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B-FREDHOLM AND DRAZIN INVERTIBLE OPERATORS
THROUGH LOCALIZED SVEP

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Abstract. Let $X$ be a Banach space and $T$ be a bounded linear operator on $X$. We denote by $S(T)$ the set of all complex $\lambda \in \mathbb{C}$ such that $T$ does not have the single-valued extension property at $\lambda$. In this note we prove equality up to $S(T)$ between the left Drazin spectrum, the upper semi-B-Fredholm spectrum and the semi-essential approximate point spectrum. As applications, we investigate generalized Weyl’s theorem for operator matrices and multiplier operators.

Keywords: B-Fredholm operator, Drazin invertible operator, single-valued extension property

MSC 2010: 47A53, 47A55, 47A10, 47A11

1. Introduction

Throughout this paper, $X$ and $Y$ are Banach spaces and $B(X, Y)$ denotes the space of all bounded linear operators from $X$ to $Y$. For $Y = X$ we write $B(X, Y) = B(X)$. For $T \in B(X)$, let $T^*$, $N(T)$, $R(T)$, $\sigma(T)$, $\sigma_s(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the adjoint, the null space, the range, the spectrum, the surjective spectrum, the point spectrum and the approximate point spectrum of $T$, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of $T$ defined by $\alpha(T) = \dim N(T)$ and $\beta(T) = \text{codim} R(T)$.

If the range $R(T)$ is closed and $\alpha(T) < \infty$ (or $\beta(T) < \infty$), then $T$ is called an upper (a lower) semi-Fredholm operator. If $T \in B(X)$ is either upper or lower semi-Fredholm, then $T$ is called a semi-Fredholm operator, and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called a Fredholm operator. An operator $T$ is called Weyl if it is Fredholm of index zero. The Weyl spectrum $\sigma_W(T)$ is defined by $\sigma_W(T) = \{ \lambda \in \mathbb{C}: T - \lambda I \text{ is not Weyl} \}$.

For $T \in B(X)$ and a nonnegative integer $n$ define $T_{[n]}$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some
integer $n$ the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (or a lower) semi-Fredholm operator, then $T$ is called an upper (a lower) semi-$B$-Fredholm operator. In this case the index of $T$ is defined to be the index of the semi-Fredholm operator $T_{[n]}$. Moreover, if $T_{[n]}$ is a Fredholm operator, then $T$ is called a $B$-Fredholm operator. A semi-$B$-Fredholm operator is an upper or a lower semi-$B$-Fredholm operator ([6], [8], [13]). The upper semi-$B$-Fredholm spectrum $σ_{UBF}(T)$, the lower semi-$B$-Fredholm spectrum $σ_{LBF}(T)$ and the $B$-Fredholm spectrum $σ_{BF}(T)$ of $T$ are defined by

$$σ_{UBF}(T) = \{ λ ∈ ℂ: T − λI \text{ is not an upper semi-B-Fredholm operator}\}$$

$$σ_{LBF}(T) = \{ λ ∈ ℂ: T − λI \text{ is not a lower semi-B-Fredholm operator}\}$$

$$σ_{BF}(T) = \{ λ ∈ ℂ: T − λI \text{ is not a B-Fredholm operator}\}.$$

We have

$$σ_{BF}(T) = σ_{UBF}(T) ∪ σ_{LBF}(T).$$

An operator $T ∈ ℬ(X)$ is said to be a $B$-Weyl operator if it is a $B$-Fredholm operator of index zero. The $B$-Weyl spectrum $σ_{BW}(T)$ of $T$ is defined by

$$σ_{BW}(T) = \{ λ ∈ ℂ: T − λI \text{ is not a B-Weyl operator}\}.$$

From [8, Lemma 4.1], $T$ is a B-Weyl operator if and only if $T = F ⊕ N$, where $F$ is a Fredholm operator of index zero and $N$ is a nilpotent operator.

