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B-FREDHOLM AND DRAZIN INVERTIBLE OPERATORS
THROUGH LOCALIZED SVEP

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Abstract. Let $X$ be a Banach space and $T$ be a bounded linear operator on $X$. We denote by $S(T)$ the set of all complex $\lambda \in \mathbb{C}$ such that $T$ does not have the single-valued extension property at $\lambda$. In this note we prove equality up to $S(T)$ between the left Drazin spectrum, the upper semi-B-Fredholm spectrum and the semi-essential approximate point spectrum. As applications, we investigate generalized Weyl’s theorem for operator matrices and multiplier operators.

Keywords: B-Fredholm operator, Drazin invertible operator, single-valued extension property

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1. Introduction

Throughout this paper, $X$ and $Y$ are Banach spaces and $\mathcal{B}(X, Y)$ denotes the space of all bounded linear operators from $X$ to $Y$. For $Y = X$ we write $\mathcal{B}(X, Y) = \mathcal{B}(X)$. For $T \in \mathcal{B}(X)$, let $T^*$, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_s(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the adjoint, the null space, the range, the spectrum, the surjective spectrum, the point spectrum and the approximate point spectrum of $T$, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of $T$ defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \text{codim } R(T)$. If the range $R(T)$ is closed and $\alpha(T) < \infty$ (or $\beta(T) < \infty$), then $T$ is called an upper (a lower) semi-Fredholm operator. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then $T$ is called a semi-Fredholm operator, and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called a Fredholm operator. An operator $T$ is called Weyl if it is Fredholm of index zero. The Weyl spectrum $\sigma_W(T)$ is defined by $\sigma_W(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \text{ is not Weyl} \}$.

For $T \in \mathcal{B}(X)$ and a nonnegative integer $n$ define $T_{[n]}$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some
integer \( n \) the range space \( R(T^n) \) is closed and \( T_{[n]} \) is an upper (or a lower) semi-Fredholm operator, then \( T \) is called an upper (or a lower) semi-B-Fredholm operator. In this case the index of \( T \) is defined to be the index of the semi-Fredholm operator \( T_{[n]} \). Moreover, if \( T_{[n]} \) is a Fredholm operator, then \( T \) is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator ([6], [8], [13]). The upper semi-B-Fredholm spectrum \( \sigma_{UBF}(T) \), the lower semi-B-Fredholm spectrum \( \sigma_{LBF}(T) \) and the B-Fredholm spectrum \( \sigma_{BF}(T) \) of \( T \) are defined by

\[
\sigma_{UBF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-B-Fredholm operator} \}, \\
\sigma_{LBF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a lower semi-B-Fredholm operator} \}, \\
\sigma_{BF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Fredholm operator} \}.
\]

We have

\[
\sigma_{BF}(T) = \sigma_{UBF}(T) \cup \sigma_{LBF}(T).
\]

An operator \( T \in \mathcal{B}(X) \) is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum \( \sigma_{BW}(T) \) of \( T \) is defined by

\[
\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator} \}.
\]

From [8, Lemma 4.1], \( T \) is a B-Weyl operator if and only if \( T = F \oplus N \), where \( F \) is a Fredholm operator of index zero and \( N \) is a nilpotent operator.

We shall denote by \( SBF^+(X) \) (or \( SBF^-(X) \)) the class of all \( T \) upper semi-B-Fredholm operators (\( T \) lower semi-B-Fredholm operators) such that \( \text{ind}(T) \leq 0 \) (\( \text{ind}(T) \geq 0 \)). The spectrum associated with \( SBF^+(X) \) is called the semi-essential approximate point spectrum and is denoted by \( \sigma_{SBF^+}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SBF^+(X) \} \), while the spectrum associated with \( SBF^-(X) \) is denoted by \( \sigma_{SBF^-}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SBF^-(X) \} \).

The ascent \( a(T) \) and the descent \( d(T) \) of \( T \) are given by \( a(T) = \inf \{ n : N(T^n) = N(T^{n+1}) \} \) and \( d(T) = \inf \{ n : R(T^n) = R(T^{n+1}) \} \), with \( \inf \emptyset = \infty \). It is well-known that if \( a(T) \) and \( d(T) \) are both finite then they are equal, see [16, Proposition 38.3].

