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2D-1D DIMENSIONAL REDUCTION IN A TOY MODEL FOR MAGNETOELASTIC INTERACTIONS

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Abstract. The paper deals with the dimensional reduction from 2D to 1D in magnetoelastic interactions. We adopt a simplified, but nontrivial model described by the Landau-Lifshitz-Gilbert equation for the magnetization field coupled to an evolution equation for the displacement. We identify the limit problem by using the so-called energy method.

Keywords: magnetoelastic materials, Landau-Lifshitz-Gilbert equation, dimensional reduction

MSC 2010: 35D30, 78A25, 35Q60, 35B40, 82D40, 74B05

1. Introduction and preliminary results

Properties of matter at nanoscale may not be as predictable as those observed at larger scales. Important changes in behavior are caused not only by continuous modification of characteristics with diminishing size, but also by the emergence of totally new phenomena. Designed and controlled fabrication and integration of nanomaterials and nanodevices is likely to be revolutionary for sciences and technology. Among the materials being most actively studied, magnetoelastic materials stand out for being used as actuators for converting electrical energy, or changes in the magnetic field, to mechanical motion. These materials consist of ferromagnetic bodies which are sensible to mechanical stress and deformations. This means that when they are subject to an external field, then mechanical stresses, due to the interaction with the field, arise within the bodies and consequent deformations of the bodies themselves can be observed (magnetorestrictive materials). Viceversa, if one deforms a ferromagnetic bodies, the consequent mechanical stress affects the state of magnetization of the body. On other words, there is interaction between magnetic and elastic processes. For the theory of magnetoelastic processes we refer for example to [2], [6]. Treatments on micromagnetics are available in [1], [5].

In this paper we are concerned with the passage from 2D to 1D in the theory of thin magnetoelastic films. Our investigation has its starting point in the work of Valente [9], where the author proposed a two-dimensional evolutive model and established the existence of weak solutions. We intend to analyze the behavior of these solutions with one diminishing edge. In order to identify the limit problem we make use of the scaling techniques which are well known in elasticity, see for example Ciarlet [3], Ciarlet-Destuynder [4].

Let us now describe the model equations. We consider a bounded open set Ω of $\mathbb{R}^2$. The generic point of Ω is denoted by (x_1, x_2). Here and throughout the paper we use bold characters to denote vector-valued functions. The calculations combine the phenomenological constitutive equations for the magnetization $\boldsymbol{M}$ and the displacement W. The nonlinear parabolic hyperbolic coupled system describing the dynamics is given by (see [9])

$$\begin{cases} \gamma^{-1}\partial_t \boldsymbol{M} - \boldsymbol{M} \times (a\Delta \boldsymbol{M} - \partial_t \boldsymbol{M} - \lambda \boldsymbol{V}) = 0, \\ \varrho \partial_{tt} W - \tau \Delta W - \lambda(\partial_{x_1}(M_1 M_3) + \partial_{x_2}(M_2 M_3)) = 0 \end{cases} \tag{1}$$

in $Q = (0,T) \times \Omega$, where the vector $\boldsymbol{V}$ is given by

$$\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{M}, \nabla W) = (M_3 \partial_{x_1} W, M_3 \partial_{x_2} W, M_1 \partial_{x_1} W + M_2 \partial_{x_2} W).$$

The first equation in (1), well known in literature, is the modified Landau-Lifshitz-Gilbert (LLG) equation. The modification lies in the presence of the term $\lambda \boldsymbol{V}$. The unknown $\boldsymbol{M}$, the magnetization vector, is a map from Ω to $\mathcal{S}^2$ (the unit sphere of $\mathbb{R}^3$). The symbol $\times$ denotes the vector cross product in $\mathbb{R}^3$. Moreover, we denote by M_i, $i = 1, 2, 3$ the components of $\boldsymbol{M}$. The constant $\gamma > 0$ represents the damping parameter while $a > 0$ is the exchange coefficient. The second equation in (1) describes the evolution of the displacement W. The parameters ϱ, λ, and τ are three positive constants. As the initial and boundary conditions we assume

