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FUNCTIGRAPHS: AN EXTENSION OF PERMUTATION GRAPHS

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Abstract. Let $G_1$ and $G_2$ be copies of a graph $G$, and let $f: V(G_1) \to V(G_2)$ be a function. Then a functigraph $C(G, f) = (V, E)$ is a generalization of a permutation graph, where $V = V(G_1) \cup V(G_2)$ and $E = E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1), v \in V(G_2), v = f(u)\}$. In this paper, we study colorability and planarity of functigraphs.

Keywords: permutation graph, generalized Petersen graph, functigraph

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1. INTRODUCTION AND DEFINITIONS

Throughout this paper, $G = (V(G), E(G))$ stands for a non-empty, simple and connected graph with order $|V(G)|$ and size $|E(G)|$. For a given graph $G$ and $S \subseteq V(G)$, we denote by $\langle S \rangle$ the subgraph of $G$ induced by $S$. The distance, $d(u, v)$, between two vertices $u$ and $v$ in $G$ is the number of edges on a shortest path between $u$ and $v$ in $G$.

A graph $G$ is planar if it can be embedded in the plane. A connected graph, with order at least 3, is outerplanar if it can be embedded in the plane so that all its vertices lie on the exterior region [2]. A vertex $v$ in a connected graph $G$ is a cut-vertex of $G$ if $G - v$ is disconnected. A (vertex) proper coloring of $G$ is an assignment of labels, traditionally called colors, to the vertices of a graph such that no two adjacent vertices share the same color. A coloring using at most $k \geq 1$ colors is called a (proper) $k$-coloring and it is equivalent to the problem of partitioning the vertex set into $k$ or fewer independent sets. The least number of colors needed to properly color a graph $G$ is the chromatic number $\chi(G)$.

Following Chartrand and Harary (see p. 434 of [2]), a permutation graph $P_\alpha$ consists of two identical disjoint copies of a labeled graph $G$, say $G_1$ and $G_2$, along
with $|V(G)| = n$ additional edges joining $V(G_1)$ and $V(G_2)$ according to a given permutation $\alpha$ on \{1, 2, \ldots, n\}. As noted by the authors, the graph $P_\alpha(G)$ depends not only on the choice of the permutation $\alpha$ but on the particular labeling of $G$ as well.

For additional graph theory terminology we refer to [3]. We now introduce the study of a functigraph. We first recall that a function graph was independently introduced and was studied by Stephen Hedetniemi in [4] which overlaps our definition in a special case.

**Definition 1.1.** Let $G_1$ and $G_2$ be two copies of a graph $G$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and let $f$ be a function from $V(G_1)$ to $V(G_2)$. We define the functigraph $C(G, f)$ to be the graph that has the vertex set

$$V(C(G, f)) = V(G_1) \cup V(G_2),$$

and the edge set

$$E(C(G, f)) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2), v = f(u)\}.$$ 

We refer to $V(G_1)$ as the domain of the function $f$, to $V(G_2)$ as the codomain of $f$, and to $f(V(G_1))$ as the range of $f$.

Note that we use the notation $C(G, f)$ to refer to functigraphs for idiosyncratic reasons. A mnemonic is that the $C$ reminds us that we have two copies of $G$ with function $f$ mapping between them.

Observe that since $G_1$ and $G_2$ are copies of the same graph, the function $f$ could be invertible. If so, then $C(G, f)$ is isomorphic to $C(G, f^{-1})$, which is a permutation graph. Indeed, functigraphs generalize permutation graphs [2]. That is, for any graph $G$, if $f$ is a permutation on $V(G)$, then the functigraph $C(G, f)$ and the permutation graph $P_\alpha(G)$ coincide. The class of permutation graphs and thus the functigraphs include several interesting families of graphs such as

1. The prisms $C(C_n, f)$ where $f : V(G_1) \rightarrow V(G_2)$ is defined by

$$f(x) = \begin{cases} 
  x + k & \text{if } 1 \leq x + k \leq n, \\
  x + k - n & \text{if } x + k > n.
\end{cases}$$

   for $0 \leq k \leq n - 1$ (see (A) of Figure 1 when $k = 1$).
2. The Petersen graph $C(C_5, f)$ where $f : V(G_1) \rightarrow V(G_2)$ is defined by

$$f(x) = \begin{cases} 
  2x & \text{if } x = 1, 2, \\
  2x - 5 & \text{if } x = 3, 4, 5.
\end{cases}$$

   See (B) of Figure 1.
(3) The hypercubes $Q_n$ which are $C(Q_{n-1}, f(x) = x), n \geq 1$.