Recall that an operator \( T \) is Drazin invertible if it has a finite ascent and descent. It is well known that \( T \) is Drazin invertible if and only if \( T = R \oplus N \) where \( R \) is invertible and \( N \) is nilpotent (see [20, Corollary 2.2]). The Drazin spectrum is defined by \( \sigma_{D}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible} \} \). From [8, Lemma 4.1] and [20, Corollary 2.2] we have

\[
\sigma_{BW}(T) \subseteq \sigma_{D}(T).
\]
Define the set \(LD(X)\) as

\[
LD(X) = \{ T \in B(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed}\}.
\]

From [21], \(LD(X)\) is a regularity and it is the dual version of the regularity \(RD(X) = \{ T \in B(X) : d(T) < \infty \text{ and } R(T^{d(T)}) \text{ is closed}\} \). An operator \(T \in B(X)\) is said to be left (or right) Drazin invertible if \(T \in LD(X) \ (T \in RD(X))\). The left Drazin spectrum \(\sigma_{\text{LD}}(T)\) and the right Drazin spectrum \(\sigma_{\text{RD}}(T)\) are defined by 
\[
\sigma_{\text{LD}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin LD(X) \} \quad \text{and} \quad \sigma_{\text{RD}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin RD(X) \}.
\]

It is not difficult to see that 
\[
\sigma_{\text{D}}(T) = \sigma_{\text{LD}}(T) \cup \sigma_{\text{RD}}(T).
\]

2. Preliminary results

An operator \(T \in B(X)\) has the single-valued extension property at \(\lambda_0 \in \mathbb{C}\) (the SVEP for short) if for every open disc \(D_{\lambda_0}\) centered at \(\lambda_0\), the only analytic function \(f : D_{\lambda_0} \to X\) which satisfies \((T - \lambda I)f(\lambda) = 0\) for all \(\lambda \in D_{\lambda_0}\) is the function \(f \equiv 0\). Trivially, every operator \(T\) has the SVEP at all points of the resolvent; also \(T\) has the SVEP at \(\lambda \in \text{iso } \sigma(T)\) (\(\text{iso } \sigma(T)\) is the set of all isolated points of \(\sigma(T)\)). We say that \(T\) has SVEP if it has SVEP at every \(\lambda \in \mathbb{C}\), [15]. We denote by \(S(T)\) the set of all \(\lambda \in \mathbb{C}\) such that \(T\) does not have the single-valued extension property at \(\lambda\). Note that (see [15], [19]) \(S(T) \subseteq \sigma_p(T)\) and \(\sigma(T) = S(T) \cup \sigma_s(T)\). In particular, if \(T\) (or \(T^*\)) has the SVEP then \(\sigma(T) = \sigma_s(T)\) (\(\sigma(T) = \sigma_a(T)\)).

Recall that if \(T - \lambda I\) has a finite ascent then it has the SVEP ([18]). Thus we have 
\[
S(T) \subseteq \sigma_{\text{LD}}(T) \text{ and } S(T^*) \subseteq \sigma_{\text{RD}}(T).
\]

In the following theorem, we prove equality up to \(S(T)\) between the left Drazin spectrum, the upper semi-B-Fredholm spectrum and the semi-essential approximate point spectrum.

**Theorem 2.1.** Let \(T \in B(X)\). Then 
\[
\sigma_{\text{LD}}(T) = \sigma_{\text{UBF}}(T) \cup S(T) = \sigma_{\text{SBF}^-}(T) \cup S(T).
\]

