Commentationes Mathematicae Universitatis Carolinae

Ryotaro Sato

Ratio Tauberian theorems for relatively bounded functions and sequences in Banach spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 52 (2011), No. 1, 77--88

Persistent URL: http://dml.cz/dmlcz/141429

Ratio Tauberian theorems for relatively bounded functions and sequences in Banach spaces

Ryotaro Sato

Abstract. We prove ratio Tauberian theorems for relatively bounded functions and sequences in Banach spaces.

Keywords: ratio Tauberian theorem, γ-th order Cesàro integral, Laplace integral, γ-th order Cesàro sum, Abel sum

Classification: 40E05, 47A35

1. Introduction

Let X be a Banach space and $u : [0, \infty) \to X$ be a locally integrable function. Let $g : [0, \infty) \to \mathbb{R}_+$ be a locally integrable function such that $\int_0^\infty g(t)\,dt > 0$, where $\mathbb{R}_+ := \{t \geq 0 : t \in \mathbb{R}\}$. We assume the condition

$$\frac{\int_0^t g(r)\,dr}{\int_0^s g(r)\,dr} \to 1 \quad \text{as} \quad t, s \to \infty \quad \text{with} \quad \frac{t}{s} \to 1,$$

and prove that if $\|u(t)\| = O(g(t))$, $t \to \infty$, then the following statements are equivalent:

(i) $x = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds \big) / \big(\int_0^t g(s)\,ds \big)$;
(ii) $x = \lim_{\lambda\downarrow 0} \big(\int_0^\infty e^{-\lambda t} u(t)\,dt \big) / \big(\int_0^\infty e^{-\lambda t} g(t)\,dt \big)$.

This solves the open problem posed in [6]. Then particular choices of the function g will be considered, leading to some generalized Tauberian theorems. Discrete analogues are obtained as well.

2. Results for functions

Let X be a Banach space and $u : [0, \infty) \to X$ be a locally integrable function. The class of all such functions will be denoted by $L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$. For $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$, $\gamma \geq 1$ and $t > 0$ we define the γ-th order Cesàro integral $\mathfrak{s}_t^\gamma(u)$ over $[0, t]$ as

$$(1) \qquad \mathfrak{s}_t^\gamma(u) := (k_\gamma * u)(t) = \int_0^t k_\gamma(t-s) u(s)\,ds,$$

where $k_\gamma(t) := t^{\gamma-1}/\Gamma(\gamma)$ for $t \in \mathbb{R}_+$. In particular we have $\mathfrak{s}_t^1(u) = \int_0^t u(s)\,ds$. The Laplace integral $\widehat{u}(\lambda)$ for $\lambda \in \mathbb{R}$ is defined as

$$\widehat{u}(\lambda) := \int_0^\infty e^{-\lambda t} u(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-\lambda t} u(t)\,dt \tag{2}$$

if the limit exists. It is known (see e.g. [1, Proposition 1.4.1]) that if $\widehat{u}(\lambda_0)$ exists then $\widehat{u}(\lambda)$ exists for all $\lambda > \lambda_0$. If μ is a locally finite positive measure on $\mathbb{R}_+$, then we use the notation $\widehat{\mu}(\lambda)$ to denote $\int_0^\infty e^{-\lambda t} d\mu(t)$ when $\int_0^\infty e^{-\lambda t} d\mu(t) < \infty$.

We begin with the following key lemma.

Lemma 2.1. *Let μ be a locally finite positive measure on $\mathbb{R}_+$ such that $\mu[0,\infty) > 0$. If*

$$\frac{\mu[0,t]}{\mu[0,s]} \to 1 \quad as \quad t, s \to \infty \quad with \quad \frac{t}{s} \to 1, \tag{C}$$

then

$$\liminf_{\lambda\downarrow 0} \frac{\mu[0,1/\lambda]}{\widehat{\mu}(\lambda)} = \liminf_{\lambda\downarrow 0} \frac{\mu[0,1/\lambda]}{\int_0^\infty e^{-\lambda t} d\mu(t)} > 0. \tag{C1}$$

Proof: By hypothesis there are two constants $G > 1$ and $\delta > 0$ such that if $t > s > G$ and $t/s \le 1+\delta$ then

$$0 \le \frac{\mu(s,t]}{\mu[0,s]} < 1.$$

Thus for $\lambda > 0$ with $1/\lambda > G$ we have $\mu(1/\lambda,\, (1+\delta)/\lambda] < 2^0 \mu[0,1/\lambda]$, and

$$\mu((1+\delta)/\lambda,\, (1+\delta)^2/\lambda] < \mu[0,(1+\delta)/\lambda] < 2^1 \mu[0,1/\lambda].$$

