Bo Du; Xueping Hu
Periodic solutions to a $p$-Laplacian neutral Rayleigh equation with deviating argument


Persistent URL: [http://dml.cz/dmlcz/141487](http://dml.cz/dmlcz/141487)

**Terms of use:**

© Institute of Mathematics AS CR, 2011

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*
PERIODIC SOLUTIONS TO A \( p \)-LAPLACIAN NEUTRAL RAYLEIGH EQUATION WITH DEVIATING ARGUMENT*

Bo Du, Huaian, Xueping Hu, Anqing

(Received September 26, 2008)

Abstract. By using the coincidence degree theory, we study a type of \( p \)-Laplacian neutral Rayleigh functional differential equation with deviating argument to establish new results on the existence of \( T \)-periodic solutions.

Keywords: deviating argument, neutral, coincidence degree theory

MSC 2010: 34B15, 34B24, 34B20

1. Introduction

In this paper we consider the \( p \)-Laplacian neutral Rayleigh functional differential equation with deviating argument of the form

\[
(\varphi_p((x(t) - cx(t - \sigma)))')' + f(x'(t)) + \alpha(t)g(x(t - \tau(t))) = e(t),
\]

where \( \varphi_p: \mathbb{R} \to \mathbb{R} \), \( \varphi_p(u) = |u|^{p-2}u \), \( p > 1 \); \( f, g \in C(\mathbb{R}, \mathbb{R}) \); \( \alpha, \tau, e \) are continuous \( T \)-periodic functions defined on \( \mathbb{R} \) with \( \alpha(t) > 0 \); \( \sigma, c \in \mathbb{R} \) are constants such that \( |c| \neq 1 \).

Neutral functional differential equations (in short NFDEs) have been used for the study of distributed networks containing lossless transmission lines and other aspects [3], [4]. In recent papers, many researchers have obtained a lot of results for the existence of periodic solutions to NFDEs. In [8], Enrico Serra studied a kind of NFDE in the form

\[
x'(t) + ax'(t - \tau) = f(t, x(t)).
\]

*Supported by National Natural Science Foundation of P. R. China (No. 10671012).
He proved the existence of at least one periodic solution for equation (1.2) (Theorem 3.1). In [6], Lu and Ge studied the existence of periodic solutions for NFDE

\[
\frac{d^2}{dt^2}(x(t) - kx(t - \tau)) = f(x'(t)) + \alpha(t)g(u(t)) + \sum_{i=1}^{n} \beta_i(t)g(x(t - \tau_i(t))) + e(t).
\]

They used Mawhin’s continuation theorem to obtain the existence of periodic solutions for equation (1.3). Liu [5] considered the first order neutral equation

\[
(u(t) + Bu(t - \tau))' = g_1(t, u(t)) - g_2(t, u(t - \tau_1)) + p(t)
\]

and Si [9] examined the kth-order neutral equation

\[
\frac{d^k}{dt^k}(x(t) + b_0x(t - h_0)) + \sum_{j=1}^{k} a_jx^{(k-j)}(t) + \sum_{j=1}^{k} a_jx^{(k-j)}(t - h_j) = f(t).
\]

However, there have been few results for the existence of periodic solutions to p-Laplacian neutral equations. The reason for it lies in the following two facts. The first is that the differential operator \( \varphi_p(u) = |u|^{p-2}u, p \neq 2 \), is no longer linear, so the theory of coincidence degree cannot be used directly; the second is that an a priori bound of solutions is not easy to achieve. In this paper we will overcome these difficulties and obtain the existence of periodic solutions to equation (1.1).