**Proof.** Let \(\lambda \notin \sigma_{\text{LD}}(T)\), without loss of generality we assume that \(\lambda = 0\). Then \(R(T^{a(T)+1})\) is closed. Hence \(R(T^{a(T)})\) is closed by [21, Lemma 12]. We shall prove that \(T_{[a(T)]}\) is upper semi-Fredholm. Let \(x \in N(T_{[a(T)]})\) then \(x \in N(T) \cap R(T^{a(T)})\). Hence \(x = T^{a(T)}y\) for some \(y \in X\). Then \(0 = Tx = T^{a(T)+1}y\). Thus
Then from \([1, \text{Theorem 3.14}]\), we conclude that \(x = 0\) and hence \(T_{[\alpha(T - \lambda I)]}\) is injective. On the other hand, \(R(T_{[\alpha(T)]}) = R(T^{\alpha(T)+1})\) is closed. Thus \(T_{[\alpha(T)]}\) is upper semi-Fredholm and hence \(0 \notin \sigma_{\text{UBF}}(T)\). Since \(S(T) \subseteq \sigma_{\text{ID}}(T)\) we have

\[
\sigma_{\text{UBF}}(T) \cup S(T) \subseteq \sigma_{\text{ID}}(T).
\]

Now let \(0 \notin [\sigma_{\text{UBF}}(T) \cup (S(T))]\), then \(T\) is an upper semi-B-Fredholm operator. Hence it follows from \([7, \text{Proposition 3.2}]\) that there exist \(n\) such that \(R(T^n)\) is closed and \(T_{[n]}\) is semi-regular. Since \(T\) has the SVEP at 0 then \(T_{[n]}\) has also the SVEP at 0. Then from \([1, \text{Theorem 3.14}]\), we conclude that \(T_{[n]}\) is injective with closed range. Let \(x \in N(T^{n+1})\), then \(TT^n x = 0\). Hence \(T^n x \in N(T) \cap R(T^n) = N(T_{[n]}) = \{0\}\). Thus \(x \in N(T^n)\), and hence \(N(T^n) = N(T^{n+1})\). So \(T\) is of finite ascent and \(a(T) \leq n\). We have \(R(T^{n+1}) = R(T_{[n]})\) is closed with \(a(T) + 1 \leq n + 1\). Hence \(R(T^{a(T)+1})\) is closed by \([21, \text{Lemma 12}]\). Thus \(T\) is left Drazin invertible. Therefore \(\sigma_{\text{ID}}(T) \subseteq \sigma_{\text{UBF}}(T) \cup S(T)\).

From \([13, \text{Lemma 2.12}]\) we have \(\sigma_{\text{SBF}_{-}}(T) \subseteq \sigma_{\text{ID}}(T)\) and since \(\sigma_{\text{UBF}}(T) \subseteq \sigma_{\text{SBF}_{-}}(T)\) we infer \(\sigma_{\text{ID}}(T) = \sigma_{\text{UBF}}(T) \cup S(T) = \sigma_{\text{SBF}_{-}}(T) \cup S(T)\).

A useful consequence of the preceding result is that under the assumption of the SVEP for \(T\), the spectra \(\sigma_{\text{ID}}(T), \sigma_{\text{UBF}}(T)\) and \(\sigma_{\text{SBF}_{-}}(T)\) are equal.

**Corollary 2.1.** If \(T \in \mathcal{B}(X)\) has the SVEP then

\[
\sigma_{\text{ID}}(T) = \sigma_{\text{UBF}}(T) = \sigma_{\text{SBF}_{-}}(T).
\]

By duality we get a similar result for the right Drazin spectrum.

**Theorem 2.2.** Let \(T \in \mathcal{B}(X)\). Then

\[
\sigma_{\text{rD}}(T) = \sigma_{\text{LBF}}(T) \cup S(T^*) = \sigma_{\text{SBF}_{+}}(T) \cup S(T^*).
\]

**Proof.** Since \(\sigma_{\text{LBF}}(T) = \sigma_{\text{UBF}}(T^*)\), \(\sigma_{\text{SBF}_{+}}(T) = \sigma_{\text{SBF}_{-}}(T^*)\) and \(\sigma_{\text{rD}}(T) = \sigma_{\text{ID}}(T^*)\) the assertion follows by Theorem 2.1.

**Corollary 2.2.** If \(T^* \in \mathcal{B}(X)\) has the SVEP then

\[
\sigma_{\text{rD}}(T) = \sigma_{\text{LBF}}(T) = \sigma_{\text{SBF}_{+}}(T).
\]

From Theorem 2.1 and Theorem 2.2 we get the following corollary.