Then for $n \ge 2$ we have inductively

$$\begin{aligned}
\mu((1+\delta)^n/\lambda,\, (1+\delta)^{n+1}/\lambda] &< \mu[0,(1+\delta)^n/\lambda] \\
&= \mu[0,1/\lambda] + \sum_{k=0}^{n-1} \mu((1+\delta)^k/\lambda,\, (1+\delta)^{k+1}/\lambda] \\
&< \Big(1 + \sum_{k=0}^{n-1} 2^k\Big) \mu[0,1/\lambda] = 2^n \mu[0,1/\lambda].
\end{aligned}$$

Hence

$$\begin{aligned}
0 < \int_0^\infty e^{-\lambda t} d\mu(t) &= \int_{[0,1/\lambda]} e^{-\lambda t} d\mu(t) + \sum_{n=0}^\infty \int_{((1+\delta)^n/\lambda,\,(1+\delta)^{n+1}/\lambda]} e^{-\lambda t}\,d\mu(t) \\
&\le \mu[0,1/\lambda] + \sum_{n=0}^\infty 2^n \mu[0,1/\lambda] e^{-(1+\delta)^n} < \infty.
\end{aligned}$$

Therefore

$$\frac{\mu[0,1/\lambda]}{\widehat{\mu}(\lambda)} = \frac{\mu[0,1/\lambda]}{\int_0^\infty e^{-\lambda t} d\mu(t)} \ge \Big(1 + \sum_{n=0}^{\infty} 2^n e^{-(1+\delta)^n}\Big)^{-1} > 0,$$

completing the proof. □

Theorem 2.2 (cf. [2, Theorem 2.2]). *Suppose $0 \neq g \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies condition* (C) *with $\mu := g(t)\,dt$. Then for any $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ with $\|u(t)\| = O(g(t))$, $t \to \infty$, the following statements are equivalent:*

(i) $x = \lim_{t\to\infty} \mathfrak{s}^1_t(u)/\mathfrak{s}^1_t(g) = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$;

(ii) $x = \lim_{t\to\infty} \mathfrak{s}^\beta_t(u)/\mathfrak{s}^\beta_t(g)$
$= \lim_{t\to\infty} \big(\int_0^t (t-s)^{\beta-1} u(s)\,ds\big)/\big(\int_0^t (t-s)^{\beta-1} g(s)\,ds\big)$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0} \widehat{u}(\lambda)/\widehat{g}(\lambda) = \lim_{\lambda\downarrow 0} \big(\int_0^\infty e^{-\lambda t} u(t)\,dt\big)/\big(\int_0^\infty e^{-\lambda t} g(t)\,dt\big)$.

Proof: "(i) ⇒ (ii) ⇒ (iii)" follows from [2, Theorem 2.1].

(iii) ⇒ (i): We first note that if $P(t) = \sum_{n=0}^N a_n t^n$ is a polynomial function such that

$$P(t) \ge d > 0 \quad \text{on} \quad [0,1], \tag{3}$$

then

$$x = \lim_{\lambda\downarrow 0} \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt}. \tag{4}$$

To see this, put $\widetilde{P}(\lambda) := \int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt$. Then

$$\frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt} = \frac{1}{\widetilde{P}(\lambda)} \sum_{n=0}^{N} a_n \Big(\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt\Big) \cdot \frac{\int_0^\infty e^{-\lambda(n+1)t} u(t)\,dt}{\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt}.$$

Here

$$\lim_{\lambda\downarrow 0} \frac{\int_0^\infty e^{-\lambda(n+1)t} u(t)\,dt}{\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt} = x \qquad \text{(by (iii))}, \tag{5}$$

and

$$0 < \frac{\int_0^\infty e^{-\lambda(n+1)t} g(t)\,dt}{\widetilde{P}(\lambda)} \le \frac{1}{d} \qquad \text{(by (3))}. \tag{6}$$

Thus

$$\lim_{\lambda\downarrow 0}\frac{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}P(e^{-\lambda t})g(t)\,dt} = \lim_{\lambda\downarrow 0}\frac{1}{\widetilde{P}(\lambda)}\sum_{n=0}^{N}a_n\Big(\int_0^\infty e^{-\lambda(n+1)t}g(t)\,dt\Big)x = x.$$

Next we write

$$\frac{\int_0^{1/\lambda}u(t)\,dt}{\int_0^{1/\lambda}g(t)\,dt} = \frac{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt}\,, \tag{7}$$

where

$$h(t) := \begin{cases} 0 & \text{if } 0 \le t < e^{-1}, \\ t^{-1} & \text{if } e^{-1} \le t \le 1. \end{cases}$$

For the proof we may assume without loss of generality that $\|u(t)\| \le g(t)$ for all $t \ge 0$. By condition (C), given an $\varepsilon > 0$, there are two constants $G > 1$ and $\delta > 0$ such that

$$0 \le \frac{\mu(s,t]}{\mu[0,s]} < \varepsilon \quad \text{if} \quad t > s > G \quad \text{and} \quad \frac{t}{s} \le 1+\delta. \tag{8}$$

It is standard to see that there exists a polynomial function $P(t) = \sum_{n=0}^{N} a_n t^n$ such that

(a) $h(t) < P(t) \le \varepsilon$ on $[0, e^{-(1+\delta)}]$,
(b) $h(t) < P(t) \le h(e^{-1}) + \varepsilon$ on $[e^{-(1+\delta)},\, e^{-1}]$,
(c) $h(t) < P(t) \le h(t) + \varepsilon$ on $[e^{-1},\, 1]$.