2. MAIN LEMMAS

Let

\[
A: C_T \to C_T, \quad (Ax)(t) = x(t) - cx(t - \sigma),
\]

where

\[
C_T = \{ \varphi: \varphi \in C(\mathbb{R}, \mathbb{R}), \varphi(t + T) = \varphi(t) \}
\]

with the norm \( |\varphi|_0 = \max_{t \in [0, T]} |\varphi(t)| \). In order to use Mawhin’s continuation theorem to obtain the existence of T-periodic solutions of equation (1.1), we rewrite equation (1.1) in the form of the two-dimensional differential system

\[
\begin{cases}
(Ax_1)'(t) = \varphi_q(x_2(t)), \\
x_2'(t) = -f([A^{-1}\varphi_q(x_2)](t)) - \alpha(t)g(x_1(t - \tau(t))) + e(t),
\end{cases}
\]

where \( q > 1 \) is a constant with \( 1/p + 1/q = 1 \). Clearly, if \( x(t) = (x_1(t), x_2(t))^\top \) is a T-periodic solution to system (2.1), then \( x_1(t) \) must be a T-periodic solution to
It suffices to show that system (2.1) has a $T$-periodic solution. Further, let

$$
(2.2) \quad L: D(L) \subset X \to X, \quad Lx = \begin{pmatrix} (Ax_1)' \\ x'_2 \end{pmatrix},
$$

and

$$
(2.3) \quad N: X \to X, \quad (Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -f([A^{-1}\varphi_q(x_2)](t)) - \alpha(t)g(x_1(t - \tau(t))) + e(t) \end{pmatrix},
$$

where $D(L) = \{x: x \in C^2(\mathbb{R}, \mathbb{R}^2), x(t + T) = x(t)\}$.

**Lemma 2.1 ([7]).** If $|c| < 1$ then $A$ has continuous inverse on $C_T$, and

1. $\|A^{-1}x\| \leq \frac{|x|}{1 - |c|} \quad \forall x \in C_T$;
2. $\int_0^T |(A^{-1}f)(t)| \, dt \leq \frac{1}{1 - |c|} \int_0^T |f(t)| \, dt \quad \forall f \in C_T$;
3. $\int_0^T |(A^{-1}f)^2(t)| \, dt \leq \frac{1}{(1 - |c|)^2} \int_0^T f^2(t) \, dt \quad \forall f \in C_T$.

By Hale’s terminology [2], a solution of the system (2.1) is $x = (x_1, x_2)^T \in C(\mathbb{R}, \mathbb{R}^2)$ such that $(Ax_1, x_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$ and the equalities in (2.1) are satisfied on $\mathbb{R}$. In general, $x$ is not from $C^1(\mathbb{R}, \mathbb{R}^2)$. Nevertheless, it is easy to see that $(Ax_1)' = Ax_1$. So a $T$-periodic solution $x$ of the system (2.1) must be from $C^1(\mathbb{R}, \mathbb{R}^2)$. According to Lemma 2.1, we can easily obtain that $\text{Ker} L = \mathbb{R}^2$, $\text{Im} L = \{x: x \in X, \int_0^T x(s) \, ds = 0\}$. So $L$ is a Fredholm operator with index zero. Let the projections $P$ and $Q$ be

$$
P: X \to \text{Ker} L, \quad Px = \frac{1}{T} \int_0^T x(s) \, ds, \quad Q: X \to X, \quad Qy = \frac{1}{T} \int_0^T y(s) \, ds.
$$

Then $\text{Im} P = \text{Ker} L$ and $\text{Ker} Q = \text{Im} L$. Let $L_P = L|_{D(L) \cap \text{Ker} P}$, $L_{P^{-1}} : \text{Im} L \to D(L) \cap \text{Ker} P$. We can easily prove that $L_P$ is invertible, $L_{P^{-1}}$. Then $L_{P^{-1}} = \int_0^T G(t, s)z(s) \, ds$,

$$
(L_{P^{-1}}z)(t) = \begin{pmatrix} (A^{-1}Fz_1)(t) \\ (Fz_2)(t) \end{pmatrix}, \quad (Fz)(t) = \int_0^T G(t, s)z(s) \, ds,
$$

where $G(t, s) = \begin{cases} \frac{s}{T}, & 0 \leq s < t \leq T, \\ \frac{s - T}{T}, & 0 \leq t \leq s \leq T, \end{cases}$.