(4) $G \times K_2$, where $G$ is any connected graph, in particular the ladder graphs $P_n \times P_2$ which are $C(P_n, f(x) = x)$.

![Figure 1. A prism and the Petersen graph](image)

In this paper, we characterize the proper colorability and planarity in $C(G, f)$ when $G$ is a cycle. In addition, we study colorability and planarity in $C(G, f)$ for an arbitrary graph $G$.

2. Colorability of functigraphs

In this section we investigate the (proper) colorability of functigraphs. Clearly, $\chi(C(G, f)) \geq \chi(G)$, for all graphs $G$. Chartrand and Frechen [1] proved that, for every graph $G$ and every permutation graph $P_\alpha(G)$ of $G$, $\chi(G) \leq \chi(P_\alpha(G)) \leq \lceil \frac{4}{3}\chi(G) \rceil$.

In this section, we generalize the result by Chartrand and Frechen for an arbitrary function $f: V(G_1) \to V(G_2)$. We begin our study by letting $G = C_n$ and then we proceed to arbitrary graphs. Let $C_n$: $v_1, v_2, \ldots, v_n, v_1$ be a cycle of length $n$. For simplicity, we refer to a vertex by the index $i$ of its label $v_i$ ($1 \leq i \leq n$).

**Proposition 2.1.** Let $G = C_n$ for $n$ even. Then $2 \leq \chi(C(G, f)) \leq 3$. Moreover, for all $i, j \in V(G_1)$, $d(i, j)$ and $d(f(i), f(j))$ have the same parity if and only if $\chi(C(G, f)) = 2$.

**Proof.** Let $G = C_n$ be an $n$-cycle for $n$ even. Then $\chi(G_1) = \chi(G_2) = 2$, so the lower bound follows.

Now we consider the upper bound. Since $G_2$ is an even cycle, there is a 2-coloring of $G_2$, say $c$, using colors 1 and 2. Let $S_t = \{x \in V(G_1): c(f(x)) = t\}$ for $t = 1, 2$. We next present a coloring of $G$, where vertices in $S_1$ are only colored 2 or 3, and
vertices in $S_2$ are colored 1 or 3. Let every other vertex in $G_1$ be colored 3, say these vertices form the set $S$. For $x \in S_1 - S$ let the color of $x$ be 2, and if $x \in S_2 - S$ let the color of $x$ be 1. Thus $\chi(C(G, f)) = 3$, giving the upper bound.

To see the characterization, note that if for all $i, j \in V(G_1)$, $d(i, j)$ and $d(f(i), f(j))$ have the same parity, then all cycles in $C(C_n, f)$ are even. This is the case if and only if $C(G, f)$ is bipartite, i.e. $\chi(C(G, f)) = 2$. For the converse, if there exist $i, j \in V(G_1)$ with $d(i, j)$ and $d(f(i), f(j))$ having different parities, then $C(G, f)$ contains an odd cycle, and so $\chi(C(G, f)) \geq 3$. \hfill $\Box$

For an odd cycle $C_n$, we have the following:

**Proposition 2.2.** Let $G = C_n$ for $n$ odd. Then $3 \leq \chi(C(G, f)) \leq 4$. Moreover, $\chi(C(G, f)) = 4$ if and only if $f$ is a constant function.

**Proof.** Since $C(G, f)$ contains an odd cycle, $\chi(C(G, f)) \geq 3$. To obtain the upper bound, we show that a coloring with at most 4 colors can be found, by considering two cases.

**Case 1.** $f$ is a constant function. Say $f : V(G_1) \to V(G_2)$ given by $f(x) = a$ for all $x \in V(G_1)$ and for some $a \in V(G_2)$ with $1 \leq a \leq n$. Define a coloring $c_1 : V(C(G, f)) \to \{1, 2, 3, 4\}$ by 3-coloring $G_1$ with colors 1, 2, and 3, color the vertex labeled $a$ by 4, and properly color the rest of $G_2$ with two colors 1 and 2. Thus $\chi(C(G, f)) \leq 4$.

**Case 2.** $f$ is not a constant function. Since $G_2$ is an odd cycle, there exists a 3-coloring of $V(G_2)$, say $c$, using colors 1, 2, and 3. Without loss of generality, color $G_2$ so that color 3 is only used once, and it is used on a vertex of $G_2$ of minimum degree in $C(G, f)$. Let $S_t = \{x \in V(G_1) : c(f(x)) = t\}$ for $t = 1, 2, 3$. Thus $|S_3| \in \{0, 1\}$. We consider two subcases.