Then

$$\begin{aligned}
&\frac{\int_0^{1/\lambda}u(t)\,dt}{\int_0^{1/\lambda}g(t)\,dt} - x \\
&\quad = \frac{\int_0^\infty e^{-\lambda t}\big(h(e^{-\lambda t}) - P(e^{-\lambda t})\big)u(t)\,dt + \int_0^\infty e^{-\lambda t}P(e^{-\lambda t})u(t)\,dt}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt} - x \\
&\quad =: I_\lambda + II_\lambda - x,
\end{aligned}$$

and

$$\begin{aligned}
I_\lambda &= \frac{\Big(\int_0^{1/\lambda} + \int_{1/\lambda}^{(1+\delta)/\lambda} + \int_{(1+\delta)/\lambda}^\infty\Big)e^{-\lambda t}\big(h(e^{-\lambda t}) - P(e^{-\lambda t})\big)u(t)\,dt}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt} \\
&=: \frac{I_\lambda(1) + I_\lambda(2) + I_\lambda(3)}{\int_0^\infty e^{-\lambda t}h(e^{-\lambda t})g(t)\,dt},
\end{aligned}$$

where

$$\|I_\lambda(1)\| < \int_0^{1/\lambda} e^{-\lambda t}\varepsilon\, g(t)\, dt \le \varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt \tag{9}$$

by (c) and the assumption that $\|u(t)\| \le g(t)$ for all $t \ge 0$. On the other hand, (b) implies

$$\|I_\lambda(2)\| < \int_{1/\lambda}^{(1+\delta)/\lambda} e^{-\lambda t}\big(h(e^{-1}) + \varepsilon\big) g(t)\, dt \le (e+\varepsilon) \int_{1/\lambda}^{(1+\delta)/\lambda} g(t)\, dt,$$

where if $\lambda > 0$ is sufficiently small, then by (8)

$$\int_{1/\lambda}^{(1+\delta)/\lambda} g(t)\, dt < \varepsilon \int_0^{1/\lambda} g(t) dt = \varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt,$$

so that

$$\|I_\lambda(2)\| < (e+\varepsilon)\, \varepsilon \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt \tag{10}$$

for all sufficiently small $\lambda > 0$. Finally (a) implies

$$\|I_\lambda(3)\| < \int_{(1+\delta)/\lambda}^\infty e^{-\lambda t}\varepsilon\, g(t)\, dt \le \varepsilon \int_0^\infty e^{-\lambda t} g(t)\, dt.$$

We apply Lemma 2.1 to infer that there exists a constant $\eta > 0$ such that

$$\liminf_{\lambda \downarrow 0} \frac{\int_0^{1/\lambda} g(t)\, dt}{\int_0^\infty e^{-\lambda t} g(t)\, dt} > \eta.$$

Thus if $\lambda > 0$ is sufficiently small, then

$$\int_0^{1/\lambda} g(t)\, dt > \eta \int_0^\infty e^{-\lambda t} g(t)\, dt,$$

so that

$$\|I_\lambda(3)\| < \frac{\varepsilon}{\eta} \int_0^{1/\lambda} g(t)\, dt = \frac{\varepsilon}{\eta} \int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt. \tag{11}$$

Consequently

$$\limsup_{\lambda \downarrow 0} \|I_\lambda\| < \varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\,. \tag{12}$$

Now we write

$$II_\lambda = \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\, dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\, dt} \cdot \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\, dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\, dt}\,.$$

Similarly as in (12) one may show that if $\lambda > 0$ is sufficiently small, then

$$1 \le \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt}{\int_0^\infty e^{-\lambda t} h(e^{-\lambda t}) g(t)\,dt} < 1 + \varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}.$$

Since $P(t) \ge d > 0$ on $[0,1]$ for some $d > 0$, it follows that

$$\lim_{\lambda \downarrow 0} \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt} = x.$$