**Corollary 2.3.** Let \(T \in \mathcal{B}(X)\). Then

\[
(2.1) \quad \sigma_{\text{D}}(T) = \sigma_{\text{BF}}(T) \cup [S(T) \cup S(T^*)] = \sigma_{\text{BW}}(T) \cup [S(T) \cup S(T^*)].
\]
In particular if \( T \) and \( T^* \) have the SVEP then

\[
\sigma_D(T) = \sigma_{BF}(T) = \sigma_{BW}(T).
\]

The equality in (2.1) may be refined for \( \sigma_D(T) \) and \( \sigma_{BW}(T) \). More precisely, we have

**Theorem 2.3.** Let \( T \in \mathcal{B}(X) \) then

\[
\sigma_D(T) = \sigma_{BW}(T) \cup [S(T) \cap S(T^*)].
\]

**Proof.** Since \( \sigma_{BW}(T) \cup (S(T) \cap S(T^*)) \subseteq \sigma_D(T) \) always holds, let \( \lambda \notin \sigma_{BW}(T) \cup (S(T) \cap S(T^*)) \). Without loss of generality we assume that \( \lambda = 0 \). Then \( T \) is a B-Fredholm operator of index zero.

**Case 1.** If \( 0 \notin S(T) \): Since \( T \) is a B-Fredholm operator of index zero, it follows from [8, Lemma 4.1] that there exists a Fredholm operator \( F \) of index zero and a nilpotent operator \( N \) such that \( T = F \oplus N \). If \( 0 \notin \sigma(F) \), then \( F \) is invertible and hence \( T \) is Drazin invertible. Now assume that \( 0 \in \sigma(F) \). Since \( T \) has the SVEP at \( 0 \), \( F \) has also the SVEP at \( 0 \). Hence it follows from [1, Theorem 3.16] that \( a(F) \) is finite. \( F \) is a Fredholm operator of index zero, hence it follows from [1, Theorem 3.4] that \( d(F) \) is also finite. Then \( a(F) = d(F) < \infty \) which implies that \( 0 \) is a pole of \( F \) and hence an isolated point of \( \sigma(F) \). Operator \( N \) is nilpotent, hence \( 0 \) is an isolated point of \( \sigma(T) \). From [8, Theorem 4.2] we get \( 0 \notin \sigma_D(T) \).

**Case 2.** If \( 0 \notin S(T^*) \), the proof goes similarly. \( \square \)

**Corollary 2.4 ([12]).** If \( T \) or \( T^* \) has the SVEP then

\[
\sigma_D(T) = \sigma_{BW}(T).
\]

Recall that \( T \) is a **Browder** operator if \( T \) is a Fredholm operator of finite ascent and descent. Let \( \sigma_B(T) \) be the **Browder spectrum** defined as the set of all \( \lambda \in \mathbb{C} \) such that \( T - \lambda I \) is not Browder. Analogously, \( T \) is a **B-Browder** operator if for some integer \( n \), \( R(T^n) \) is closed and \( T_{[n]} \) is Browder. Let \( \sigma_{BB}(T) \) be the **B-Browder spectrum**. In [1, Corollary 3.53] it is proved that if \( T \) or \( T^* \) has the SVEP, then

\[
\sigma_W(T) = \sigma_B(T).
\]

From [7, Theorem 3.6] we have \( \sigma_D(T) = \sigma_{BB}(T) \), hence by Corollary 2.4, if \( T \) or \( T^* \) has the SVEP then

\[
\sigma_{BW}(T) = \sigma_{BB}(T).
\]
Theorem 2.4. Let $T \in \mathcal{B}(X)$ and let $f$ be an analytic function on some open neighborhood of $\sigma(T)$ which is nonconstant on any connected component of $\sigma(T)$. Then

$$f(\sigma_{BW}(T) \cup [S(T) \cap S(T^*)]) = \sigma_{BW}(f(T)) \cup [S(f(T)) \cap S(f(T^*))].$$

Proof. According to [21] the Drazin spectrum satisfies the spectral mapping theorem for such a function $f$, hence the result follows at once from Theorem 2.3. □

It is well known that if $T$ has the SVEP then $f(T)$ has also the SVEP [19]. Now we retrieve the result proved in [2], [23]: $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ whenever $T$ or $T^*$ has the SVEP. Note that in [2], [23] the condition “$f$ is nonconstant on any connected component of $\sigma(T)$” is dropped.