$$W(0,\cdot) = W_0, \;\; \partial_t W(0,\cdot) = W_1, \;\; \boldsymbol{M}(0,\cdot) = M_0, \;\; |\boldsymbol{M}_0| = 1 \;\text{ in } \Omega, \tag{2}$$

$$W = 0, \;\; \partial_\nu \boldsymbol{M} = 0 \;\text{ on } \Sigma = (0,T) \times \partial\Omega, \tag{3}$$

where ν is the outer unit normal at the boundary $\partial\Omega$.

We introduce the functional $\mathcal{E}(t)$ defined as

$$\mathcal{E}(t) = \frac{a}{2} \int_\Omega |\nabla \boldsymbol{M}|^2 \,\mathrm{d}\Omega + \frac{\tau}{4} \int_\Omega |\nabla W|^2 \,\mathrm{d}\Omega + \frac{\varrho}{2} \int_\Omega |\partial_t W|^2 \,\mathrm{d}\Omega \tag{4}$$

and put

$$\mathcal{E}(0)=\frac{a}{2}\int_\Omega|\nabla \boldsymbol{M}_0|^2\,\mathrm{d}\Omega+\frac{\tau}{4}\int_\Omega|\nabla W_0|^2\,\mathrm{d}\Omega+\frac{\varrho}{2}\int_\Omega|W_1|^2\,\mathrm{d}\Omega. \tag{5}$$

The following result has been proved (see [9]).

Theorem 1 ([9])**.** *Given $W_0\in H^1(\Omega)$, $W_1\in L^2(\Omega)$, and $\boldsymbol{M}_0\in H^1(\Omega)$ with $|\boldsymbol{M}_0|=1$ a.e. in Ω, there exists a weak solution $(\boldsymbol{M},W)$ to the problem (1)–(3) in the sense that*

- *$\boldsymbol{M}\in H^1(Q)$ with $|\boldsymbol{M}|=1$ a.e. in Q, $W\in L^2(0,T;H^1_0(\Omega))$ and $\partial_t W\in L^2(0,T;L^2(\Omega))$;*
- *for each couple $(\boldsymbol{p},g)$ such that $\boldsymbol{p}\in C^\infty(\overline{Q})$ vanishes at $t=0$ and $t=T$, and $g\in H^1(Q)\cap C_0(Q)$, one has*

$$\int_Q\Big(\gamma^{-1}\partial_t\boldsymbol{M}\cdot\boldsymbol{p}+a\sum_{j=1}^{2}\boldsymbol{M}\times\partial_{x_j}\boldsymbol{M}\cdot\partial_{x_j}\boldsymbol{p} \tag{6}$$
$$+(\boldsymbol{M}\times\partial_t\boldsymbol{M})\cdot\boldsymbol{p}+\lambda(\boldsymbol{M}\times\boldsymbol{V})\cdot\boldsymbol{p}\Big)\,\mathrm{d}\Omega\,\mathrm{d}t=0,$$

$$\int_Q(-\varrho\partial_t W\partial_t g+\tau\nabla W\cdot\nabla g+\lambda M_3(M_1\partial_{x_1}g+M_2\partial_{x_2}g))\,\mathrm{d}\Omega\,\mathrm{d}t=0, \tag{7}$$

where the dot product operation denotes the Euclidean scalar product on $\mathbb{R}^3$.

Moreover, there exist two constants c_1 and c_2 such that if $\boldsymbol{M}$ and W are solutions of the problem (1)–(3) the estimate

$$\mathcal{E}(t)+\int_0^t\int_\Omega|\partial_t\boldsymbol{M}|^2\,\mathrm{d}\Omega\,\mathrm{d}t\leqslant c_1\mathcal{E}(0)+c_2 \tag{8}$$

holds for all $t\in(0,T)$.