Thus, in order to prove that equation (1.1) has a $T$-periodic solution, it suffices to show that system (2.1) has a $T$-periodic solution. Now we set

$$
X = Y = \{x = (x_1, x_2)^T \in C(\mathbb{R}, \mathbb{R}^2), \ x(t + T) = x(t)\}
$$

with the norm $\|x\| = \max\{|x_1|, |x_2|\}$. Clearly $X$ and $Y$ are two Banach spaces.
Lemma 2.2 ([1]). Suppose that $X$ and $Y$ are two Banach spaces, and $L: D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N: \bar{\Omega} \to Y$ is $L$-compact on $\bar{\Omega}$. Let the following conditions hold:
1. $Lx \neq \lambda Nx \forall x \in \partial \Omega \cap D(L) \forall \lambda \in (0, 1)$,
2. $Nx \notin \text{Im} L \forall x \in \partial \Omega \cap \text{Ker} L$,
3. $\deg \{ JQN, \Omega \cap \text{Ker} L, 0 \} \neq 0$,
where $J: \text{Im} Q \to \text{Ker} L$ is an isomorphism. Then equation $Lx = Nx$ has a solution in $\bar{\Omega} \cap D(L)$.

3. Main results

For the sake of convenience, we list the following conditions which will be needed in our study of equation (1.1).

(H$_1$) There is a constant $K > 0$ such that $|f(x)| \leq K \forall x \in \mathbb{R}$.
(H$_2$) There is a constant $D > 0$ such that
\[
\begin{cases}
  g(x) < -\frac{|e|_0}{\alpha_m} - \frac{K}{\alpha_m} & \text{for } x > D, \\
  g(x) > \frac{K}{\alpha_m} & \text{for } x < -D,
\end{cases}
\]
where $\alpha_m = \min_{t \in [0, T]} \alpha(t)$ and $K$ is defined by (H$_1$).
(H$_3$) There is a constant $r$ such that
\[
\limsup_{x \to -\infty} \frac{|g(x)|}{|x|^{p-1}} \leq r \in [0, \infty).
\]
(H'$_3$) There is a constant $r$ such that
\[
\limsup_{x \to +\infty} \frac{|g(x)|}{|x|^{p-1}} \leq r \in [0, \infty).
\]

Theorem 3.1. Suppose that $|c| < 1$, $\int_0^T e(s) \, ds = 0$, and (H$_1$)–(H$_3$) are all satisfied. Then equation (1.1) has at least one $T$-periodic solution if
\[
\frac{2(1 + |c|)r\alpha_M T^p}{|1 - |c||^p} < 1,
\]
where $\alpha_M = \max_{t \in [0, T]} \alpha(t)$. 

256
Proof. Consider the operator equation

\[ Lx = \lambdaNx, \quad \lambda \in (0, 1), \]

where \( L \) and \( N \) are defined by (2.2) and (2.3), respectively. Let \( \Omega_1 = \{x: x \in D(L), Lx = \lambdaNx, \lambda \in (0, 1)\} \). If \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \Omega_1 \) then \( x \) must satisfy

\[
\begin{cases}
(Ax_1)'(t) = \lambda \varphi_2(x_2(t)), \\
x_2'(t) = -\lambda f([A^{-1}\varphi_2(x_2)](t)) - \lambda \alpha(t)g(x_1(t - \tau(t))) + \lambda e(t).
\end{cases}
\tag{3.1}
\]