We shall denote by $SBF_+(X)$ (or $SBF^+(X)$) the class of all $T$ upper semi-$B$-Fredholm operators ($T$ lower semi-$B$-Fredholm operators) such that $\text{ind}(T) ≤ 0$ ($\text{ind}(T) ≥ 0$). The spectrum associated with $SBF_+(X)$ is called the semi-essential approximate point spectrum and is denoted by $σ_{SBF^+}(T) = \{ λ ∈ ℂ: T − λI ∉ SBF^+(X)\}$, while the spectrum associated with $SBF^+(X)$ is denoted by $σ_{SBF^+}(T) = \{ λ ∈ ℂ: T − λI ∉ SBF^+(X)\}$.

The ascent $a(T)$ and the descent $d(T)$ of $T$ are given by $a(T) = \inf\{n: N(T^n) = N(T^{n+1})\}$ and $d(T) = \inf\{n: R(T^n) = R(T^{n+1})\}$, with $\inf \emptyset = ∞$. It is well-known that if $a(T) \text{ and } d(T)$ are both finite then they are equal, see [16, Proposition 38.3].

Recall that an operator $T$ is Drazin invertible if it has a finite ascent and descent. It is well known that $T$ is Drazin invertible if and only if $T = R ⊕ N$ where $R$ is invertible and $N$ is nilpotent (see [20, Corollary 2.2]). The Drazin spectrum is defined by $σ_D(T) = \{ λ ∈ ℂ: T − λI \text{ is not Drazin invertible}\}$. From [8, Lemma 4.1] and [20, Corollary 2.2] we have

$$σ_{BW}(T) \subseteq σ_D(T).$$
Define the set $LD(X)$ as

$$LD(X) = \{ T \in B(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed} \}.$$  

From [21], $LD(X)$ is a regularity and it is the dual version of the regularity $RD(X) = \{ T \in B(X) : d(T) < \infty \text{ and } R(T^{d(T)}) \text{ is closed} \}$. An operator $T \in B(X)$ is said to be left (or right) Drazin invertible if $T \in LD(X) \ (T \in RD(X))$. The left Drazin spectrum $\sigma_{lD}(T)$ and the right Drazin spectrum $\sigma_{rD}(T)$ are defined by $\sigma_{lD}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin LD(X) \}$ and $\sigma_{rD}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin RD(X) \}$. It is not difficult to see that

$$\sigma_{D}(T) = \sigma_{lD}(T) \cup \sigma_{rD}(T).$$

2. Preliminary results

An operator $T \in B(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (the SVEP for short) if for every open disc $D_{\lambda_0}$ centered at $\lambda_0$, the only analytic function $f : D_{\lambda_0} \rightarrow X$ which satisfies $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. Trivially, every operator $T$ has the SVEP at all points of the resolvent; also $T$ has the SVEP at $\lambda \in \text{iso} \sigma(T)$ (iso $\sigma(T)$ is the set of all isolated points of $\sigma(T)$). We say that $T$ has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$, [15]. We denote by $S(T)$ the set of all $\lambda \in \mathbb{C}$ such that $T$ does not have the single-valued extension property at $\lambda$. Note that (see [15], [19]) $S(T) \subseteq \sigma_p(T)$ and $\sigma(T) = S(T) \cup \sigma_s(T)$. In particular, if $T$ (or $T^*$) has the SVEP then $\sigma(T) = \sigma_s(T)$ ($\sigma(T) = \sigma_a(T)$).

Recall that if $T - \lambda I$ has a finite ascent then it has the SVEP ([18]). Thus we have

$$S(T) \subseteq \sigma_{lD}(T) \text{ and } S(T^*) \subseteq \sigma_{rD}(T).$$

In the following theorem, we prove equality up to $S(T)$ between the left Drazin spectrum, the upper semi-B-Fredholm spectrum and the semi-essential approximate point spectrum.