The rest of the paper is organized as follows. In the next section we consider the dimensional reduction from 2D to 1D. We introduce the natural scaling for the problem and prove uniform bounds for the solutions, with respect to the vanishing parameter, which allows us to identify the limit problem. The last section concludes the paper and provides future directions for this work.

2. Dimensional reduction from 2D to 1D

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. We consider flat magnetoelastic domains represented by $\Omega^\varepsilon = (0,1) \times (0,\varepsilon)$. We shall be interested in getting the asymptotic behavior of the solutions when $\varepsilon \to 0$.

2.1. Scaling and uniform bounds

Let $(\boldsymbol{M}, W)$ be a solution of the problem posed in Ω^ε. We introduce the change of variables $(x_1, x_2) = (x, \varepsilon y)$ with $(x, y) = X \in \Omega = (0,1) \times (0,1)$. For functions $\boldsymbol{R}(x_1, x_2)$ and $S(x_1, x_2)$ defined in Ω^ε we introduce the functions $\boldsymbol{r}^\varepsilon(x,y)$ and $s^\varepsilon(x,y)$ defined on Ω by setting

$$\boldsymbol{R}(x_1, x_2) = \boldsymbol{r}^\varepsilon(x,y); \quad S(x_1,x_2) = s^\varepsilon(x,y). \tag{9}$$

Let $(\boldsymbol{m}^\varepsilon, w^\varepsilon)$ be the fields associated with $(\boldsymbol{M}, W)$. The scaled equations satisfied by $(\boldsymbol{m}^\varepsilon, w^\varepsilon)$ are

$$\gamma^{-1}\partial_t \boldsymbol{m}^\varepsilon - \boldsymbol{m}^\varepsilon \times \Big(a\Big(\partial_{xx}\boldsymbol{m}^\varepsilon + \frac{1}{\varepsilon^2}\partial_{yy}\boldsymbol{m}^\varepsilon\Big) - \partial_t \boldsymbol{m}^\varepsilon - \lambda \widetilde{\boldsymbol{V}}^\varepsilon\Big) = 0, \tag{10}$$

$$\varrho \partial_{tt} w^\varepsilon - \tau\Big(\partial_{xx} w^\varepsilon + \frac{1}{\varepsilon^2}\partial_{yy} w^\varepsilon\Big) - \lambda \partial_x(m_1^\varepsilon m_3^\varepsilon) - \frac{\lambda}{\varepsilon}\partial_y(m_2^\varepsilon m_3^\varepsilon) = 0, \tag{11}$$

where $\widetilde{\boldsymbol{V}}^\varepsilon$ is the vector defined by

$$\widetilde{\boldsymbol{V}}^\varepsilon = \Big(m_3^\varepsilon \partial_x w^\varepsilon, \frac{1}{\varepsilon} m_3^\varepsilon \partial_y w^\varepsilon, m_1^\varepsilon \partial_x w^\varepsilon + \frac{1}{\varepsilon} m_2^\varepsilon \partial_y w^\varepsilon\Big). \tag{12}$$

The associated energy $\mathcal{E}^\varepsilon(t)$, defined in (4), becomes

$$\begin{aligned} \mathcal{E}^\varepsilon(t) = \frac{a}{2}\int_\Omega |\partial_x \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}\Omega + \frac{a}{2\varepsilon^2}\int_\Omega |\partial_y \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}\Omega + \frac{\tau}{4\varepsilon^2}\int_\Omega |\partial_y w^\varepsilon|^2 \,\mathrm{d}\Omega \\ + \frac{\tau}{4}\int_\Omega |\partial_x w^\varepsilon|^2 \,\mathrm{d}\Omega + \frac{\varrho}{2}\int_\Omega |\partial_t w^\varepsilon|^2 \,\mathrm{d}\Omega. \end{aligned} \tag{13}$$