**Subcase 2.1.** $|S_3| = 0$. First we assume that either $S_1 = \emptyset$ or $S_2 = \emptyset$, say the former. Then $V(G_1) = S_2$ and one can properly color $G_1$ with colors 1, 3, and 4. Thus $\chi(C(G, f)) \leq 4$. Next we assume that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Then find two consecutive vertices in $G_1$, say $k$ and $k + 1$ such that $k \in S_1$ and $k + 1 \in S_2$, or $k \in S_2$ and $k + 1 \in S_1$, say the former. Then complete the 3-coloring $c_2$ of $C(G, f)$ by coloring every other vertex of the cycle $G_1$ starting with $k + 2$ with color 3, say these vertices form the set $S$. For $i \in V(G_1) - S$ let the color of $i$ be

$$c_2(i) = \begin{cases} 
2 & \text{if } i \in S_1, \\
1 & \text{if } i \in S_2.
\end{cases}$$

And so $\chi(C(G, f)) = 3$ in this case.

**Subcase 2.2.** $|S_3| = 1$. Then let $V(G_1) \cap S_3 = \{k\}$. Then either $k + 1 \in S_1$ or $k + 1 \in S_2$, say the former. Now define a coloring $c_2 : V(C(G, f)) \to \{1, 2, 3\}$ by
letting every other vertex in \(G_1\) be colored 3 starting with \(k+1\), say these vertices form the set \(S\). For \(x \in S_1 - S\) let the color of \(x\) be 2, if \(x \in S_2 - S\) let the color of \(x\) be 1, and the color of \(k\) is

\[
c_2(k) = \begin{cases} 
1 & \text{if } k - 1 \in S_1, \\
2 & \text{if } k - 1 \in S_2.
\end{cases}
\]

And so, \(\chi(C(G, f)) = 3\).

We now prove the characterization. Assume that \(f(x) = \alpha\) for all \(x \in V(G_1)\) and for some \(\alpha\) with \(1 \leq \alpha \leq n\). Since the vertex labeled \(a\) in \(G_2\) is adjacent to every vertex of the odd cycle \(G_1\), we have that \(\chi(C(G, f)) \geq 4\), and by the proof above we have that \(\chi(C(G, f)) \leq 4\), thus \(\chi(C(G, f)) = 4\).

For the converse, assume to the contrary that \(f\) is not a constant function. Then the coloring \(c_2\) above proves that \(\chi(C(G, f)) = 3\), a contradiction. \(\square\)

We now give bounds for the chromatic number of the functigraph \(C(G, f)\) in terms of the chromatic number of the graph \(G\), for any graph \(G\) and for all functions \(f\). The upper bound of the theorem below is a special case of a result independently proved by Hedetniemi in [4].

**Theorem 2.3.** If \(\chi(G) = \alpha\), then \(\alpha \leq \chi(C(G, f)) \leq \alpha + \lceil \frac{1}{2} \alpha \rceil\). Both bounds are sharp.

**Proof.** Since \(C(G, f)\) contains a copy of \(G\), \(\chi(C(G, f)) \geq \alpha\). For the upper bound let \(G_1\) and \(G_2\) be the two copies of \(G\) in \(C(G, f)\). Let \(c^*\) be a coloring of \(G_2\) with the color classes \(1, 2, \ldots, \alpha\) such that \(W_i = \{w \in V(G_2): c^*(w) = i\}\) for \(1 \leq i \leq \alpha\) with \(V(G_2) = \bigcup_{i=1}^{\alpha} W_i\). And let \(S_i = \{v \in V(G_1): c^*(f(v)) = i\}\) for \(1 \leq i \leq \alpha\). Since \(G_1\) is also \(\alpha\)-partite, it follows that \(V(G_1) = \bigcup_{i=1}^{\alpha} U_i\), where \(U_i\) is the independent set corresponding to \(W_i\) \((1 \leq i \leq \alpha)\). We construct the coloring \(c\) of \(C(G, f)\) by \(c\): \(V(C(G, f)) \to \{1, 2, \ldots, \lceil \frac{1}{2} \alpha \rceil\}\), where

\[
c(v) = \begin{cases} 
i & \text{if } v \in W_i \; (1 \leq i \leq \alpha), \\
 j + \lceil \frac{1}{2} \alpha \rceil & \text{if } v \in U_j \cap S_j \; (1 \leq j \leq \lceil \frac{1}{2} \alpha \rceil), \\
j & \text{if } v \in U_j - S_j \; (1 \leq j \leq \lceil \frac{1}{2} \alpha \rceil), \\
2\alpha + 1 - k & \text{if } v \in U_k \; ([\frac{1}{2} \alpha] + 1 \leq k \leq \alpha).
\end{cases}
\]

And so \(\chi(C(G, f)) \leq \alpha + \lceil \frac{1}{2} \alpha \rceil\).