3. Applications

3.1. Perturbations.

Lemma 3.1. Let $T \in \mathcal{B}(X)$. Let $N \in \mathcal{B}(X)$ be a nilpotent operator such that $TN = NT$. Then

$$\mathcal{S}(T + N) = \mathcal{S}(T).$$

Proof. See for instance [5, Lemma 2.1]. □

Lemma 3.2. Let $T \in \mathcal{B}(X)$. If $N \in \mathcal{B}(X)$ is a nilpotent operator which commutes with $T$ then

$$\sigma_{ID}(T + N) = \sigma_{ID}(T).$$

Proof. Assume that $\lambda = 0 \notin \sigma_{ID}(T)$. Then $\alpha(T)$ is finite and $R(T^{\alpha(T) + 1})$ is closed. Let $m$ be the nonnegative integer such that $N^m = 0 \neq N^{m-1}$. Let $s = \max(\alpha(T), m)$. Then

$$(T + N)^{2s} = \sum_{k=0}^{2s} \binom{k}{2s} T^k N^{2s-k}$$

$$= \binom{0}{2s} N^{2s} + \ldots + \binom{s}{2s} T^s N^s + \binom{s + 1}{2s} T^{s+1} N^{s-1} + \ldots + \binom{2s}{2s} T^{2s}$$

$$= \binom{s + 1}{2s} T^{s+1} N^{s-1} + \ldots + \binom{2s}{2s} T^{2s}$$

$$= T^s \left[ \binom{s + 1}{2s} T^1 N^{s-1} + \ldots + \binom{2s}{2s} T^{1} \right].$$
Now let $x \in N(T)^{2s} = N(T)^s$ that is $(T)^{2s}x = 0$. Then it follows from the above equality that $(T + N)^{2s}x = 0$. Hence $N(T)^{2s} \subseteq N(T + N)^{2s}$. With the same argument for $T + N$ and $-N$ we have $N(T + N)^{2s} \subseteq N(T)^{2s}$. Thus $N(T)^{2s} = N(T + N)^{2s}$. Since $N(T^s) = N(T^{2s}) = N(T^{2s+1})$, we get $N(T + N)^{2s} = N(T + N)^{2s+1}$. Therefore $T + N$ is of finite ascent. On the other hand, $R(T + N)^{2s} \subseteq R(T^s)$ is closed. Hence by [21, Lemma 12] $R(T^s)$ is closed. Thus $N(T)^{2s} = N(T + N)^{2s}$.

The next result follows from Theorem 2.1, Lemma 3.1 and Lemma 3.2.

**Theorem 3.1.** Let $T \in \mathcal{B}(X)$. Let $N \in \mathcal{B}(X)$ be a nilpotent operator which commutes with $T$. Then

$$\sigma_{\text{SBF}^-}(T + N) \cup S(T) = \sigma_{\text{SBF}^-}(T) \cup S(T).$$

The following corollary which is proved in [3] gives an affirmative answer to the question posed by Berkani-Amouch [9] in the case when $T$ has the SVEP.

**Corollary 3.1.** Let $T \in \mathcal{B}(X)$ have the SVEP. Let $N \in \mathcal{B}(X)$ be a nilpotent operator which commutes with $T$. Then

$$\sigma_{\text{SBF}^-}(T + N) = \sigma_{\text{SBF}^-}(T).$$

### 3.2. Generalized Weyl’s theorem for operator matrices.

Berkani [8, Theorem 4.5] has shown that every normal operator $T$ acting on a Hilbert space $H$ satisfies

$$\sigma(T) \setminus E(T) = \sigma_{\text{BW}}(T),$$

where $E(T)$ is the set of all isolated eigenvalues of $T$. We say that the *generalized Weyl’s theorem* holds for $T$ if equality (3.1) holds. This gives a generalization of the classical Weyl’s theorem. Recall that $T \in \mathcal{B}(X)$ obeys *Weyl’s theorem* if

$$\sigma(T) \setminus E_0(T) = \sigma_{\text{W}}(T)$$

where $E_0(T)$ denotes the set of the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. By [13, Theorem 3.9] the generalized Weyl’s theorem implies Weyl’s theorem and generally the reverse is not true.