The energy equation as well as the saturation constraint on magnetization (see (2)) remain unchanged which is written as

$$|\boldsymbol{m}^\varepsilon(t,X)|^2 = |\boldsymbol{m}_0^\varepsilon(X)|^2 = 1 \tag{14}$$

for almost every (t, X). The following estimates hold true for all $t \geqslant 0$:

$$\mathcal{E}^\varepsilon(t) + \int_0^t \int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \,\mathrm{d}\Omega\,\mathrm{d}t \leqslant c_1 \mathcal{E}^\varepsilon(0) + c_2. \tag{15}$$

To get uniform bounds for the solutions we discuss the admissibility criterion for the initial data. An initial data $(\boldsymbol{m}_0^\varepsilon, w_0^\varepsilon)$ are said to be admissible if we have

$$\mathcal{E}^\varepsilon(0) < \infty. \tag{16}$$

The admissibility criterion reads

$$\begin{aligned} \frac{a}{2}\int_\Omega |\partial_x \boldsymbol{m}_0^\varepsilon|^2 \,\mathrm{d}\Omega + \frac{a}{2\varepsilon^2}\int_\Omega |\partial_y \boldsymbol{m}_0^\varepsilon|^2 \,\mathrm{d}\Omega + \frac{\tau}{4\varepsilon^2}\int_\Omega |\partial_y w_0^\varepsilon|^2 \,\mathrm{d}\Omega \\ + \frac{\tau}{4}\int_\Omega |\partial_x w_0^\varepsilon|^2 \,\mathrm{d}\Omega + \frac{\varrho}{2}\int_\Omega |w_1^\varepsilon|^2 \,\mathrm{d}\Omega < \infty. \end{aligned} \tag{17}$$

Thus, since $|\boldsymbol{m}_0^\varepsilon|^2 = 1$ a.e., to satisfy the criterion we assume that there exists $C > 0$ independent of ε such that

$$\begin{cases} |\partial_x \boldsymbol{m}_0^\varepsilon|_{L^2(\Omega)} \leqslant C, \ \ |\partial_y \boldsymbol{m}_0^\varepsilon|_{L^2(\Omega)} \leqslant C\varepsilon, \ \ |\boldsymbol{m}_0^\varepsilon(x,y)|^2 = 1 \text{ a.e.}, \\ |\partial_x w_0^\varepsilon|_{L^2(\Omega)} \leqslant C, \ \ |\partial_y w_0^\varepsilon|_{L^2(\Omega)} \leqslant C\varepsilon, \ \ |w_1^\varepsilon|_{L^2(\Omega)} \leqslant C. \end{cases} \tag{18}$$

Condition (18) means that the couple $(\boldsymbol{m}_0^\varepsilon, w_0^\varepsilon)$ is essentially independent of the variable y and its strong limit $(\boldsymbol{m}_0, w_0)$ is independent of y.

Remark 1. If the initial data are not admissible, then we expect that the initial layer occurs when ε tends to zero.

2.2. Passing to the limit

Let $(\boldsymbol{m}^\varepsilon, w^\varepsilon)$ be a solution of the problem associated with an admissible initial data $(\boldsymbol{m}_0^\varepsilon, w_0^\varepsilon)$. We have

$$\begin{cases} \boldsymbol{m}_0^\varepsilon \rightharpoonup \boldsymbol{m}_0 \text{ weakly-}* \text{ in } L^\infty(\Omega) \text{ and weakly in } H^1(\Omega), \\ w_0^\varepsilon \rightharpoonup w_0 \text{ weakly in } H^1(\Omega). \end{cases} \tag{19}$$

Moreover, $\boldsymbol{m}_0(x,y) = \boldsymbol{m}_0(x)$ is independent of y. For subsequences, the solutions verify the convergences

$$\begin{cases} \boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m} \text{ weakly-}* \text{ in } L^\infty(\mathbb{R}^+ \times \Omega) \cap L^\infty(\mathbb{R}^+, H^1(\Omega)), \\ w^\varepsilon \rightharpoonup w \text{ weakly in } L^2(0,T,H_0^1(\Omega)) \end{cases} \tag{20}$$

and

$$\begin{cases} \partial_y \boldsymbol{m}^\varepsilon \to 0 \text{ strongly in } L^\infty(\mathbb{R}^+, L^2(\Omega)), \\ \partial_y w^\varepsilon \to 0 \text{ strongly in } L^\infty(\mathbb{R}^+, L^2(\Omega)), \\ \partial_t \boldsymbol{m}^\varepsilon \rightharpoonup \partial_t \boldsymbol{m} \text{ weakly in } L^2(\mathbb{R}^+, L^2(\Omega)), \\ \partial_t w^\varepsilon \rightharpoonup \partial_t w \text{ weakly in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{21}$$