To see the sharpness of the lower bound, let \(G\) be an \(a\)-partite graph with \(V(G) = \{V_1, V_2, \ldots, V_a\}\) being a partition of \(V(G)\) into independent sets, and \(f\) be the identity
function. Let $G_1$ and $G_2$ be the two copies of $G$, with $V(G_1) = \{V_1^1, V_2^1, \ldots, V_a^1\}$ and $V(G_2) = \{V_1^2, V_2^2, \ldots, V_a^2\}$ be the corresponding partitions into independent sets of $V(G_1)$ and $V(G_2)$. Then define a coloring $c : V(C(G,f)) \to \{1,2,\ldots,a\}$ given by

$$c(v) = \begin{cases} i & \text{if } v \in V_i^1 \ (1 \leq i \leq a), \\ j + 1 & \text{if } v \in V_j^2 \ (1 \leq j \leq a - 1), \\ 1 & \text{if } v \in V_a^2. \end{cases}$$

Thus $\chi(C(G,f)) = \chi(G) = a$.

To see the sharpness of the upper bound, let (1) $G$ be a complete $a$-partite graph $K_{a,a,\ldots,a}$, (2) $G_1$ be a copy of $G$ with the partition of $V(G_1)$ into independent classes as $V(G_1) = \bigcup_{i=1}^a U_i$, where $U_i = \{u_{i1}, u_{i2}, \ldots, u_{ia}\}$, (3) $G_2$ be another copy of $G$ with the partition of $V(G_2)$ into independent classes as $V(G_2) = \bigcup_{j=1}^a W_j$, where $W_j = \{w_{j1}, w_{j2}, \ldots, w_{ja}\}$, and (4) $f(u_{ij}) = w_{jj}$, for all $i, j, 1 \leq i, j \leq a$.

Then define the coloring $c : V(C(G,f)) \to \{1,2,\ldots,[\frac{3}{2}a]\}$ given by

$$c(v) = \begin{cases} j & \text{if } v \in W_j \ (1 \leq j \leq a), \\ i + \lfloor \frac{1}{2}a \rfloor & \text{if } v = u_{ij} \ (i = j, 1 \leq i \leq \lfloor \frac{1}{2}a \rfloor), \\ i & \text{if } v = u_{ij} \ (i \neq j, 1 \leq i \leq \lfloor \frac{1}{2}a \rfloor), \\ 2a + 1 - i & \text{if } v \in U_i \ (\lfloor \frac{1}{2}a \rfloor + 1 \leq i \leq a). \end{cases}$$

Thus $\chi(C(G,f)) \leq \lfloor \frac{3}{2}a \rfloor$. We claim that $\chi(C(G,f)) = \lfloor \frac{3}{2}a \rfloor$. Assume, to the contrary, that $\chi(C(G,f)) = l < \lfloor \frac{3}{2}a \rfloor$. Then $l < \frac{3}{2}a$. Since we have that $\langle\{w_{11}, w_{22}, \ldots, w_{aa}\}\rangle \cong K_a$, we then obtain $\chi(\langle\{w_{11}, w_{22}, \ldots, w_{aa}\}\rangle) = a$. Without loss of generality, we assign $c(w_{jj}) = j$ for $1 \leq j \leq a$. Suppose that $r$ of the sets $U_i$ have the same color assigned to each vertex of set. Since each vertex of $U_i$ $(1 \leq i \leq a)$ is adjacent to a different vertex of the clique $\{w_{11}, w_{22}, \ldots, w_{aa}\}$, it follows that the $r$ colors must be distinct from the $a$ colors already used. Then $\chi(C(G,f)) \geq a + r$. Since at least two colors are used for each of the remaining $a - r$ sets of $U_i$, $r + 2(a - r) \leq l$, and so $r \geq 2a - l$. Therefore, $\chi(C(G,f)) \geq a + r \geq a + 2a - l = 3a - l > 3a - \frac{3}{2}a = \frac{3}{2}a > l = \chi(C(G,f))$, which is a contradiction.