From the first equation of (3.1) we get \( x_2(t) = \varphi_p(\lambda^{-1}(Ax_1)'(t)) \), and combining it with the second equation of (3.1) yields

\[
(\varphi_p((Ax_1)'(t)))' + \lambda^p f\left(\frac{1}{\lambda}x_1'(t)\right) + \lambda^p \alpha(t)g(x_1(t - \tau(t))) = \lambda^p e(t).
\tag{3.2}
\]

Let \( t_0 \) be the point, where \( Ax_1 \) achieves its maximum on \([0, T]\), i.e., \((Ax_1)(t_0) = \max_{t \in [0, T]}(Ax_1)(t)\). Then \((Ax_1)'(t_0) = 0\) and \( x_2(t_0) = \varphi_p(\lambda^{-1}(Ax_1)'(t_0)) = 0\) for all \( \lambda \in (0, 1) \). Furthermore, we assume that \( t_0 < T \). Then we have

\[
x_2'(t_0) \leq 0.
\tag{3.3}
\]

In fact, if \( x_2'(t_0) > 0 \) then there exists a constant \( \delta > 0 \) such that \( x_2'(t) > 0 \) for \( t \in [t_0, t_0 + \delta] \), and then \( x_2(t) > x_2(t_0) = 0 \) for \( t \in [t_0, t_0 + \delta] \). So \((Ax_1)'(t) = \lambda \varphi_2(x_2(t)) > 0\) for \( t \in [t_0, t_0 + \delta] \) and then \((Ax_1)(t) > (Ax_1)(t_0)\), which contradicts the assumption on \( t_0 \). This proves (3.3). From the second equation of (3.1) we have

\[
-\lambda f([A^{-1}\varphi_2(x_2)](t_0)) - \lambda \alpha(t_0)g(x_1(t_0 - \tau(t_0))) + \lambda e(t_0) \leq 0.
\]

Hence,

\[
g(x_1(t_0 - \tau(t_0))) \geq -\frac{|e|_0}{\alpha_m} - \frac{K}{\alpha_m}.
\]

Assumption \((H_2)\) implies

\[
x_1(t_0 - \tau(t_0)) \leq D.
\tag{3.4}
\]

Integrating both sides of (3.2) over \([0, T]\), we get

\[
\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)dt + \int_0^T \alpha(t)g(x_1(t - \tau(t)))dt = 0.
\tag{3.5}
\]
From the integral mean value theorem and (3.5) we know that there exists a constant 
\( t_1 \in [0, T] \) such that

\[
\alpha(t_1)g(x_1(t_1 - \tau(t_1))) + f\left(\frac{1}{\chi}x_1'(t_1)\right) = 0.
\]

Then we have

\[
g(x_1(t_1 - \tau(t_1))) \leq \frac{K}{\alpha_m}.
\]

Assumption (H2) implies

(3.6)

\[ x_1(t_1 - \tau(t_1)) \geq -D. \]

From (3.4) and (3.6) it is easy to prove that there exists a constant \( \xi \in [0, T] \) such that

(3.7)

\[ |x_1(\xi)| \leq D. \]

In fact, by (3.4) we know \( x_1(t_0 - \tau(t_0)) \in [-D, D] \), or \( x_1(t_0 - \tau(t_0)) < -D \).

1. If \( x_1(t_0 - \tau(t_0)) \in [-D, D] \) then \( t_0 - \tau(t_0) = k\pi + \xi, k \in \mathbb{Z}, \xi \in [0, T] \). This proves (3.7).

2. If \( x_1(t_0 - \tau(t_0)) < -D \) then by (3.6) and the fact that \( x_1(t) \) is continuous on \( \mathbb{R} \), there is a point \( t_2 \) between \( t_0 - \tau(t_0) \) and \( t_1 - \tau(t_1) \) such that \( |x_1(t_2)| \leq D \).