Theorem 2.1. Let $T \in B(X)$. Then

$$\sigma_{lD}(T) = \sigma_{UBF}(T) \cup S(T) = \sigma_{SBF^-}(T) \cup S(T).$$

Proof. Let $\lambda \notin \sigma_{lD}(T)$, without loss of generality we assume that $\lambda = 0$. Then $R(T^{a(T)+1})$ is closed. Hence $R(T^{a(T)})$ is closed by [21, Lemma 12]. We shall prove that $T_{[a(T)]]}$ is upper semi-Fredholm. Let $x \in N(T_{[a(T)]]})$ then $x \in N(T) \cap R(T^{a(T)})$. Hence $x = T^{a(T)} y$ for some $y \in X$. Then $0 = Tx = T^{a(T)+1} y$. Thus
$y \in N(T^{a(T)+1}) = N(T^{a(T)})$. Therefore $x = 0$ and hence $T_{[\alpha(T - \lambda I)\cap]}$ is injective.
On the other hand, $R(T_{[\alpha(T)\cap]}) = R(T^{a(T)+1})$ is closed. Thus $T_{[\alpha(T)\cap]}$ is upper semi-Fredholm and hence $0 \notin \sigma_{\text{UBF}}(T)$. Since $S(T) \subseteq \sigma_{\text{ID}}(T)$ we have

$$\sigma_{\text{UBF}}(T) \cup S(T) \subseteq \sigma_{\text{ID}}(T).$$

Now let $0 \notin [\sigma_{\text{UBF}}(T) \cup (S(T))]$, then $T$ is an upper semi-B-Fredholm operator. Hence it follows from [7, Proposition 3.2] that there exist $n$ such that $R(T^n)$ is closed and $T_{[n]}$ is semi-regular. Since $T$ has the SVEP at $0$ then $T_{[n]}$ has also the SVEP at $0$. Then from [1, Theorem 3.14], we conclude that $T_{[n]}$ is injective with closed range.

Let $x \in N(T^{n+1})$, then $TT^nx = 0$. Hence $T^nx \in N(T) \cap R(T^n) = N(T_{[n]}) = \{0\}$. Thus $x \in N(T^n)$, and hence $N(T^n) = N(T^{n+1})$. So $T$ is of finite ascent and $a(T) \leq n$. We have $R(T^{n+1}) = R(T_{[n]})$ is closed with $a(T) + 1 \leq n + 1$. Hence $R(T^{a(T)+1})$ is closed by [21, Lemma 12]. Thus $T$ is left Drazin invertible. Therefore

$$\sigma_{\text{ID}}(T) \subseteq \sigma_{\text{UBF}}(T) \cup S(T).$$

From [13, Lemma 2.12] we have $\sigma_{\text{SBF} \pm}(T) \subseteq \sigma_{\text{ID}}(T)$ and since $\sigma_{\text{UBF}}(T) \subseteq \sigma_{\text{SBF} \pm}(T)$ we infer $\sigma_{\text{ID}}(T) = \sigma_{\text{UBF}}(T) \cup S(T) = \sigma_{\text{SBF} \pm}(T) \cup S(T)$.

A useful consequence of the preceding result is that under the assumption of the SVEP for $T$, the spectra $\sigma_{\text{ID}}(T)$, $\sigma_{\text{UBF}}(T)$ and $\sigma_{\text{SBF} \pm}(T)$ are equal.

**Corollary 2.1.** If $T \in \mathcal{B}(X)$ has the SVEP then

$$\sigma_{\text{ID}}(T) = \sigma_{\text{UBF}}(T) = \sigma_{\text{SBF} \pm}(T).$$

By duality we get a similar result for the right Drazin spectrum.

**Theorem 2.2.** Let $T \in \mathcal{B}(X)$. Then

$$\sigma_{\text{rD}}(T) = \sigma_{\text{LBF}}(T) \cup S(T^*) = \sigma_{\text{SBF} \pm}(T) \cup S(T^*).$$

**Proof.** Since $\sigma_{\text{LBF}}(T) = \sigma_{\text{UBF}}(T^*)$, $\sigma_{\text{SBF} \pm}(T) = \sigma_{\text{SBF} \pm}(T^*)$ and $\sigma_{\text{rD}}(T) = \sigma_{\text{ID}}(T^*)$ the assertion follows by Theorem 2.1.

**Corollary 2.2.** If $T^* \in \mathcal{B}(X)$ has the SVEP then

$$\sigma_{\text{rD}}(T) = \sigma_{\text{LBF}}(T) = \sigma_{\text{SBF} \pm}(T).$$

From Theorem 2.1 and Theorem 2.2 we get the following corollary.