Below is an example that shows the construction of the coloring of $C(G,f)$ for $G = K_{3,3,3}$ for the sharpness of the the upper bound above.

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Chartrand and Harary [2] proved a result analogous to Kuratowski’s theorem for outerplanar graphs: a connected graph $G$ is outerplanar if and only if it contains no subgraph which is homeomorphic to $K_4$ or $K_{2,3}$. Further, Chartrand and Harary characterized planar permutation graphs that contain no cut-vertices. The result states that the permutation graph $P_\alpha(G)$ of a nonseparable graph $G$ is planar if and only if $G$ is outerplanar and $\alpha$ is dihedral. (See Ch. 4 of [5] for the dihedral groups.)

We begin this section with a characterization of planar functigraphs $C(C_n, f)$. We further characterize planar functigraphs $C(G, f)$ for an arbitrary graph $G$, and thus generalize the results obtained by Chartrand and Harary on permutation graphs to functigraphs.

We first consider $G = C_n$. Let $C_n: v_1, v_2, \ldots, v_n, v_1$ be a cycle of length $n$. For simplicity, we refer to each vertex of the cycle by its index $i$ of its label $v_i$ ($1 \leq i \leq n$). Let $G_1$ and $G_2$ be already embedded in the plane, with both labeled counterclockwise as in (A) of Figure 3. Let $f: V(G_1) \to V(G_2)$ be a function and let $\sigma: V(G_2) \to V(G_2')$ be an identity function, where $G_2'$ is a copy of $G_2$ with clockwise orientation. Then the composition $g := \sigma \circ f$ maps (B) of Figure 3 to (A) of Figure 3. Thus we only need to consider when $V(G_1)$ and $V(G_2)$ are labeled counterclockwise as in (A) of Figure 3.

For the rest of the paper, the function $f$ such that $f(i) = a_i$ for all $i = 1, 2, \ldots, n$ will be denoted by $f = (a_1, a_2, \ldots, a_n)$. Note that if all $a_i$’s are distinct and $\bigcup_{i=1}^n \{a_i\} = \bigcup_{j=1}^n \{j\}$, then $f: V(G_1) \to V(G_2)$ is a bijection and thus $C(G, f)$ is a permutation graph.

Example. Let $G = C_5$ be a cycle of length 5. Let $G_1$ and $G_2$ be copies of $G$ with labelings of vertices assigned as in Figure 4. Then $f = (2, 2, 3, 3, 1)$ means

Figure 2. An example of $\chi(C(K_3,3,3,f)) = 5$
that $f(1) = 2$, $f(2) = 2$, $f(3) = 3$, $f(4) = 3$, and $f(5) = 1$. Note that $C(G, f)$ is isomorphic to $C(G, \tilde{f})$ for $\tilde{f} = (1, 1, 2, 2, 5)$.

Notice that, by allowing relabeling, we can assume that there exists a vertex in $G_1$ that is mapped to vertex $n$ in $G_2$. If $f(n) = a_n \neq n$ then there exists an automorphism $\sigma$: $i \rightarrow i + n - a_n$ such that $\sigma \circ f(n) = n$.

We next characterize planar functigraphs when $G$ is a cycle. One can easily check that $C(C_3, f)$ is planar for any function $f$: there are three distinct (non-isomorphic) cases. Thus, we consider $G = C_n$ for $n \geq 4$.

We define a function $f$ of a functigraph $C(C_n, f)$ to be semi-monotonic, and we denote it by $f \in SM_n$, if and only if $f = (a_1, a_2, \ldots, a_n)$ satisfies

(1) there exists $k \in V(G_1)$ ($1 \leq k \leq n$) such that

\[ 1 \leq a_k \leq a_{k+1} \leq \ldots \leq a_{n-1} \leq a_n \leq a_1 \leq a_2 \leq \ldots \leq a_{k-1} \leq n \]

or

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(2) there exists \( k \in V(G_1) \) (\( 1 \leq k \leq n \)) such that

\[
1 \leq a_k \leq a_{k-1} \leq \ldots \leq a_2 \leq a_1 \leq a_n \leq a_{n-1} \leq \ldots \leq a_{k+1} \leq n.
\]

A graph \( H \) is called a subdivision of a graph \( G \) if one or more vertices of degree 2 are inserted into one or more edges of \( G \) (see p.236 of [3]). Now we recall Kuratowski’s Theorem: a graph \( G \) is planar if and only if \( G \) does not contain \( K_5 \), \( K_{3,3} \), or a subdivision of \( K_5 \) or \( K_{3,3} \) as a subgraph.