For $A \in \mathcal{B}(X), B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ we denote by $M_C$ the operator defined on $X \oplus Y$ by

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$
In general the fact that the generalized Weyl's theorem holds for $A$ and $B$ does not imply that the generalized Weyl's theorem holds for $M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Indeed, let $I_1$ and $I_2$ be the identities on $\mathbb{C}$ and $l_2$, respectively. Let $S_1$ and $S_2$ be defined on $l_2$ by

$$S_1(x_1, x_2, \ldots) = (0, \frac{1}{3}x_1, \frac{1}{3}x_2, \ldots), \quad S_2(x_1, x_2, \ldots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \ldots).$$

Let $T_1 = I_1 \oplus S_1$, $T_2 = S_2 - I_2$, $A = T_2 - I_1$, and $B = T_2^2$, then from [23, Example 1] we have $A$ and $B$ obey the generalized Weyl's theorem but $M_0$ does not obey it. It also may happen that $M_C$ obeys the generalized Weyl’s theorem while $M_0$ does not obey it. Let $A$ be the unilateral unweighted shift operator. For $B = A^*$ and $C = I - AA^*$, we have that $M_C$ is unitary without eigenvalues. Hence $M_C$ satisfies the generalized Weyl’s theorem (see [10, Remark 3.5]). But $\sigma_W(M_0) = \{\lambda : |\lambda| = 1\}$ and $\sigma(M_0) \setminus E_0(M_0) = \{\lambda : |\lambda| \leq 1\}$. Hence $M_0$ does not satisfy the Weyl’s theorem and so by [13, Theorem 3.9] it does not satisfy the generalized Weyl’s theorem either.

A bounded linear operator $T$ is said to be isloid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$.

**Proposition 3.1.** Let $A$ and $B$ be isoloids. Assume that $\sigma_{BW}(M_0) = \sigma_{BW}(A) \cup \sigma_{BW}(B)$. If $A$ and $B$ obey the generalized Weyl’s theorem, then $M_0$ obeys the generalized Weyl’s theorem.

**Proof.** Since $A$ and $B$ are isoloids, we have

$$E(M_0) = [E(A) \cap \varrho(B)] \cup [\varrho(A) \cap E(B)] \cup [E(A) \cap E(B)].$$

Now if $A$ and $B$ obey the generalized Weyl’s theorem, then

$$E(M_0) = [\sigma(A) \cup \sigma(B)] \setminus [\sigma_{BW}(A) \cup \sigma_{BW}(B)]$$

$$= \sigma(M_0) \setminus \sigma_{BW}(M_0).$$

Then $M_0$ obeys the generalized Weyl’s theorem. \qed

**Lemma 3.3.** Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ have the SVEP. Then

$$\sigma_{BW}(M_C) = \sigma_{BW}(A) \cup \sigma_{BW}(B)$$

for all $C \in \mathcal{B}(Y, X)$.

**Proof.** Since $A$ and $B$ have the SVEP, then it follows from [17, Proposition 3.1] that $M_C$ also has the SVEP. Hence $\sigma_{BW}(M_C) = \sigma_D(M_C)$ by Corollary 2.4. Also since $A$ and $B$ have the SVEP, it follows from [24, Corollary 2.1] that $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$. Therefore $\sigma_{BW}(M_C) = \sigma_{BW}(A) \cup \sigma_{BW}(B)$ by Corollary 2.4. \qed
Theorem 3.2. Let $A$ and $B$ be isoloids with the SVEP. If $A$ and $B$ obey the generalized Weyl’s theorem, then $M_C$ obeys the generalized Weyl’s theorem for every $C \in B(Y, X)$.