**Corollary 2.3.** Let $T \in \mathcal{B}(X)$. Then

\begin{equation}
\sigma_{\text{D}}(T) = \sigma_{\text{BF}}(T) \cup [S(T) \cup S(T^*)] = \sigma_{\text{BW}}(T) \cup [S(T) \cup S(T^*)].
\end{equation}
In particular if $T$ and $T^*$ have the SVEP then

$$
\sigma_D(T) = \sigma_{BF}(T) = \sigma_{BW}(T).
$$

The equality in (2.1) may be refined for $\sigma_D(T)$ and $\sigma_{BW}(T)$. More precisely, we have

**Theorem 2.3.** Let $T \in B(X)$ then

$$
\sigma_D(T) = \sigma_{BW}(T) \cup [S(T) \cap S(T^*)].
$$

**Proof.** Since $\sigma_{BW}(T) \cup (S(T) \cap S(T^*)) \subseteq \sigma_D(T)$ always holds, let $\lambda \notin \sigma_{BW}(T) \cup (S(T) \cap S(T^*))$. Without loss of generality we assume that $\lambda = 0$. Then $T$ is a B-Fredholm operator of index zero.

**Case 1.** If $0 \notin S(T)$: Since $T$ is a B-Fredholm operator of index zero, it follows from [8, Lemma 4.1] that there exists a Fredholm operator $F$ of index zero and a nilpotent operator $N$ such that $T = F \oplus N$. If $0 \notin \sigma(F)$, then $F$ is invertible and hence $T$ is Drazin invertible. Now assume that $0 \in \sigma(F)$. Since $T$ has the SVEP at 0, $F$ has also the SVEP at 0. Hence it follows from [1, Theorem 3.16] that $a(F)$ is finite. $F$ is a Fredholm operator of index zero, hence it follows from [1, Theorem 3.4] that $d(F)$ is also finite. Then $a(F) = d(F) < \infty$ which implies that 0 is a pole of $F$ and hence an isolated point of $\sigma(F)$. Operator $N$ is nilpotent, hence 0 is an isolated point of $\sigma(T)$. From [8, Theorem 4.2] we get $0 \notin \sigma_D(T)$.

**Case 2.** If $0 \notin S(T^*)$, the proof goes similarly. \qed

**Corollary 2.4 ([12]).** If $T$ or $T^*$ has the SVEP then

$$
\sigma_D(T) = \sigma_{BW}(T).
$$

Recall that $T$ is a Browder operator if $T$ is a Fredholm operator of finite ascent and descent. Let $\sigma_B(T)$ be the Browder spectrum defined as the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Browder. Analogously, $T$ is a B-Browder operator if for some integer $n$, $R(T^n)$ is closed and $T_{[n]}$ is Browder. Let $\sigma_{BB}(T)$ be the B-Browder spectrum. In [1, Corollary 3.53] it is proved that if $T$ or $T^*$ has the SVEP, then

$$
\sigma_W(T) = \sigma_B(T).
$$

From [7, Theorem 3.6] we have $\sigma_D(T) = \sigma_{BB}(T)$, hence by Corollary 2.4, if $T$ or $T^*$ has the SVEP then

$$
\sigma_{BW}(T) = \sigma_{BB}(T).
$$

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Theorem 2.4. Let $T \in \mathcal{B}(X)$ and let $f$ be an analytic function on some open neighborhood of $\sigma(T)$ which is nonconstant on any connected component of $\sigma(T)$. Then

$$f(\sigma_{BW}(T) \cup [\mathcal{S}(T) \cap \mathcal{S}(T^*)]) = \sigma_{BW}(f(T)) \cup [\mathcal{S}(f(T)) \cap \mathcal{S}(f(T^*))].$$

Proof. According to [21] the Drazin spectrum satisfies the spectral mapping theorem for such a function $f$, hence the result follows at once from Theorem 2.3. □

It is well known that if $T$ has the SVEP then $f(T)$ has also the SVEP [19]. Now we retrieve the result proved in [2], [23]: $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ whenever $T$ or $T^*$ has the SVEP. Note that in [2], [23] the condition "$f$ is nonconstant on any connected component of $\sigma(T)$" is dropped.