**Proposition 3.1.** Let \( G = C_n \) be a cycle of length \( n \geq 4 \). Let \( G_1 \) and \( G_2 \) be copies of \( G \) with cyclic labelings. Without loss of generality, we assume that \( f(n) = n \). Further, by relabeling the vertices so that two adjacent vertices 1 and \( n \) in \( G_1 \) get mapped to two different vertices in \( G_2 \), we may assume that \( f(1) \neq n \) if \( f \) is not a constant function. Then \( C(G, f) \) is planar if and only if \( f \) is semi-monotonic (\( f \) could be a constant function).

**Proof.** (\( \Leftarrow \)) It is easy to check.  
(\( \Rightarrow \)) Let \( C(G, f) \) be planar for \( G = C_n \) with \( n \geq 4 \). Assume, to the contrary, that \( f \) is not semi-monotonic. Then, without loss of generality, we may assume that there exist vertices \( r, s, t, u \) in \( G_1 \) such that \( 1 \leq r < s < t < u \leq n \) and \( 1 \leq a_r \leq a_t < a_s \leq a_u \leq n \). We consider three cases.

**Case 1.** \( |f(V(G_1))| = 2 \): Note that \( C(G, f) \) contains a subdivision of \( H_1 \cong K_{3,3} \) (see Figure 5) as a subgraph.

![Figure 5](image)

**Case 2.** \( |f(V(G_1))| = 3 \): Notice that \( C(G, f) \) contains a subdivision of either \( H_1, H_2, \) or \( H_2^* \) as a subgraph. Moreover, \( H_2 \) also contains a subdivision of \( K_{3,3} \) as a subgraph, where the two bipartite sets are \( \{r, t, a_s\} \) and \( \{s, u, a_r\} \). Also note that \( H_2^* \) contains a subdivision of \( K_{3,3} \) as a subgraph, where the two bipartite sets are \( \{r, t, a_u\} \) and \( \{s, u, a_t\} \) (see Figure 6).
Case 3. $|f(V(G_1))| \geq 4$: Notice that $C(G, f)$ contains a subdivision of either $H_1$, $H_2$, $H_2^*$, or $H_3$ as a subgraph. Moreover $H_3$ contains a subdivision of $H_3' \cong K_{3,3}$ as a subgraph, where the two bipartite sets are $\{t, a_r, a_s\}$ and $\{s, a_t, a_u\}$ (see Figure 7).

Thus, by Kuratowski’s Theorem, $C(G, f)$ is not planar, which is a contradiction to the planarity hypothesis. □

The following is an immediate result of Proposition 3.1. The sufficient direction of the corollary below, for any graph $G$, was independently proved by Hedetniemi in the study of function graph in [4].

**Corollary 3.2.** Let $G$ be a graph without cut-vertices. Then the functigraph $C(G, f)$ is planar if and only if $G$ is outerplanar and $f$ is semi-monotonic ($f$ could be a constant function).

**Proof.** $(\Leftarrow)$ It is easy to check.

$(\Rightarrow)$ Assume, to the contrary, that either $G$ is not outerplanar or $f$ is not semi-monotonic. If $G$ is not outerplanar, then there is a vertex $v \in V(G)$ such that $v$ doesn’t lie on the exterior region. Then the edge $vf(v)$ will cross one of the edges of $G$, a contradiction to the planarity hypothesis of $C(G, f)$. Thus $G$ is outerplanar, and we can assume that every nonseparable outerplanar graph $G$ is cyclically labeled. If $f \notin SM_n$, then $C(G, f)$ is not planar, a contradiction. □
Remark. The condition of $G$ having no cut-vertex is necessary. To see this, let $G = P_6$ be a path of length 5, with cut-vertices, as in Figure 8. Define $f: V(G_1) \to V(G_2)$ by

$$f(x) = \begin{cases} 
\frac{x + 6}{2} & \text{if } x \text{ is even,} \\
\frac{x + 1}{2} & \text{if } x \text{ is odd.}
\end{cases}$$

Then $C(G, f)$ is planar and $G$ is outerplanar, but $f = (1, 4, 2, 5, 3, 6)$ is not semi-monotonic.

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