Let \( t_2 = k\pi + \xi, k \in \mathbb{Z}, \) and \( \xi \in [0, T] \). This also proves (3.7). Hence, we get

(3.8)

\[
|x_1|_0 = \max_{t \in [0, T]} \left| x_1(\xi) + \int_{\xi}^{t} x_1'(s) \, ds \right|
\leq |x_1(\xi)| + \int_{0}^{T} |x_1'(s)| \, ds \leq D + \int_{0}^{T} |x_1'(s)| \, ds.
\]

Let

\[
E_1 = \{ t \in [0, T]: x_1(t - \tau(t)) < -\varrho \},
\]

\[
E_2 = \{ t \in [0, T]: |x_1(t - \tau(t))| \leq \varrho \},
\]

\[
E_3 = \{ t \in [0, T]: x_1(t - \tau(t)) > \varrho \},
\]

where \( \varrho > D > 0 \) is a given constant. Integrating the two sides of (3.2) on \( [0, T] \), we get

\[
\int_{0}^{T} \alpha(t)g(x_1(t - \tau(t))) \, dt = -\int_{0}^{T} f\left(\frac{1}{\chi}x_1'(t)\right) \, dt.
\]

258
Therefore, using \((H_1)\) and \((H_2)\), we obtain

\[
(3.9) \quad \int_{E_3} \alpha(t) |g(x_1(t - \tau(t)))| \, dt = - \int_{E_3} \alpha(t) g(x_1(t - \tau(t))) \, dt \\
= \int_{E_1 \cup E_2} \alpha(t) g(x_1(t - \tau(t))) \, dt + \int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right) \, dt \\
\leq \int_{E_1 \cup E_2} \alpha(t) |g(x_1(t - \tau(t)))| \, dt + KT.
\]

Since \((2(1 + |c|)r\alpha_MT^p)/(1 + |c|^p) < 1\), there exists a constant \(\varepsilon > 0\) such that

\[
(3.10) \quad \frac{2(1 + |c|)(r + \varepsilon)\alpha_MT^p}{1 - |c|^p} < 1.
\]

For such \(\varepsilon\), by assumption \((H_3)\), there exists a constant \(\varrho > 0\) such that \(\varrho > D\) and

\[
(3.11) \quad |g(u)| \leq (r + \varepsilon)|u|^{p-1} \quad \text{for} \quad u < -\varrho.
\]

From \((3.9)\) and \((3.11)\) we get

\[
(3.12) \quad \int_0^T \alpha(t) |g(x_1(t - \tau(t)))| \, dt = \int_{E_1 \cup E_2 \cup E_3} \alpha(t) |g(x_1(t - \tau(t)))| \, dt \\
\leq 2 \int_{E_1 \cup E_2} \alpha(t) |g(x_1(t - \tau(t)))| \, dt + KT \\
\leq 2(r + \varepsilon)\alpha_MT|x_1|^{p-1} + 2g_\varrho\alpha_MT + KT,
\]

where \(g_\varrho = \max_{t \in E_2} |g(x_1(t - \tau(t)))|\). On the other hand, multiplying the two sides of equation \((3.2)\) by \((Ax_1)(t)\), integrating them over \([0, T]\) and combining it with \((3.12)\), we arrive at

\[
(3.13) \quad \int_0^T |(Ax_1)'(t)|^p \, dt \\
\leq (1 + |c|) |x_1|_0 \left(\int_0^T \left| f\left(\frac{1}{\lambda}x_1'(t)\right) \right| \, dt + \int_0^T \alpha(t) |g(x_1(t - \tau(t)))| \, dt + T|e|_0 \right) \\
\leq (1 + |c|) |x_1|_0 \int_0^T \alpha(t) |g(x_1(t - \tau(t)))| \, dt + (1 + |c|) |x_1|_0(T|e|_0 + KT) \\
\leq 2(1 + |c|)(r + \varepsilon)\alpha_MT|x_1|^p + (1 + |c|)(2g_\varrho\alpha_MT + 2KT + T|e|_0)|x_1|_0.
\]
For simplicity, let \( k_1 = 2(1 + |c|)(r + \varepsilon)\alpha_MT \), \( k_2 = (1 + |c|)(2g_o\alpha_MT + 2KT + T|e|_0) \).