Hence

$$\begin{aligned} \|II_\lambda - x\| &\le \left\| \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt} - x \right\| \\ &\quad + \Big(\varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\Big) \left\| \frac{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) u(t)\,dt}{\int_0^\infty e^{-\lambda t} P(e^{-\lambda t}) g(t)\,dt} \right\| \\ &\longrightarrow \Big(\varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\Big)\|x\| \end{aligned} \tag{13}$$

as $\lambda \downarrow 0$. Combining this with (12) yields

$$\limsup_{\lambda \downarrow 0} \left\| \frac{\int_0^{1/\lambda} u(t)\,dt}{\int_0^{1/\lambda} g(t)\,dt} - x \right\| < \Big(\varepsilon + (e+\varepsilon)\varepsilon + \frac{\varepsilon}{\eta}\Big)(1 + \|x\|) \tag{14}$$

which completes the proof, since $\varepsilon > 0$ is arbitrary. □

Theorem 2.3 (cf. [2, Theorem 4.2], [5, Proposition 3.4])**.** *Let $\alpha \ge 0$. Suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{\alpha-1})$, $t \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{t\to\infty} \big(\Gamma(\alpha+1)/t^\alpha\big) \int_0^t u(s)\,ds$;

(ii) $x = \lim_{t\to\infty} \big(\Gamma(\alpha+\beta)/\Gamma(\beta)t^{\alpha+\beta-1}\big) \int_0^t (t-s)^{\beta-1} u(s)\,ds$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0} \lambda^\alpha\, \widehat{u}(\lambda) = \lim_{\lambda\downarrow 0} \lambda^\alpha \int_0^\infty e^{-\lambda t} u(t)\,dt$.

Proof: "(i) ⇒ (ii) ⇒ (iii)" follows from [2, Theorem 4.1].

(iii) ⇒ (i): Suppose $\alpha > 0$. Then define $g(t) := k_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t \in \mathbb{R}_+$ and $\mu := g(t)\,dt$. It follows that $\|u(t)\| = O(g(t))$, $t \to \infty$, that $\widehat{g}(\lambda) = \lambda^\alpha$ for all $\lambda > 0$, and that $\mu[0,t] = \int_0^t k_\alpha(s)\,ds = (k_1 * k_\alpha)(t) = k_{\alpha+1}(t) = t^\alpha/\Gamma(\alpha+1)$. Hence μ satisfies condition (C), and so (i) follows from Theorem 2.2.

Next suppose $\alpha = 0$. Since $\|u(t)\| = O(t^{-1})$, $t \to \infty$, it follows from standard calculations (see e.g. [8, pp. 204, 206]) that the function $U(t) := \int_0^t u(s)\,ds$ is bounded and feebly oscilating (i.e. $\|U(t) - U(s)\| \to 0$ as t and $s \to \infty$ in such a way that $t/s \to 1$). Thus (i) follows from [5, Proposition 3.4]. The proof is complete. □

Remark. The special case $\alpha = 1$ of Theorem 2.3 states that, under the assumption that u is bounded, the Cesàro limit $\lim_{t\to\infty}(1/t)\int_0^t u(s)\,ds$ exists if and only if the Abel limit $\lim_{\lambda\downarrow 0}\lambda\,\widehat{u}(\lambda)$ exists and they are both equal. This is a classical Tauberian theorem (see e.g. [1, Theorem 4.2.7]). The special case $\alpha = 0$ of Theorem 2.3 states that, under the assumption that $\|u\| = O(t^{-1})$, $t\to\infty$, the limit $\lim_{t\to\infty}\int_0^t u(s)\,ds$ exists if and only if the limit $\lim_{\lambda\downarrow 0}\widehat{u}(\lambda)$ exists and they are both equal. This is another classical Tauberian theorem (see e.g. [1, Theorem 4.2.9]).

Theorem 2.4. *Suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{t\to\infty}\big(1/\log t\big)\int_0^t u(s)\,ds$;

(ii) $x = \lim_{t\to\infty}\big(1/t^{\beta-1}\log t\big)\int_0^t (t-s)^{\beta-1}u(s)\,ds$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0}\big(1/-\log\lambda\big)\widehat{u}(\lambda) = \lim_{\lambda\downarrow 0}\big(1/-\log\lambda\big)\int_0^\infty e^{-\lambda t}u(t)\,dt$.

Proof: Let

$$g(t) := \begin{cases} 0 & \text{if } 0 \le t < 1, \\ t^{-1} & \text{if } t \ge 1. \end{cases}$$

An approximation argument yields that

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_0^t (t-s)^{\gamma-1}g(s)\,ds}{t^{\gamma-1}\log t} &= \lim_{t\to\infty}\frac{\int_1^t (t-s)^{\gamma-1}s^{-1}\,ds}{t^{\gamma-1}\log t} \\ &= \lim_{t\to\infty}\frac{\int_{1/t}^1 (1-s)^{\gamma-1}s^{-1}\,ds}{\log t} = 1\end{aligned} \tag{15}$$

for all $\gamma \ge 1$ and that

$$\lim_{\lambda\downarrow 0}\frac{\widehat{g}(\lambda)}{-\log\lambda} = \lim_{\lambda\downarrow 0}\frac{\int_1^\infty e^{-\lambda t}t^{-1}\,dt}{-\log\lambda} = \lim_{\lambda\downarrow 0}\frac{\int_\lambda^\infty e^{-t}t^{-1}\,dt}{-\log\lambda} = 1. \tag{16}$$

Since $\|u(t)\| = O(g(t))$, $t\to\infty$, and the measure $\mu := g(t)\,dt$ satisfies condition (C), the desired result follows from Theorem 2.2. □

Remark. If X is a Banach lattice with positive cone X_+ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then statements (i), (ii) and (iii) in Theorem 2.4 are also equivalent. This follows from [2, Theorem 2.2]. (We note that if $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then statement (ii) in Theorem 2.4 implies that $\widehat{u}(\lambda)$ exists for all $\lambda > 0$ (see [3, Lemma 2.5]).)