3. Applications

3.1. Perturbations.

Lemma 3.1. Let $T \in \mathcal{B}(X)$. Let $N \in \mathcal{B}(X)$ be a nilpotent operator such that $TN = NT$. Then

$$\mathcal{S}(T + N) = \mathcal{S}(T).$$

Proof. See for instance [5, Lemma 2.1]. □

Lemma 3.2. Let $T \in \mathcal{B}(X)$. If $N \in \mathcal{B}(X)$ is a nilpotent operator which commutes with $T$ then

$$\sigma_{ID}(T + N) = \sigma_{ID}(T).$$

Proof. Assume that $\lambda = 0 \notin \sigma_{ID}(T)$. Then $a(T)$ is finite and $R(T^{a(T)+1})$ is closed. Let $m$ be the nonnegative integer such that $N^m = 0 \neq N^{m-1}$. Let $s = \max(a(T), m)$. Then

$$(T + N)^{2s} = \sum_{k=0}^{2s} \binom{k}{2s} T^k N^{2s-k}$$

$$= \left(\frac{0}{2s}\right) N^{2s} + \ldots + \left(\frac{s}{2s}\right) T^s N^s + \left(\frac{s+1}{2s}\right) T^{s+1} N^{s-1} + \ldots + \left(\frac{2s}{2s}\right) T^{2s}$$

$$= \left(\frac{s+1}{2s}\right) T^{s+1} N^{s-1} + \ldots + \left(\frac{2s}{2s}\right) T^{2s}$$

$$= T^s \left[ \left(\frac{s+1}{2s}\right) T^1 N^{s-1} + \ldots + \left(\frac{2s}{2s}\right) T^s \right].$$
Now let $x \in N(T)^{2s} = N(T)^s$ that is $(T)^{2s}x = 0$. Then it follows from the above equality that $(T+N)^{2s}x = 0$. Hence $N(T)^{2s} \subseteq N(T+N)^{2s}$. With the same argument for $T + N$ and $-N$ we have $N(T+N)^{2s} \subseteq N(T)^{2s}$. Thus $N(T)^{2s} = N(T+N)^{2s}$. Since $N(T^*) = N(T^{2s}) = N(T^{2s+1})$, we get $N(T+N)^{2s} = N(T+N)^{2s+1}$. Therefore $T+N$ is of finite ascent. On the other hand, $R(T+N)^{2s} \subseteq R(T^*)$ is closed. Hence by [21, Lemma 12] $R(T+N)^{2s+1}$ is closed. Thus $0 \notin \sigma_{ID}(T+N)$. Hence $\sigma_{ID}(T+N) \subseteq \sigma_{ID}(T)$. With the same argument for $T + N$ and $-N$ we get $\sigma_{ID}(T) \subseteq \sigma_{ID}(T+N)$.

The next result follows from Theorem 2.1, Lemma 3.1 and Lemma 3.2.

**Theorem 3.1.** Let $T \in \mathcal{B}(X)$. Let $N \in \mathcal{B}(X)$ be a nilpotent operator which commutes with $T$. Then

$$\sigma_{\text{SBF}^+}(T+N) \cup S(T) = \sigma_{\text{SBF}^-}(T) \cup S(T).$$

The following corollary which is proved in [3] gives an affirmative answer to the question posed by Berkani-Amouch [9] in the case when $T$ has the SVEP.

**Corollary 3.1.** Let $T \in \mathcal{B}(X)$ have the SVEP. Let $N \in \mathcal{B}(X)$ be a nilpotent operator which commutes with $T$. Then

$$\sigma_{\text{SBF}^+}(T+N) = \sigma_{\text{SBF}^-}(T).$$

### 3.2. Generalized Weyl’s theorem for operator matrices.

Berkani [8, Theorem 4.5] has shown that every normal operator $T$ acting on a Hilbert space $H$ satisfies

$$\sigma(T) \setminus E(T) = \sigma_{\text{BW}}(T),$$

where $E(T)$ is the set of all isolated eigenvalues of $T$. We say that the *generalized Weyl’s theorem* holds for $T$ if equality (3.1) holds. This gives a generalization of the classical Weyl’s theorem. Recall that $T \in \mathcal{B}(X)$ obeys *Weyl’s theorem* if

$$\sigma(T) \setminus E_0(T) = \sigma_{W}(T)$$

where $E_0(T)$ denotes the set of the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. By [13, Theorem 3.9] the generalized Weyl’s theorem implies Weyl’s theorem and generally the reverse is not true.