From (3.8) and (3.13) we have

\[
(3.14) \int_0^T |(Ax_1)'(t)|^p \, dt \leq k_1 |x_1|_0^p + k_2 |x_1|_0
\]

\[
\leq k_1 \left( D + \int_0^T |x_1'(t)| \, dt \right)^p + k_2 \int_0^T |x_1'(t)| \, dt + Dk_2.
\]

By applying the second part of Lemma 2.1 and the Hölder inequality, we get

\[
(3.15) \int_0^T |x_1'(t)| \, dt = \int_0^T |(A^{-1}Ax_1')(t)| \, dt
\]

\[
\leq \frac{\int_0^T |(Ax_1')(t)| \, dt}{|1 - |c||} \leq \frac{T^{1/q} (\int_0^T |(Ax_1')(t)|^p \, dt)^{1/p}}{|1 - |c||}.
\]

**Case 1.** If \( \int_0^T |(Ax_1')(t)| \, dt = 0 \) then \( \int_0^T |x_1'(t)| \, dt = 0 \), by (3.8), \( |x_1|_0 \leq D \).

**Case 2.** If \( \int_0^T |(Ax_1')(t)| \, dt > 0 \) then by (3.14) and (3.15) we have

\[
(3.16) \int_0^T |(Ax_1')(t)|^p \, dt \leq k_1 \left( D + \frac{\int_0^T |(Ax_1')(t)| \, dt}{|1 - |c||} \right)^p + k_2 \frac{\int_0^T |(Ax_1')(t)| \, dt}{|1 - |c||} + Dk_2.
\]

Clearly,

\[
(3.17) \left( D + \frac{\int_0^T |(Ax_1')(t)| \, dt}{|1 - |c||} \right)^p
\]

\[
= \frac{1}{|1 - |c||^p} \left( \int_0^T |(Ax_1')(t)| \, dt \right)^p \left( 1 + \frac{D|1 - |c||}{\int_0^T |(Ax_1')(t)| \, dt} \right)^p.
\]

By classical elementary inequalities, we see that there is a constant \( h(p) > 0 \) which is dependent on \( p \) only, such that

\[
(3.18) (1 + u)^p < 1 + (1 + p)u \quad \forall u \in (0, h(p)].
\]

If \( (D|1 - |c||)/\int_0^T |(Ax_1')(t)| \, dt > h \) then \( \int_0^T |(Ax_1')(t)| \, dt < (D|1 - |c||)/h \). By (3.8) and (3.15), \( |x_1|_0 < D + D/h \). If \( (D|1 - |c||)/\int_0^T |(Ax_1')(t)| \, dt \leq h \) then by (3.17) and (3.18) we have

\[
(3.19) \left( D + \frac{\int_0^T |(Ax_1')(t)| \, dt}{|1 - |c||} \right)^p
\]

\[
\leq \frac{1}{|1 - |c||^p} \left( \int_0^T |(Ax_1')(t)| \, dt \right)^p \left( 1 + \frac{(p + 1)D|1 - |c||}{\int_0^T |(Ax_1')(t)| \, dt} \right)^p
\]

\[
\leq \frac{(\int_0^T |(Ax_1')(t)| \, dt)^p}{|1 - |c||^p} + (p + 1)D|1 - |c||^{1-p} \left( \int_0^T |(Ax_1')(t)| \, dt \right)^{p-1}.
\]
By (3.16) and (3.19),