Fact 2.5. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$. Consider the following three statements:*

(i) $x = \lim_{t\to\infty}\int_0^t u(s)\,ds$;

(ii) $\widehat{u}(\lambda)$ *exists for all* $\lambda > 0$ *and* $x = \lim_{t\to\infty}\big(1/t^{\beta-1}\big)\int_0^t (t-s)^{\beta-1}u(s)\,ds$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{\lambda\downarrow 0}\widehat{u}(\lambda) = \lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}u(t)\,dt$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof: Letting $g(t) := \chi_{[0,1]}(t)$ we have

$$\lim_{t\to\infty} \frac{\int_0^t (t-s)^{\gamma-1} g(s)\,ds}{t^{\gamma-1}} = \lim_{t\to\infty} \frac{\int_0^1 (t-s)^{\gamma-1}\,ds}{t^{\gamma-1}} = 1$$

for all $\gamma \geq 1$ and

$$\lim_{\lambda\downarrow 0} \widehat{g}(\lambda) = \lim_{\lambda\downarrow 0} \int_0^1 e^{-\lambda t}\,dt = 1.$$

Thus the desired result follows from [2, Theorem 2.1]. □

Remarks. (a) If $\int_0^\infty \|u(t)\|\,dt < \infty$, then clearly both (i) and (iii) in Fact 2.5 hold. In general (iii) does not imply (i). (For example let $u(t) := \sin t$.) If $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$, or if X is a Banach lattice and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X_+)$, then (iii) implies (i). (See Theorem 2.3 and [2, Theorem 4.2], respectively.)

(b) There exists a continuous function $u : [0,\infty) \to \mathbb{R}$ such that $\inf\{\lambda \in \mathbb{R} : \widehat{u}(\lambda) \text{ exists}\} = 1$ and also such that $\lim_{t\to\infty} (1/t) \int_0^t (t-s)u(s)\,ds$ $(\in \mathbb{R})$ exists (see the Remark over Theorem 2.4 in [3], or [7, Example 5]). Thus the hypothesis that $\widehat{u}(\lambda)$ exists for all $\lambda > 0$ cannot be omitted from (ii) in Fact 2.5.

3. Results for sequences

Let $\{x_n\} := \{x_n\}_{n=0}^\infty$ be a sequence in a Banach space X. For $\gamma \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, we define the γ-th order Cesàro sum $\mathfrak{s}_n^\gamma(\{x_i\})$ as

$$\mathfrak{s}_n^\gamma(\{x_i\}) := \sum_{k=0}^n \binom{n-k+\gamma-1}{n-k} x_k, \tag{17}$$

where $\binom{r}{0} := 1$ and $\binom{r}{n} := r(r-1)\dots(r-n+1)/n!$ for $r \in \mathbb{R}$ and $n \geq 1$. Thus $\mathfrak{s}_0^\gamma(\{x_i\}) = x_0$ for all $\gamma \in \mathbb{R}$, $\mathfrak{s}_n^0(\{x_i\}) = x_n$ and $\mathfrak{s}_n^1(\{x_i\}) = \sum_{k=0}^n x_k$ for all $n \in \mathbb{N}_0$. The Abel sum $\widehat{\{x_i\}}(r)$ of $\{x_n\}$ is defined as

$$\widehat{\{x_i\}}(r) := \sum_{n=0}^\infty r^n x_n, \qquad 0 < r < \Big(\limsup_{n\to\infty} \|x_n\|^{1/n}\Big)^{-1}. \tag{18}$$

Clearly $\widehat{\{x_i\}}(r)$ exists for all $0 < r < 1$ if and only if $\limsup_{n\to\infty} \|x_n\|^{1/n} \leq 1$. Let $\{a_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers such that $\sum_{n=0}^\infty a_n > 0$. We define $u(t) := x_{[t]}$ and $g(t) := a_{[t]}$ for $t \geq 0$, where $[t]$ denotes the largest integer less than or equal to t. Then we have the following

Lemma 3.1. (i) $x = \lim_{n\to\infty} \big(\sum_{k=0}^n x_k\big)/\big(\sum_{k=0}^n a_k\big)$ *if and only if* $x = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$.