For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$ we denote by $M_C$ the operator defined on $X \oplus Y$ by

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$
In general the fact that the generalized Weyl’s theorem holds for $A$ and $B$ does not imply that the generalized Weyl’s theorem holds for $M_0 = [A \ 0 \ 0 \ B]$. Indeed, let $I_1$ and $I_2$ be the identities on $\mathbb{C}$ and $l_2$, respectively. Let $S_1$ and $S_2$ be defined on $l_2$ by

$$S_1(x_1, x_2, \ldots) = (0, \frac{1}{3}x_1, \frac{1}{3}x_2, \ldots), \quad S_2(x_1, x_2, \ldots) = (0, \frac{1}{2}x_1, \frac{1}{2}x_2, \ldots).$$

Let $T_1 = I_1 \oplus S_1$, $T_2 = S_2 - I_2$, $A = T_2^2$ and $B = T_1^2$, then from [23, Example 1] we have $A$ and $B$ obey the generalized Weyl’s theorem but $M_0$ does not obey it.

It also may happen that $M_C$ obeys the generalized Weyl’s theorem while $M_0$ does not obey it. Let $A$ be the unilateral unweighted shift operator. For $B = A^*$ and $C = I - AA^*$, we have that $M_C$ is unitary without eigenvalues. Hence $M_C$ satisfies the generalized Weyl’s theorem (see [10, Remark 3.5]). But $\sigma_W(M_0) = \{\lambda: |\lambda| = 1\}$ and $\sigma(M_0) \setminus E_0(M_0) = \{\lambda: |\lambda| \leq 1\}$. Hence $M_0$ does not satisfy the Weyl’s theorem and so by [13, Theorem 3.9] it does not satisfy the generalized Weyl’s theorem either.

A bounded linear operator $T$ is said to be **isoloid** if every isolated point of $\sigma(T)$ is an eigenvalue of $T$.

**Proposition 3.1.** Let $A$ and $B$ be isoloids. Assume that $\sigma_{BW}(M_0) = \sigma_{BW}(A) \cup \sigma_{BW}(B)$. If $A$ and $B$ obey the generalized Weyl’s theorem, then $M_0$ obeys the generalized Weyl’s theorem.

**Proof.** Since $A$ and $B$ are isoloids, we have

$$E(M_0) = [E(A) \cap \varrho(B)] \cup [\varrho(A) \cap E(B)] \cup [E(A) \cap E(B)].$$

Now if $A$ and $B$ obey the generalized Weyl’s theorem, then

$$E(M_0) = [\sigma(A) \cup \sigma(B)] \setminus [\sigma_{BW}(A) \cup \sigma_{BW}(B)]$$

$$= \sigma(M_0) \setminus \sigma_{BW}(M_0).$$

Then $M_0$ obeys the generalized Weyl’s theorem. \qed

**Lemma 3.3.** Let $A \in B(X)$ and $B \in B(Y)$ have the SVEP. Then

$$\sigma_{BW}(M_C) = \sigma_{BW}(A) \cup \sigma_{BW}(B)$$

for all $C \in B(Y, X)$.

**Proof.** Since $A$ and $B$ have the SVEP, then it follows from [17, Proposition 3.1] that $M_C$ also has the SVEP. Hence $\sigma_{BW}(M_C) = \sigma_{D}(M_C)$ by Corollary 2.4. Also since $A$ and $B$ have the SVEP, it follows from [24, Corollary 2.1] that $\sigma_{D}(M_C) = \sigma_{D}(A) \cup \sigma_{D}(B)$. Therefore $\sigma_{BW}(M_C) = \sigma_{BW}(A) \cup \sigma_{BW}(B)$ by Corollary 2.4. \qed
Theorem 3.2. Let $A$ and $B$ be isoloids with the SVEP. If $A$ and $B$ obey the generalized Weyl’s theorem, then $M_C$ obeys the generalized Weyl’s theorem for every $C \in B(Y, X)$.