Hence, the couple $(\boldsymbol{m}, w)$ is independent of the variable y. By Aubin's compactness results, we have

$$(\boldsymbol{m}^\varepsilon, w^\varepsilon) \to (\boldsymbol{m}, w) \text{ strongly in } L^2_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega)). \tag{22}$$

Moreover, the Sobolev imbedding theorem $W^{1,2}(Q) \to L^q(Q)$ $(2 \leqslant q \leqslant 6)$ yields the compactness result

$$m_i^\varepsilon m_j^\varepsilon \to m_i m_j \text{ strongly in } L^2(Q), \quad i,j = 1,2,3. \tag{23}$$

Recall that $Q = (0,T) \times \Omega$ with $\Omega = (0,1) \times (0,1)$.

In order to pass to the limit we look at the variational formulation of the scaled problem (10)–(11) by using oscillating test functions. Let $\boldsymbol{\psi}^\varepsilon(t,x,y)$ and $g^\varepsilon(t,x,y)$ be regular test functions depending on ε. Multiplying equation (10) by $\boldsymbol{\psi}^\varepsilon$, equation (11) by g^ε and integrating by parts, we get the weak formulations

$$\begin{aligned}\gamma^{-1}\int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t &+ \int_Q (\boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon) \cdot \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \\ &= -\int_Q a(\boldsymbol{m}^\varepsilon \times \partial_x \boldsymbol{m}^\varepsilon) \cdot \partial_x \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \\ &\quad - \frac{1}{\varepsilon^2}\int_Q a(\boldsymbol{m}^\varepsilon \times \partial_y \boldsymbol{m}^\varepsilon) \cdot \partial_y \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t - \lambda \int_Q \boldsymbol{m}^\varepsilon \times \widetilde{\boldsymbol{V}}^\varepsilon \cdot \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t\end{aligned} \tag{24}$$

and

$$\begin{aligned}-\varrho\int_Q \partial_t w^\varepsilon \partial_t g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \tau\int_Q \partial_x w^\varepsilon \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \frac{\tau}{\varepsilon^2}\int_Q \partial_y w^\varepsilon \partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t& \\ + \lambda\int_Q (m_1^\varepsilon m_3^\varepsilon)\partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t + \frac{\lambda}{\varepsilon}\int_Q (m_2^\varepsilon m_3^\varepsilon)\partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t &= 0.\end{aligned} \tag{25}$$

To pass to the limit we need the following convergence result:

Lemma 1. *Define* $\Theta^\varepsilon := \varepsilon^{-1}\partial_y w^\varepsilon$. *Then*

$$\Theta^\varepsilon \rightharpoonup \Theta = -\frac{\lambda}{\tau} m_2 m_3 + K \textit{ weakly-}*, \tag{26}$$

where K *is a function of the variable* x.

Proof. We multiply (11) first by ε then by a function $g \in \mathcal{D}(Q)$. Integrating by parts, one gets

$$\begin{aligned}\varepsilon\Big(-\varrho\int_Q \partial_t w^\varepsilon \partial_t g \,\mathrm{d}\Omega\,\mathrm{d}t + \tau\int_Q \partial_x w^\varepsilon \partial_x g \,\mathrm{d}\Omega\,\mathrm{d}t + \lambda\int_Q (m_1^\varepsilon m_3^\varepsilon)\partial_x g \,\mathrm{d}\Omega\,\mathrm{d}t\Big)& \\ + \int_Q \Big(\frac{\tau}{\varepsilon}\partial_y w^\varepsilon + \lambda(m_2^\varepsilon m_3^\varepsilon)\Big)\partial_y g \,\mathrm{d}\Omega\,\mathrm{d}t &= 0.\end{aligned} \tag{27}$$