\[
(3.20) \quad \int_0^T |(Ax'_1)(t)|^p \, dt \\
\leq \frac{k_1}{|1 - |c||} \left( \int_0^T |(Ax'_1)(t)| \, dt \right)^p \\
\quad + k_1(p + 1)D |1 - |c||^{1-p} \left( \int_0^T |(Ax'_1)(t)| \, dt \right)^{p-1} \\
\quad + k_2 \int_0^T |(Ax'_1)(t)| \, dt / |1 - |c|| + Dk_2 \\
\leq \frac{k_1}{|1 - |c||} T^{p/q} \int_0^T |(Ax'_1)(t)|^p \, dt \\
\quad + k_1(p + 1)D |1 - |c||^{1-p} T^{(p-1)/q} \left( \int_0^T |(Ax'_1)(t)|^p \, dt \right)^{(p-1)/p} \\
\quad + \frac{k_2}{|1 - |c||} T^{1/q} \left( \int_0^T |(Ax'_1)(t)|^p \, dt \right)^{1/p} + Dk_2.
\]

In view of the definition of the number \( k_1 \), by virtue of (3.10), (3.20), \((p - 1)/p < 1\) and \(1/p < 1\), there is a constant \( M_1 > 0 \) such that \( \int_0^T |(Ax'_1)(t)|^p \, dt \leq M_1 \). It follows from (3.15) that \( \int_0^T |x'_1(t)| \, dt \leq (T^{1/q}(M_1)^{1/p})/(|1 - |c||) := M_2 \). By (3.8) we get

\[ |x_1|_0 \leq D + M_2 := M_3. \]

Consequently, in both cases 1 and 2, we have \( |x_1|_0 \leq M_3 \). In view of the first equation of (3.1) we have \( \int_0^T |x_2(t)|^{q-2} x_2(t) \, dt = 0 \). By the integral mean value theorem there exists a constant \( \eta \in [0, T] \) such that \( x_2(\eta) = 0 \). Hence, \( |x_2|_0 \leq \int_0^T |x'_2(t)| \, dt \). By the second equation of (3.1) we get

\[
\int_0^T |x'_2(t)| \, dt \leq \int_0^T \left| f \left( \frac{1}{\lambda} x'_1(t) \right) \right| \, dt + \int_0^T \alpha_M |g(x_1(t - \tau(t)))| \, dt + \int_0^T |e(t)| \, dt \\
\leq KT + T\alpha_M g_{M_3} + T|e|_0,
\]

where \( g_{M_3} = \max_{|u| < M_3} |g(u)| \). So we obtain

\[ |x_2|_0 \leq KT + T\alpha_M g_{M_3} + T|e|_0 =: M_4. \]

We have proved that if \( x = (x_1, x_2)^T \in D(L), Lx = \lambda N x, \lambda \in (0, 1) \), then \( |x_1|_0 \leq M_3 \) and \( |x_2|_0 \leq M_4 \). Let \( M = \max\{M_3, M_4\} \) and

\[ \Omega = \{ x = (x_1, x_2)^T \in X : |x_1|_0 \leq M, |x_2|_0 \leq M \}. \]

261
Then $M > D$ and it is clear that the assumption (1) of Lemma 2.2 is satisfied. Moreover, for any $x = (x_1, x_2)^T \in X$ we have

$$QN x = \begin{pmatrix} \frac{1}{T} \int_0^T \varphi_q(x_2(t)) \, dt \\ \frac{1}{T} \int_0^T \left( - f([A^{-1} \varphi_q(x_2)](t)) - \alpha(t) g(x_1(t - \tau(t))) \right) \, dt \end{pmatrix}.$$  