(ii) *Suppose* $\widehat{\{x_i\}}(r)$ *and* $\widehat{\{a_i\}}(r)$ *exist for all* $0 < r < 1$. *Then*

$$x = \lim_{r\uparrow 1} \frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \lim_{r\uparrow 1} \frac{\sum_{n=0}^{\infty} r^n x_n}{\sum_{n=0}^{\infty} r^n a_n}$$

if and only if

$$x = \lim_{\lambda\downarrow 0} \frac{\widehat{u}(\lambda)}{\widehat{g}(\lambda)} = \lim_{\lambda\downarrow 0} \frac{\int_0^\infty e^{-\lambda t} u(t)\, dt}{\int_0^\infty e^{-\lambda t} g(t)\, dt}.$$

Proof: (i) Putting $\delta(t) := t - [t]$ we have $0 \le \delta(t) < 1$, and

$$\frac{\int_0^t u(s)\, ds}{\int_0^t g(s)\, ds} = \frac{\big(1-\delta(t)\big)\sum_{k=0}^{[t]-1} x_k + \delta(t)\sum_{k=0}^{[t]} x_k}{\big(1-\delta(t)\big)\sum_{k=0}^{[t]-1} a_k + \delta(t)\sum_{k=0}^{[t]} a_k},$$

so that the first condition of (i) implies the second condition. The converse implication is obvious.

(ii) By an elementary calculation we have

$$\frac{\int_0^\infty e^{-\lambda t} u(t)\, dt}{\int_0^\infty e^{-\lambda t} g(t)\, dt} = \frac{\sum_{n=0}^\infty e^{-\lambda n} x_n}{\sum_{n=0}^\infty e^{-\lambda n} a_n}, \qquad \lambda > 0,$$

whence (ii) follows. □

Theorem 3.2 (cf. [2, Theorem 3.2])**.** *Let* $\{a_n\}_{n=0}^\infty$ *be a sequence of nonnegative real numbers such that* $\sum_{n=0}^\infty a_n > 0$. *Suppose*

$$\text{(D)} \qquad \frac{\sum_{k=0}^m a_k}{\sum_{k=0}^n a_k} \to 1 \quad \text{as} \quad m, n \to \infty \quad \text{with} \quad \frac{m}{n} \to 1.$$

Then for any sequence $\{x_n\}_{n=0}^\infty$ *in* X, *with* $\|x_n\| = O(a_n)$, $n \to \infty$, *the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \mathfrak{s}_n^1(\{x_i\})/\mathfrak{s}_n^1(\{a_i\}) = \lim_{n\to\infty} \big(\sum_{k=0}^n x_k\big)/\big(\sum_{k=0}^n a_k\big)$;

(ii) $x = \lim_{n\to\infty} \mathfrak{s}_n^\beta(\{x_i\})/\mathfrak{s}_n^\beta(\{a_i\})$ *for some/all* $\beta > 1$;

(iii) $x = \lim_{r\uparrow 1} \widehat{\{x_i\}}(r)/\widehat{\{a_i\}}(r) = \lim_{r\uparrow 1} \big(\sum_{n=0}^\infty r^n x_n\big)/\big(\sum_{n=0}^\infty r^n a_n\big)$.

Proof: Condition (D) implies that the function $g(t) = a_{[t]}$ satisfies condition (C) with $\mu := g(t)\, dt$. Hence $\widehat{\{a_i\}}(r)$ and $\widehat{\{x_i\}}(r)$ exist for all $0 < r < 1$. Then "(i) ⇒ (ii) ⇒ (iii)" follows from [2, Theorem 3.1].

(iii) ⇒ (i): By Lemma 3.1 and Theorem 2.2 we have

$$x = \lim_{r\uparrow 1} \frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \lim_{\lambda\downarrow 0} \frac{\widehat{u}(\lambda)}{\widehat{g}(\lambda)} = \lim_{t\to\infty} \frac{\mathfrak{s}_t^1(u)}{\mathfrak{s}_t^1(g)} = \lim_{n\to\infty} \frac{\mathfrak{s}_n^1(\{x_i\})}{\mathfrak{s}_n^1(\{a_i\})},$$

which completes the proof. □

Theorem 3.3 (cf. [2, Theorem 4.4], [5, Proposition 3.6])**.** *Let $\alpha \geq 0$. Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\|x_n\| = O(n^{\alpha-1})$, $n \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \big(\Gamma(\alpha+1)/(n+1)^{\alpha}\big) \sum_{k=0}^{n} x_k$;
(ii) $x = \lim_{n\to\infty} \big(\Gamma(\alpha+\beta)/(n+1)^{\alpha+\beta-1}\big)\mathfrak{s}_n^{\beta}(\{x_i\})$ *for some/all* $\beta > 1$;
(iii) $x = \lim_{r\uparrow 1} (1-r)^{\alpha}\widehat{\{x_i\}}(r) = \lim_{r\uparrow 1} (1-r)^{\alpha} \sum_{n=0}^{\infty} r^n x_n$.

Proof: "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)" follows from [2, Theorem 4.3].