Proof. It follows from Proposition 3.1 and Lemma 3.3 that

$$E(M_0) = \sigma(M_0) \setminus \sigma_{BW}(M_0) = \sigma(M_C) \setminus \sigma_{BW}(M_C).$$

Hence it is enough to show that $E(M_0) = E(M_C)$. Let $\lambda \in E(M_C)$. Then $\lambda \in \sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B)$. Hence $\lambda \in \sigma_p(M_0)$. Since $\lambda \in \text{iso} \sigma(M_C) = \text{iso} \sigma(M_0)$ we have $\lambda \in E(M_0)$. Now let $\lambda \in E(M_0)$. If $\lambda \in \sigma(A)$ then $\lambda \in \text{iso} \sigma(A)$. Since $A$ is an isoloid, we have $\lambda \in \sigma_p(A) \subseteq \sigma_p(M_C)$. Hence $\lambda \in E(M_C)$. If $\lambda \in \sigma(B) \setminus \sigma(A)$, then $\lambda \in \sigma_p(B)$. Since $A$ is invertible, we conclude that $\lambda \in \sigma_p(M_C)$. Thus $\lambda \in E(M_C)$. Therefore $E(M_0) = E(M_C)$. □

Let $\pi(T)$ be the set of all poles of the resolvent of $T$. Recall from [14] that $T$ is a polaroid if $\text{iso} \sigma(T) \subseteq \pi(T)$. Since $\pi(T) \subseteq E(T)$ holds without restriction on $T$, then if $T$ is a polaroid then $E(T) = \pi(T)$.

Corollary 3.2. Let $A$ and $B$ be polaroids with the SVEP. Then $M_C$ obeys the generalized Weyl’s theorem for every $C \in B(Y, X)$.

Proof. $A$ and $B$ are polaroids hence $E(A) = \pi(A)$ and $E(B) = \pi(B)$. Since $A$ and $B$ have the SVEP, we have by [4] that $A$ and $B$ satisfy the generalized Weyl’s theorem. Hence we complete the proof by Theorem 3.2. □

3.3. Multipliers on a commutative Banach algebra. Let $\mathcal{A}$ be a semi-simple commutative Banach algebra. A mapping $T: \mathcal{A} \rightarrow \mathcal{A}$ is called a multiplier if

$$T(x)y = xT(y) \quad \text{for all } x, y \in \mathcal{A}.$$ 

By semi-simplicity of $\mathcal{A}$, every multiplier is a bounded linear operator on $\mathcal{A}$. Also the semi-simplicity of $\mathcal{A}$ implies that every multiplier has the SVEP (see [1], [19]).

By [1, Theorem 4.36], for every multiplier $T$ on a semi-simple commutative Banach algebra $\mathcal{A}$, $E(T) = \pi(T)$ and since $T$ has the SVEP we get from [4]

Proposition 3.2. Every multiplier on a semi-simple commutative Banach algebra $\mathcal{A}$ obeys the generalized Weyl’s theorem.
From Corollary 2.4 we have

**Proposition 3.3** ([11]). Let $T$ be a multiplier on a semi-simple commutative Banach algebra $A$. Then the following assertions are equivalent:

i) $T$ is B-Fredholm of index zero.

ii) $T$ is Drazin invertible.

Now if we assume in addition that $A$ is regular and Tauberian (see [19] for definition) then every multiplier $T$ has the weak decomposition property ($\delta_w$) and then $T^*$ has also the SVEP (see [22] for definition and details). Hence we get from Corollary 2.3

**Proposition 3.4.** Let $T$ be a multiplier on a semi-simple regular Tauberian commutative Banach algebra $A$. Then the following assertions are equivalent:

i) $T$ is B-Fredholm.

ii) $T$ is Drazin invertible.

For $G$ a locally compact abelian group, let $L^1(G)$ be the space of $\mathbb{C}$-valued functions on $G$ integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on $G$. We recall that $L^1(G)$ is a regular semi-simple Tauberian commutative Banach algebra. Then we have

**Corollary 3.3.** Let $G$ be a locally compact abelian group, $\mu \in M(G)$ and $X = L^1(G)$. Then every convolution operator $T_\mu : X \to X, T_\mu(k) = \mu \star k$ is B-Fredholm if and only if it is Drazin invertible.

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**References**


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