Since $\text{Ker } L = \mathbb{R}^2$ and $\text{Im } L = \text{Ker } Q$, if $QN x = 0$ for some $x = (x_1, x_2)^T \in \partial \Omega \cap \text{Ker } L$, then $x_2 \equiv 0$, $|x_1| \equiv M$, and

$$g(x_1) = - \frac{f(0)}{\frac{1}{T} \int_0^T \alpha(t) \, dt}.$$  

By assumptions (H$_1$) and (H$_2$), one has $M = |x_1| \leq D$, which is a contradiction. So $QN x \neq 0$ for all $x \in \partial \Omega \cap \text{Ker } L$ and thus the assumption (2) of Lemma 2.2 is satisfied. It remains to verify condition (3) of Lemma 2.2. In order to prove it, let

$$J : \text{Im } Q \rightarrow \text{Ker } L, \quad J(x_1, x_2)^T = (x_2, x_1)^T,$$

and $H(x, \mu) = \mu x + (1 - \mu) J Q N x$ for $(x, \mu) \in X \times [0, 1]$. Then we have

$$H(x, \mu) = \begin{pmatrix} \mu x_1 + \frac{(1 - \mu)}{T} \int_0^T \left( - f([A^{-1} \varphi_q(x_2)](t)) - \alpha(t) g(x_1(t - \tau(t))) \right) \, dt \\ \mu x_2 + \frac{(1 - \mu)}{T} \int_0^T \varphi_q(x_2(t)) \, dt \end{pmatrix}.$$  

It is not difficult to verify that, using (H$_2$), for any $x \in \partial \Omega \cap \text{Ker } L$ and $\mu \in [0, 1]$, we have $H(x, \mu) \neq 0$. Therefore,

$$\text{deg}\{J Q N, \Omega \cap \text{Ker } L, 0\} = \text{deg}\{H(\cdot, 0), \Omega \cap \text{Ker } L, 0\} = \text{deg}\{H(\cdot, 1), \Omega \cap \text{Ker } L, 0\} = \text{deg}\{I, \Omega \cap \text{Ker } L, 0\} \neq 0.$$  

Therefore, by using Lemma 2.2, we see that equation $L x = N x$ has a solution $x = (x_1, x_2)^T$ in $\Omega$, i.e., equation (1.1) has a $T$-periodic solution $x_1$.  

262
Corollary 3.2. Suppose that $|c| < 1$, $\int_0^T e(s) \, ds = 0$ and that (H$_1$), (H$_2$), and (H$_3'$) are satisfied. Then equation (1.1) has at least one $T$-periodic solution provided

$$2(1 + |c|)r\alpha_MT^p / (|1 - |c||^p) < 1.$$ 

As an application, we consider the following NFDE:

$$\varphi_3((x(t) - 0.1x(t - \pi))')' + 5 \sin x'(t) + (1 + \frac{1}{2} \sin t)g(x(t - \frac{1}{2} \cos t)) = 20 \cos t,$$

where

$$g(u) = \begin{cases} -\frac{1}{1000}u^2, & u > 10, \\ 5 - \frac{51}{100}u, & u \in [-10, 10], \\ 10 + \frac{1}{1000}u^2, & u < -10. \end{cases}$$

Clearly, equation (3.21) is a particular case of (1.1) in which

$$p = 3, \ c = 0.1, \ \sigma = \pi, \ \alpha(t) = 1 + 1/2 \sin t, \ \tau(t) = 1/2 \cos t,$$

$$e(t) = 20 \cos t, \ f(u) = 5 \sin u.$$ 

Then we have $T = 2\pi$, $\alpha_M = 3/2$ and $r = 1/1000$, and thus

$$\frac{2(1 + |c|)r\alpha_MT^p}{(1 - |c||^p} = \frac{1.1 \times 3 \times (2\pi)^3}{0.9^3 \times 1000} < 1.$$ 

Here assumptions (H$_1$)–(H$_3$) are satisfied. By using Theorem 3.1, we conclude that equation (3.21) has at least one $2\pi$-periodic solution.

Acknowledgement. The author is very grateful to the referees for their helpful suggestions.

References


*Authors’ addresses:* B. Du, Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu, 223300, P. R. China, e-mail: dubo7307@163.com; X. Hu, Department of Mathematics, Anqing Normal College, Anqing, Anhui, 246011, P. R. China, e-mail: hxpprob@yahoo.com.cn.