Proof. It follows from Proposition 3.1 and Lemma 3.3 that

$$E(M_0) = \sigma(M_0) \setminus \sigma_{BW}(M_0) = \sigma(M_C) \setminus \sigma_{BW}(M_C).$$

Hence it is enough to show that $E(M_0) = E(M_C)$. Let $\lambda \in E(M_C)$. Then $\lambda \in \sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B)$. Hence $\lambda \in \sigma_p(M_0)$. Since $\lambda \in \text{iso}\sigma(M_C) = \text{iso}\sigma(M_0)$ we have $\lambda \in E(M_0)$. Now let $\lambda \in E(M_0)$. If $\lambda \in \sigma(A)$ then $\lambda \in \text{iso}\sigma(A)$. Since $A$ is an isoloid, we have $\lambda \in \sigma_p(A) \subseteq \sigma_p(M_C)$. Hence $\lambda \in E(M_C)$. If $\lambda \in \sigma(B) \setminus \sigma(A)$, then $\lambda \in \sigma_p(B)$. Since $A$ is invertible, we conclude that $\lambda \in \sigma_p(M_C)$. Thus $\lambda \in E(M_C)$. Therefore $E(M_0) = E(M_C)$.

Let $\pi(T)$ be the set of all poles of the resolvent of $T$. Recall from [14] that $T$ is a polaroid if $\text{iso}\sigma(T) \subseteq \pi(T)$. Since $\pi(T) \subseteq E(T)$ holds without restriction on $T$, then if $T$ is a polaroid then $E(T) = \pi(T)$.

Corollary 3.2. Let $A$ and $B$ be polaroids with the SVEP. Then $M_C$ obeys the generalized Weyl’s theorem for every $C \in B(Y, X)$.

Proof. $A$ and $B$ are polaroids hence $E(A) = \pi(A)$ and $E(B) = \pi(B)$. Since $A$ and $B$ have the SVEP, we have by [4] that $A$ and $B$ satisfy the generalized Weyl’s theorem. Hence we complete the proof by Theorem 3.2.

3.3. Multipliers on a commutative Banach algebra. Let $\mathcal{A}$ be a semi-simple commutative Banach algebra. A mapping $T : \mathcal{A} \to \mathcal{A}$ is called a multiplier if

$$T(x)y = xT(y) \text{ for all } x, y \in \mathcal{A}.$$

By semi-simplicity of $\mathcal{A}$, every multiplier is a bounded linear operator on $\mathcal{A}$. Also the semi-simplicity of $\mathcal{A}$ implies that every multiplier has the SVEP (see [1], [19]).

By [1, Theorem 4.36], for every multiplier $T$ on a semi-simple commutative Banach algebra $\mathcal{A}$, $E(T) = \pi(T)$ and since $T$ has the SVEP we get from [4]

Proposition 3.2. Every multiplier on a semi-simple commutative Banach algebra $\mathcal{A}$ obeys the generalized Weyl’s theorem.
From Corollary 2.4 we have

**Proposition 3.3** ([11]). Let $T$ be a multiplier on a semi-simple commutative Banach algebra $A$. Then the following assertions are equivalent:

i) $T$ is B-Fredholm of index zero.

ii) $T$ is Drazin invertible.

Now if we assume in addition that $A$ is regular and Tauberian (see [19] for definition) then every multiplier $T$ has the weak decomposition property ($\delta_w$) and then $T^*$ has also the SVEP (see [22] for definition and details). Hence we get from Corollary 2.3

**Proposition 3.4.** Let $T$ be a multiplier on a semi-simple regular Tauberian commutative Banach algebra $A$. Then the following assertions are equivalent:

i) $T$ is B-Fredholm.

ii) $T$ is Drazin invertible.

For $G$ a locally compact abelian group, let $L^1(G)$ be the space of $\mathbb{C}$-valued functions on $G$ integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on $G$. We recall that $L^1(G)$ is a regular semi-simple Tauberian commutative Banach algebra. Then we have

**Corollary 3.3.** Let $G$ be a locally compact abelian group, $\mu \in M(G)$ and $X = L^1(G)$. Then every convolution operator $T_\mu : X \rightarrow X$, $T_\mu(k) = \mu \star k$ is B-Fredholm if and only if it is Drazin invertible.

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**References**


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