta}]\big)\,\mathrm{d}x \\
&\quad+ \lim_{\delta\to 0^+} \int_\Omega \zeta(x)\big(\varrho_\delta^\Theta\mathbf{u}_\delta\cdot\mathcal{R}[1_\Omega\varrho_\delta\mathbf{u}_\delta] - \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta) : \mathcal{R}[1_\Omega\varrho_\delta^\Theta]\big)\,\mathrm{d}x \\
&\quad- \int_\Omega \zeta(x)\big(\overline{\varrho^\Theta}\mathbf{u}\cdot\mathcal{R}[1_\Omega\varrho\mathbf{u}] - \varrho(\mathbf{u}\otimes\mathbf{u}) : \mathcal{R}[1_\Omega\overline{\varrho^\Theta}]\big)\,\mathrm{d}x.
\end{aligned} \tag{5.10}$$

Now we apply Lemma 4 with

$$\begin{aligned}
v_\delta = \varrho_\delta^\Theta &\rightharpoonup \overline{\varrho^\Theta} \quad \text{in } L^{2/\Theta}(\mathbb{R}^2;\mathbb{R}), \\
\mathbf{U}_\delta = \varrho_\delta\mathbf{u}_\delta &\rightharpoonup \varrho\mathbf{u} \quad \text{in } L^q(\mathbb{R}^2;\mathbb{R}^2),\ \ q < 2,
\end{aligned}$$

arbitrarily close to 2. It yields similarly to the above

$$\mathbf{u}_\delta \cdot \big[\varrho_\delta^\Theta \mathcal{R}[1_\Omega \varrho_\delta \mathbf{u}_\delta] - \mathcal{R}[1_\Omega \varrho_\delta^\Theta]\varrho_\delta \mathbf{u}_\delta\big] \rightharpoonup \mathbf{u}\big[\overline{\varrho^\Theta}\mathcal{R}[1_\Omega \varrho \mathbf{u}] - \mathcal{R}[1_\Omega \overline{\varrho^\Theta}]\varrho \mathbf{u}\big] \tag{5.11}$$

in $L^1(\mathbb{R}^2;\mathbb{R})$. Next we write

$$\begin{aligned}
&\int_\Omega \zeta(x)\mu(\vartheta_\delta)(\nabla \mathbf{u}_\delta + (\nabla \mathbf{u}_\delta)^T) : \mathcal{R}[1_\Omega \varrho_\delta^\Theta]\,\mathrm{d}x \\
&\quad = \int_\Omega \mathcal{R} : \big[\zeta(x)\mu(\vartheta_\delta)(\nabla \mathbf{u}_\delta + (\nabla \mathbf{u}_\delta)^T)\big]\varrho_\delta^\Theta\,\mathrm{d}x \\
&\quad = \int_\Omega \zeta(x) 2\mu(\vartheta_\delta)\operatorname{div}\mathbf{u}_\delta \varrho_\delta^\Theta\,\mathrm{d}x + \int_\Omega \mathcal{R} : \big[\zeta(x)\mu(\vartheta_\delta)(\nabla \mathbf{u}_\delta + (\nabla \mathbf{u}_\delta)^T)\big]\varrho_\delta^\Theta\,\mathrm{d}x \\
&\qquad - \int_\Omega \zeta(x)\mu(\vartheta_\delta)\mathcal{R} : [\nabla \mathbf{u}_\delta + (\nabla \mathbf{u}_\delta)^T]\varrho_\delta^\Theta\,\mathrm{d}x
\end{aligned}$$

and apply exactly as above Lemma 5 with $w = \zeta(x)\mu(\vartheta) \sim (1+\vartheta)$, $r < 2$ and $z_i = \partial_j u_i + \partial_i u_j$, $j = 1,2,3$, $p = 2$ to get (5.9). □

Note that due to Lemma 2 the continuity equation is satisfied also in the renormalized sense. Thus for $0 < \Theta < 1$

$$\operatorname{div}(\varrho_\delta^\Theta \mathbf{u}_\delta) + (\Theta - 1)\varrho_\delta^\Theta \operatorname{div}\mathbf{u}_\delta = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2) \tag{5.12}$$

and passing to the limit

$$\operatorname{div}(\overline{\varrho_\delta^\Theta}\mathbf{u}_\delta) + (\Theta - 1)\overline{\varrho_\delta^\Theta \operatorname{div}\mathbf{u}_\delta} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{5.13}$$

Repeating the computations from [12, Lemma 4.39] we arrive at (here we also need that $\varrho \in L^2(\Omega;\mathbb{R})$!)

$$\operatorname{div}((\overline{\varrho^\Theta})^{1/\Theta}\mathbf{u}) = \frac{1-\Theta}{\Theta}(\overline{\varrho^\Theta})^{1/\Theta - 1}(\overline{\varrho_\delta^\Theta \operatorname{div}\mathbf{u}_\delta} - \overline{\varrho_\delta^\Theta}\operatorname{div}\mathbf{u}). \tag{5.14}$$

Therefore, using (5.9) and (4.17), $0 < \Theta < 1$,

$$\operatorname{div}\big(((\overline{\varrho^\Theta})^{1/\Theta} - \varrho)\mathbf{u}\big) = \frac{1-\Theta}{\Theta}(\overline{\varrho^\Theta})^{1/\Theta - 1}\frac{\overline{p_1(\varrho,\vartheta)\varrho^\Theta} - \overline{p_1(\varrho,\vartheta)}\,\overline{\varrho^\Theta}}{\mu(\vartheta) + \xi(\vartheta)} \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{5.15}$$

where all functions are extended by 0 outside of Ω. Hence, testing (5.15) by $\psi \equiv 1$ reads

$$\int_\Omega \frac{\overline{p_1(\varrho,\vartheta)\varrho^\Theta} - \overline{p_1(\varrho,\vartheta)}\,\overline{\varrho^\Theta}}{\mu(\vartheta) + \xi(\vartheta)}(\overline{\varrho^\Theta})^{1/\Theta - 1} = 0. \tag{5.16}$$

It is not difficult to see (cf. [12, Subsection 4.9.4]) that $\varrho_\delta \to 0$ in $L^1(\{\overline{\varrho^\Theta} = 0\})$. Thus, due to the strong convergence of the temperature and monotonicity of the mapping

$$t \mapsto \frac{t^2}{1+t} \ln^\alpha(1+t)$$

we arrive at

$$\overline{\frac{\varrho^2}{1+\varrho} \ln^\alpha(1+\varrho)\varrho^\Theta} + \vartheta\overline{\varrho^{1+\Theta}} = \overline{\frac{\varrho^2}{1+\varrho} \ln^\alpha(1+\varrho)}\,\overline{\varrho^\Theta} + \vartheta\varrho\overline{\varrho^\Theta}. \tag{5.17}$$

This immediately implies due to Lemma 7 (recall that $\vartheta > 0$ a.e. in Ω due to the a priori estimate (4.3))

$$\overline{\varrho^\Theta} = \varrho^\Theta \quad \text{a.e. in } \Omega; \tag{5.18}$$

thus also

$$\overline{\varrho^{1+\Theta}} = \varrho^{1+\Theta} \quad \text{a.e. in } \Omega, \tag{5.19}$$

yielding the strong convergence of the density. The proof of Theorem 1 is complete. □

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