(iii) $\Rightarrow$ (i): Suppose $\alpha > 0$. Then define $a_n := \binom{n+\alpha-1}{n}$ for $n \geq 0$. It follows (cf. [9, pp. 76–77]) that $(1-r)^{-\alpha} = \sum_{n=0}^{\infty} r^n a_n$ for $0 < r < 1$, and $a_n = n^{\alpha-1}(1+o(1))/\Gamma(\alpha)$, $n \to \infty$. Thus $\|x_n\| = O(a_n)$, $n \to \infty$. Since

$$\sum_{k=0}^{n} a_k = \binom{n+\alpha}{n} = \frac{n^{\alpha}}{\Gamma(\alpha+1)}\big(1+o(1)\big), \quad n \to \infty,$$

$\{a_n\}_{n=0}^{\infty}$ satisfies condition (D). Hence (i) follows from Theorem 3.2.

Next suppose $\alpha = 0$. Then the function $u(t) = x_{[t]}$ satisfies $\|u(t)\| = O(t^{-1})$, $t \to \infty$, and (iii) implies that $x = \lim_{\lambda\downarrow 0} \int_0^{\infty} e^{-\lambda t} u(t)\,dt$. Hence, by Theorem 2.3, $x = \lim_{t\to\infty} \int_0^t u(s)\,ds = \lim_{n\to\infty} \sum_{k=0}^{n} x_k$. This completes the proof. □

Remark. The special cases $\alpha = 1$ and $\alpha = 0$ of Theorem 3.3 are classical results for sequences corresponding to $\alpha = 1$ and $\alpha = 0$ of Theorem 2.3, respectively. (See e.g. [4, Theorem 3.1], [1, Theorem 4.2.17].)

Theorem 3.4. *Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\|x_n\| = O(n^{-1})$, $n \to \infty$. Then the following statements are equivalent:*

(i) $x = \lim_{n\to\infty} \big(1/\log(n+1)\big) \sum_{k=0}^{n} x_k$;
(ii) $x = \lim_{n\to\infty} \big(\Gamma(\beta)/(n+1)^{\beta-1}\log(n+1)\big)\, \mathfrak{s}_n^{\beta}(\{x_i\})$ *for some/all* $\beta > 1$;
(iii) $x = \lim_{\lambda\downarrow 0} \big(1/-\log\lambda\big)\widehat{\{x_i\}}(e^{-\lambda}) = \lim_{\lambda\downarrow 0} \big(1/-\log\lambda\big) \sum_{n=0}^{\infty} e^{-\lambda n} x_n$.

Proof: Define $a_0 := 0$ and $a_n := n^{-1}$ for $n \geq 1$. Hence $\|x_n\| = O(a_n)$, $n \to \infty$, and $\sum_{k=0}^{n} a_k = \log n + O(1)$, $n \to \infty$. It follows that $\{a_n\}_{n=0}^{\infty}$ satisfies condition (D). If $\beta > 1$, then

$$\mathfrak{s}_n^{\beta}(\{a_i\}) = \sum_{k=1}^{n} \binom{n-k+\beta-1}{n-k}\frac{1}{k}. \tag{19}$$

Since

$$\binom{n+\beta-1}{n} = \frac{n^{\beta-1}}{\Gamma(\beta)}\big(1+o(1)\big), \quad n \to \infty,$$

it follows by an approximation argument that

$$\begin{aligned}\mathfrak{s}_n^\beta(\{a_i\}) &= \int_1^n \frac{(n-s)^{\beta-1}}{\Gamma(\beta)} s^{-1}\,ds \cdot \bigl(1+o(1)\bigr) \\ &= \frac{n^{\beta-1}\log n}{\Gamma(\beta)} \cdot \bigl(1+o(1)\bigr), \quad n\to\infty \qquad \text{(by (15))}.\end{aligned}$$

Similarly

$$\begin{aligned}\widehat{\{a_i\}}(e^{-\lambda}) &= \sum_{n=1}^\infty e^{-\lambda n} n^{-1} = \int_1^\infty e^{-\lambda t} t^{-1}\,dt \cdot \bigl(1+o(1)\bigr) \\ &= -\log\lambda \cdot \bigl(1+o(1)\bigr), \quad \lambda\downarrow 0 \qquad \text{(by (16))}.\end{aligned}$$

Hence the desired result follows from Theorem 3.2. □

Remark. If X is a Banach lattice and $\{x_n\}_{n=0}^\infty \subset X_+$, then statements (i), (ii) and (iii) in Theorem 3.4 are also equivalent. This follows from [2, Theorem 3.2]. (We note that statement (ii) in Theorem 3.4 implies that $\widehat{\{x_i\}}(r)$ exists for all $0<r<1$.)