Hence, passing to the limit, by using convergences (21), (22), and (23), we deduce that the weak-$*$ limit Θ of the sequence Θ^ε satisfies $\partial_y(\tau\Theta + \lambda m_2 m_3) = 0$, which allows to get (26). $\square$

R e m a r k 2. In the sequel and without loss of generality we will assume that $K \equiv 0$.

Now we are able to pass to the limit. We set $Q_{\mathcal{T}} = \mathbb{R}^+ \times (0,1)$. We choose in the above weak formulations test functions of the form

$$\begin{cases} \boldsymbol{\psi}^\varepsilon(t,x,y) = \boldsymbol{\psi}_0(t,x) + \varepsilon\boldsymbol{\psi}(t,x,\varepsilon y), \\ g^\varepsilon(t,x,y) = g_0(t,x) + \varepsilon g(t,x,\varepsilon y). \end{cases} \tag{28}$$

We pass to the limit in each term of (24) by using the convergence results (21), (22), and (23). Hence, we first get

$$\begin{gathered} \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to \int_{Q_{\mathcal{T}}} \partial_t \boldsymbol{m} \cdot \boldsymbol{\psi}_0 \,\mathrm{d}x\,\mathrm{d}t, \\ \int_Q \partial_t(\boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon) \cdot \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to \int_{Q_{\mathcal{T}}} (\boldsymbol{m} \times \partial_t \boldsymbol{m}) \cdot \boldsymbol{\psi}_0 \,\mathrm{d}x\,\mathrm{d}t. \end{gathered} \tag{29}$$

Next, we have

$$\int_Q a(\boldsymbol{m}^\varepsilon \times \partial_x \boldsymbol{m}^\varepsilon) \cdot \partial_x \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to \int_{Q_{\mathcal{T}}} (\boldsymbol{m} \times \partial_x \boldsymbol{m}) \cdot \partial_x \boldsymbol{\psi}_0 \,\mathrm{d}x\,\mathrm{d}t. \tag{30}$$

We also get

$$\frac{1}{\varepsilon^2} \int_Q a(\boldsymbol{m}^\varepsilon \times \partial_y \boldsymbol{m}^\varepsilon) \cdot \partial_y \boldsymbol{\psi}^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to 0. \tag{31}$$

Recall that

$$\widetilde{\boldsymbol{V}}^\varepsilon = \Big(m_3^\varepsilon \partial_x w^\varepsilon, \frac{1}{\varepsilon} m_3^\varepsilon \partial_y w^\varepsilon, m_1^\varepsilon \partial_x w^\varepsilon + \frac{1}{\varepsilon} m_2^\varepsilon \partial_y w^\varepsilon \Big). \tag{32}$$

We pass to the limit in the last term of (24) by using the convergence of Lemma 1. In passing to the limit, we use the following facts: $\partial_y \boldsymbol{\psi}^\varepsilon = \varepsilon^2 (\partial_y \boldsymbol{\psi}^\varepsilon)(\varepsilon y)$ and $\partial_x \boldsymbol{\psi}^\varepsilon = \partial_x \boldsymbol{\psi}_0 + \varepsilon \partial_x \boldsymbol{\psi}(\varepsilon y)$.