Fact 3.5. *Let $\{x_n\}_{n=0}^\infty$ be a sequence in X. Consider the following three statements:*

(i) $x = \lim_{n\to\infty} \sum_{k=0}^n x_k$;
(ii) $x = \lim_{n\to\infty} \bigl(\Gamma(\beta)/(n+1)^{\beta-1}\bigr)\, \mathfrak{s}_n^\beta(\{x_i\})$ *for some/all* $\beta>1$;
(iii) $x = \lim_{r\uparrow 1} \widehat{\{x_i\}}(r) = \lim_{r\uparrow 1} \sum_{n=0}^\infty r^n x_n$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof: By letting $a_0 := 1$ and $a_n := 0$ for $n \geq 1$, the desired result follows as in Fact 2.5. We may omit the details. □

Remark. In general (iii) does not imply (i) in Fact 3.5. (For example let $x_n := (-1)^n$.) If $\{x_n\}$ satisfies $\|x_n\| = O(n^{-1})$, $n\to\infty$, or if X is a Banach lattice and $\{x_n\} \subset X_+$, then (iii) implies (i). (See Theorem 3.3 and [2, Theorem 4.4], respectively.)

4. A counterexample

The following example shows that condition (D) is essential in Theorem 3.2. (See also Example 3 in [6].)

Example. Define $\{a_n\}_{n=0}^\infty$ by

$$a_n := \begin{cases} n & \text{if } n \in \{2^k, 2^k+1\} \text{ for some } k\geq 1, \\ 0 & \text{otherwise.}\end{cases}$$

Thus $\{a_n\}$ does not satisfy condition (D). Next define $\{x_n\}_{n=0}^{\infty}$ by

$$x_n := \begin{cases} n & \text{if } \ n = 2^k \ \text{ for some } \ k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $x_n = O(a_n)$, $n \to \infty$. An elementary calculation yields

$$\frac{1}{2} = \liminf_{n\to\infty} \frac{\sum_{k=0}^{n} x_k}{\sum_{k=0}^{n} a_k} < \limsup_{n\to\infty} \frac{\sum_{k=0}^{n} x_k}{\sum_{k=0}^{n} a_k} = \frac{2}{3},$$

so that $\lim_{n\to\infty} \mathfrak{s}_n^1(\{x_i\})/\mathfrak{s}_n^1(\{a_i\})$ does not exist. Nevertheless we have

$$\frac{\widehat{\{x_i\}}(r)}{\widehat{\{a_i\}}(r)} = \frac{\sum_{n=1}^{\infty} 2^n r^{2^n}}{(1+r)\sum_{n=1}^{\infty} 2^n r^{2^n} + r\sum_{n=1}^{\infty} r^{2^n}} \to \frac{1}{2} \quad \text{as } \ r \uparrow 1.$$

Remark. Let $0 \neq g \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that $\widehat{g}(\lambda)$ exists for all $\lambda > 0$ and that $x = \lim_{\lambda\downarrow 0} \widehat{u}(\lambda)/\widehat{g}(\lambda)$ implies $x = \lim_{t\to\infty} \big(\int_0^t u(s)\,ds\big)/\big(\int_0^t g(s)\,ds\big)$ for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ with $\|u(t)\| = O(g(t))$, $t \to \infty$. Then in view of Theorem 2.2 it would be natural to ask the following question: Does the measure $\mu := g(t)\,dt$ satisfy condition (C) of Lemma 2.1? The author could not solve this problem.

References

[1] Arendt W., Batty C.J.K., Hieber M., Neubrander F., *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics Vol. 96, Birkhäuser, Basel, 2001.

[2] Chen J.-C., Sato R., *Ratio limit theorems and Tauberian theorems for vector-valued functions and sequences*, J. Math. Anal. Appl. **367** (2010), 108–115.

[3] Chen J.-C., Sato R., Shaw S.-Y., *Growth orders of Cesàro and Abel means of functions in Banach spaces*, Taiwanese J. Math. **14** (2010), 1201–1248.

[4] Emilion R., *Mean-bounded operators and mean ergodic theorems*, J. Funct. Anal. **61** (1985), 1–14.

[5] Li Y.-C., Sato R., Shaw S.-Y., *Convergence theorems and Tauberian theorems for functions and sequences in Banach spaces and Banach lattices*, Israel J. Math. **162** (2007), 109–149.

[6] Li Y.-C., Sato R., Shaw S.-Y., *Ratio Tauberian theorems for positive functions and sequences in Banach lattices*, Positivity **11** (2007), 433–447.

[7] Sato R., *On means of Banach-space-valued functions*, Math. J. Okayama Univ., to appear.

[8] Widder D.V., *An Introduction to Transform Theory*, Academic Press, New York and London, 1971.

[9] Zygmund A., *Trigonometric Series. Vol. I*, Cambridge University Press, Cambridge, 1959.

Department of Mathematics, Okayama University, Okayama, 700-8530 Japan

Current address:
19-18, Higashi-hongo 2-chome, Midori-ku, Yokohama, 226-0002 Japan

E-mail: satoryot@math.okayama-u.ac.jp

(*Received* July 18, 2010, *revised* December 1, 2010)