Similarly we pass to the limit in the weak formulation (25). The convergences (21) and (22) allow to get

$$\begin{gathered} \int_Q \partial_t w^\varepsilon \partial_t g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to \int_{Q_{\mathcal{T}}} \partial_t w \partial_t g_0 \,\mathrm{d}x\,\mathrm{d}t, \\ \int_Q \partial_x w^\varepsilon \partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to \int_{Q_{\mathcal{T}}} \partial_x w \partial_x g_0 \,\mathrm{d}x\,\mathrm{d}t. \end{gathered} \tag{33}$$

Next, we have

$$\int_Q (m_1^\varepsilon m_3^\varepsilon)\partial_x g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to \int_{Q_{\mathcal{T}}} m_1 m_3 \partial_x g_0 \,\mathrm{d}x\,\mathrm{d}t. \tag{34}$$

We also get

$$\frac{1}{\varepsilon^2}\int_Q \partial_y w^\varepsilon \partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to 0, \tag{35}$$

and

$$\frac{1}{\varepsilon}\int_Q (m_2^\varepsilon m_3^\varepsilon)\partial_y g^\varepsilon \,\mathrm{d}\Omega\,\mathrm{d}t \to 0. \tag{36}$$

We have proved the result.

Theorem 2. *Let $(\boldsymbol{m}^\varepsilon, w^\varepsilon)$ be a solution of the problem associated with the admissible initial data $(\boldsymbol{m}_0^\varepsilon, w_0^\varepsilon)$. Then we have $(\boldsymbol{m}^\varepsilon, w^\varepsilon) \to (\boldsymbol{m}, w)$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega))$, $\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m}$ weakly-$*$ in $L^\infty(\mathbb{R}^+, H^1(\Omega))$ and $w^\varepsilon \rightharpoonup w$ weakly in $L^2(\mathbb{R}^+, H_0^1(\Omega))$. The couple $(\boldsymbol{m}, w)$ is independent of the variable y and in $\mathbb{R}^+ \times (0,1)$ satisfies $|\boldsymbol{m}(t,x)|^2 = 1$ and the one dimensional coupled system*

$$\begin{cases} \gamma^{-1}\partial_t \boldsymbol{m} + \boldsymbol{m}\times\partial_t\boldsymbol{m} = -\boldsymbol{m}\times(a\partial_{xx}\boldsymbol{m} + \lambda\widetilde{\boldsymbol{V}}), \\ \varrho\partial_{tt}w - \tau\partial_{xx}w - \lambda\partial_x(m_1 m_3) = 0, \end{cases} \tag{37}$$

where

$$\widetilde{\boldsymbol{V}} = \Big(m_3\partial_x w, -\frac{\lambda}{\tau}m_2 m_3^2, m_1\partial_x w - \frac{\lambda}{\tau}m_2^2 m_3\Big). \tag{38}$$

The associated initial and boundary conditions are given by

$$w(0,x) = w_0,\ \ \partial_t w(0,x) = w_1,\ \ \boldsymbol{m}(0,x) = \boldsymbol{m}_0,\ \ |\boldsymbol{m}_0| = 1 \text{ in } (0,1), \tag{39}$$

$$w(t,j) = 0,\ \ \partial_x\boldsymbol{m}(t,j) = 0 \text{ for } j = 0,1, \tag{40}$$

where w_1 is the weak limit of w_1^ε in $L^2(\Omega)$.

3. Concluding remarks

The limiting behavior obtained in this work concerns the simplified two-dimensional system. It can be used as a toy model for introducing the mathematical approach which can be adapted to more realistic models. It would be interesting to consider the general model which consists of the three-dimensional case with total energy (see [9])

$$\begin{aligned}(41)\qquad \mathcal{E}(t) = \frac{1}{2}\int_\Omega & a|\nabla \boldsymbol{M}|^2 + \tau_1|\nabla U|^2 + \tau_2(\operatorname{div} U)^2 \\ &+ \lambda_1\delta_{klij}\nabla_i U_j M_k M_l + \lambda_2|\boldsymbol{M}|^2 \operatorname{div} U + 2\lambda_3(\nabla U_i \cdot \boldsymbol{M})M_i,\end{aligned}$$

where $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. The parameters τ_1, τ_2, λ_1, λ_2, and λ_3 are positive constants. As a direction for future research one may try to establish an existence result for the last model and justify classical dimensional reductions. We finally mention that the effect of roughness on magnetoelastic materials is also of interest.

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